Exponential mixing for randomly forced nonlinear Schrödinger equations

Shengquan Xiang

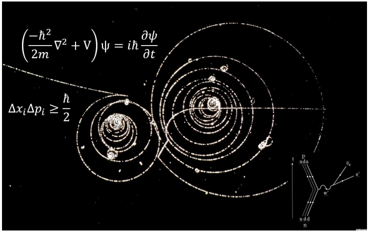
Joint work with Y. Chen, Z. Zhang and J.-C. Zhao

Quantissima sur Oise, Cergy 2025

北京大學

Motivation

Mixing for random dispersive PDEs: wave, KdV, Schrödinger and others?



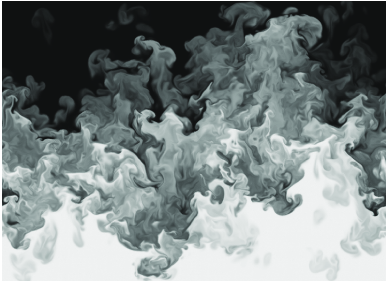
Outline of the presentation

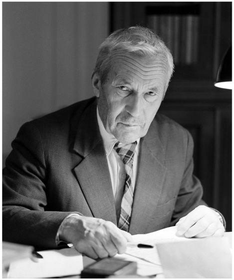
1. Introduction and results
2. An improved probabilistic framework
3. Exponential mixing for NLS

Turbulence and randomness

Statistical behaviors of Navier–Stokes equation:

$$\partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = \eta^\omega$$





"Whether a random process has a unique attracting equilibrium?"

$$\partial_t u - \nu\Delta u + u \cdot \nabla u + \nabla p = \eta^\omega$$

Definition (Mixing)

Markov process $(u(t))$ is *mixing* if

① there is a unique invariant measure:

$$\exists\,!\ \mu \in \mathcal{P}(X) \quad \text{s.t.} \quad \mathcal{D}(u(t)) = \mu \quad \text{if} \quad \mathcal{D}(u(0)) = \mu$$

② convergence to this measure:

$$\mathcal{D}(u(t)) \to \mu \quad \text{as} \quad t \to \infty \quad \text{for any} \quad u(0) = u_0 \in X$$

If the convergence is exponential, then *exponential mixing*

Quick review

$$\partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = \eta^\omega$$

- Full white noise: Flandoli–Maslowski (1995)

$$\eta^\omega(t) = \sum_{j=1}^{\infty} b_j e_j \beta_j(t), \quad e_j : \text{ Fourier modes}$$

- Degenerate white noise:

E–Mattingly–Sinai (2001), Mattingly (2002), Kuksin–Shirikyan (2001)

$$\eta^\omega(t) = \sum_{j=1}^{N} b_j e_j \beta_j(t), \quad N = N(\nu)$$

Quick review

$$\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p = \eta^\omega$$

- Highly degenerate white noise: Hairer–Mattingly (2006, 2008, 2011)

$$\eta^\omega(t) = \sum_{j=1}^{4} b_j e_j \beta_j(t)$$

- Highly degenerate bounded noise: Kuksin–Nersesyan–Shirikyan (2020)

$$\eta^\omega(t) = \sum_{j=1}^{4} b_j e_j \xi_j^\omega(t)$$

- Localized bounded noise: Shirikyan (2015, 2021)

$$\eta^\omega(t) = \chi(x) \sum_{j=1}^{N} b_j e_j \xi_j^\omega(t)$$

Comparing with parabolic PDEs, few result to dispersive PDEs.

- 3D damped wave, exponential mixing (Martirosyan 2014)

$$\partial_{tt}u - \Delta u + a\partial_t u + |u|^{2-\varepsilon}u = \eta^\omega, \quad u_0 \in H^{1+\varepsilon}$$

- 1D damped NLS, polynomial mixing (Debussche–Odasso 2005)

$$iu_t + u_{xx} + iau - |u|^{p-1}u = \eta^\omega, x \in (0,1)$$

- Unique invariant measure: Tolomeo, Ekren, Kukavica, Ziane, *et. al.*

Randomly forced waves

Randomly forced wave with localized damping:

$$\partial_{tt} u - \Delta u + a(x)\partial_t u + u^3 = \chi \eta^\omega , \quad x \in D \subset \mathbb{R}^3$$

- damping can be localized, sharp

  noise reduction, design optimization

- noise can be localized

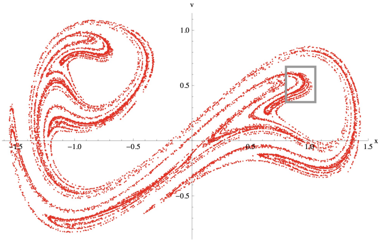
  acoustics, economy, radar

- bounded noise

  stochastic resonance, viscoelastic noise-induced transition

(Duffing equation)

$$\partial_{tt} x + \alpha x + \delta \partial_t x + \beta x^3 = \gamma \cos(\omega t)$$



Main theorem (Liu–Wei–X.–Zhang–Zhao 2024)

Assume $\eta^\omega(t,x)$ *is bounded in* $L^\infty H^2$, *statistically* T*-periodic.*

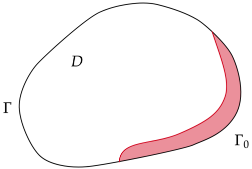
$$\partial_{tt}u - \Delta u + a(x)\partial_t u + u^3 = \chi\eta^\omega$$

Then $\exists\,!$ *invariant measure* μ_* *s.t.* $\forall\, u[0]$ *with law* $\nu \in \mathcal{P}(\mathcal{H})$

$$\|\mathcal{D}(u[t]) - \mu_*\|_L^* \lesssim e^{-\gamma t}\left(1 + \int_{\mathcal{H}} \|v\|^2 \nu(dv)\right)$$

- Geometry condition:

$a(x) \geq a_0 > 0$ near Γ_0



- Noise structure:

$$\eta^\omega = \eta_n^\omega \text{ on } [nT,(n+1)T)$$
$$\eta_n^\omega = \sum_{j,k} b_{jk}\theta_{jk}^n e_j\alpha_k$$

$$\begin{cases} \theta_{jk}^n \text{ indep bounded r.v.s} \\ e_j(x)\alpha_k(t) \text{ space-time modes} \\ b_{jk} \neq 0 \quad \forall\, j,k \leq N \end{cases}$$

Main theorem (Chen–X.–Zhang–Zhao 2025)

Assume $\eta^\omega(t,x)$ is bounded in $L^\infty H^2$, statistically T-periodic.

$$iu_t + u_{xx} + ia(x)u - |u|^{p-1}u = \chi\eta^\omega$$

Then $\exists\,!$ invariant μ_* s.t. $\forall\, u(0)$ with law $\nu \in \mathcal{P}(H^1)$

$$\|\mathcal{D}(u[t]) - \mu_*\|_L^* \lesssim e^{-\gamma t}\left(1 + \int_{H^1} \|v\|^2 \nu(dv)\right)$$

- first exponential mixing result
- $\mathbb{T}^1$ domain, p is odd
- expo. mixing in H^s with $s > 1$ also holds

Outline of the presentation

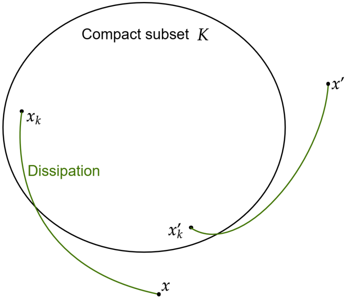
Idea for Navier-Stokes' mixing

- A probabilistic result: Harris-type framework
- Two deterministic properties: stability, controllability

Harris-type framework for Navier-Stokes

compactness + irreducibility + coupling condition

(RDS) $$x_n = F(x_{n-1}, \eta_n) \quad \text{with} \quad x_0 = x \in X$$



- Compact absorbing set K:

$$x_n \in K, \quad \forall\, n \geq k$$

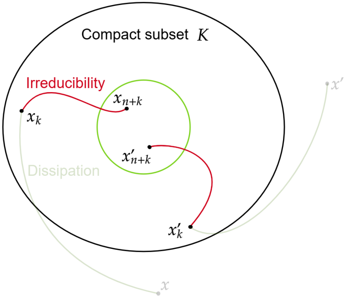
Navier-Stokes: a consequence of

smoothing

Harris-type framework for Navier-Stokes

compactness + **irreducibility** + coupling condition

(RDS) $$x_n = F(x_{n-1}, \eta_n) \quad \text{with} \quad x_0 = x \in K$$



- Irreducibility: stability property

  For any $\varepsilon > 0$, there exists n, p s.t.

  $$\mathbb{P}_x(\|x_n\| \leq \varepsilon) \geq p \quad \text{for any } x \in K$$

Global stability:

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0 \end{cases}$$

Can we find $C > 0$ and $\gamma > 0$ such that for any initial condition x_0?

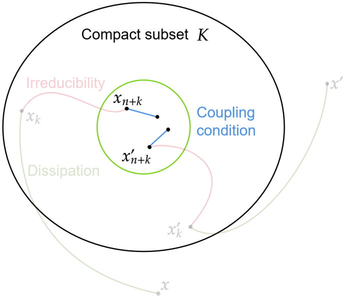
$$\|x(t)\| \leq C\|x_0\|e^{-\gamma t}$$

Straightforward to 2D Navier-Stokes, but can be very challenging to other systems.

Harris-type framework for Navier-Stokes

compactness + irreducibility + **coupling condition**

(RDS) $\qquad x_n = F(x_{n-1}, \eta_n)$ with $x_0 = x \in K$



- Coupling condition: keep converging

$$\mathbb{P}(\|\mathcal{R} - \mathcal{R}'\| > \frac{1}{2}\|x - x'\|) \leq C\|x - x'\|$$

where $(\mathcal{R}, \mathcal{R}')$ is a coupling of (x_1, x_1')

- Coupling condition $\Leftarrow$ control

Foias–Prodi (contraction)
Shirikyan (localized control)

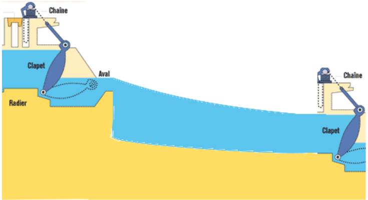
- Related: asymptotic strong Feller

Hairer–Mattingly (Malliavin calculus)

Control problem

"If we can act on a system, what can we make it do?"

$$\dot{x}(t) = f(x(t)) + \eta(t)$$



$$\longrightarrow \begin{cases} \partial_t H + \partial_x(UH) = 0, \\ \partial_t U + \partial_x(\frac{U^2}{2} + gH) - F(x, H, U) = 0, \\ UH(t, 0) = \eta_1(t), \\ UH(t, L) = \eta_2(t) \end{cases}$$

Control: the dams $\rightarrow$ $\eta_1(t)$ and $\eta_2(t)$ are controls that can be chosen

A control property

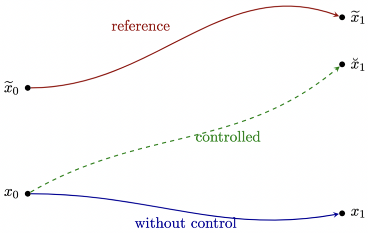
Compare two trajectories $\tilde{x}$ and x:

$$\frac{d}{dt}\tilde{x}(t) = f(\tilde{x}(t)) + h(t) \qquad \text{with } \tilde{x}(0) = \tilde{x}_0,$$

$$\frac{d}{dt}x(t) = f(x(t)) + h(t) + \eta(t) \qquad \text{with } x(0) = x_0,$$

Construct control η:

to make trajectories closer



Comparing with parabolic PDEs, extra challenges to dispersive PDEs:

- no compact absorbing set
- delicate global stability analysis
- loss of derivatives for controllability

Compactness is required...

Controlled system:

$$i\partial_t u + \Delta u + ia(x)u - |u|^{p-1}u = h(t,x) + \chi \mathscr{P}_N \eta(t,x), \quad t \in [0,T]$$

where $h =$ given force $\mathscr{P}_N =$ finite-dim. proj.

Theorem

Let $\tilde{u} = H^{1+s}$-uncontrolled sol. $(\eta \equiv 0)$. Then $\forall\, u(0) \in H^1$ closed to $\tilde{u}(0)$, there is a control η s.t.

$$\|u(T) - \tilde{u}(T)\| \leq \varepsilon \|u(0) - \tilde{u}(0)\|$$

with $u =$ controlled sol.

Compactness from dynamics of dispersive equations

- Parabolic equation bounded domain $\xrightarrow{\text{smoothing}}$ Compactness

Thanks to dynamical systems' view

- Dispersive equations $\xrightarrow{\text{dynamics}}$ Exponential asymptotic compactness

★ Expo. asymptotic compactness (AC): promoting regularity eventually

Example: $\partial_{tt}u - \Delta u + a\partial_t u = f$ with $u[\cdot] = (u, \partial_t u) \in H_0^1 \times L^2$

Exponential asymptotic compactness

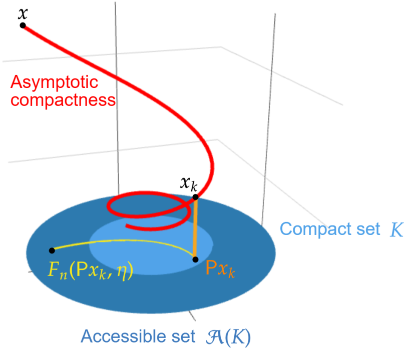
There exists $R > 0$ s.t. $u(t)$ converges to $B_{H^2}(R)$ exponentially in H^1 (not in H^2!)

An improved probabilistic framework

Theorem (Liu–Wei–X.–Zhang–Zhao 2024)

$$\left.\begin{array}{c}\text{Expo. asymptotic compactness}\\ \text{Irreducibility on compact set}\\ \text{Coupling condition on compact set}\end{array}\right\} \Longrightarrow \text{Expo. mixing}$$

- Asymptotic compactness:
  Dynamics with force converge to a compact subset
- Irreducibility on compact set:
  Stability property
- Coupling condition on compact set:
  Control property

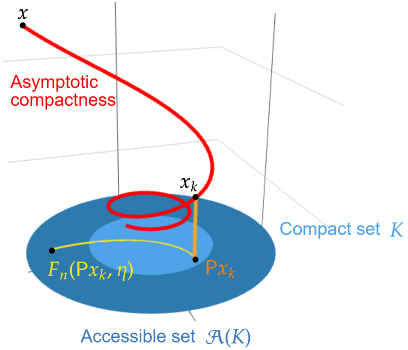


An improved probabilistic framework

Theorem (Liu–Wei–X.–Zhang–Zhao 2024, Chen–Liu–X.–Zhang 2024)

$$\left.\begin{array}{c}\text{Expo. asymptotic compactness}\\ \text{Irreducibility on compact set}\\ \text{Coupling condition on compact set}\end{array}\right\} \Longrightarrow \text{Expo. mixing} + \text{LDP}$$

- Asymptotic compactness:
  Dynamics with force converge to a compact subset
- Irreducibility on compact set:
  Stability property
- Coupling condition on compact set:
  Control property



Outline of the presentation

Part 1. Asymptotic compactness

$$\boxed{iu_t + u_{xx} + ia(x)u = |u|^{p-1}u + f(t,x), \quad u|_{t=0} = u_0 \in H^1}$$

Proposition (Chen–X.–Zhang–Zhao, 2025)

Given $R > 0$ *and* $\sigma \in (0, 1/2)$, $\exists \mathscr{B}_{1+\sigma} \subset H^{1+\sigma}$ *bounded, s.t.*

$$\operatorname{dist}_{H^1}(u(t), \mathscr{B}_{1+\sigma}) \leq C(\|u_0\|_{H^1})e^{-\kappa t},$$

for any $u_0 \in H^1$ *and* $f \in B_{L^\infty H^{1+\sigma}}(R)$.

Part 1. Asymptotic compactness

$$\boxed{iu_t + u_{xx} + ia(x)u = |u|^{p-1}u + f(t,x), \quad u|_{t=0} = u_0 \in H^1}$$

Proposition (Chen–X.–Zhang–Zhao, 2025)

Given $R > 0$ *and* $\sigma \in (0, 1/2)$, $\exists \mathscr{B}_{1+\sigma} \subset H^{1+\sigma}$ *bounded, s.t.*

$$\mathrm{dist}_{H^1}(u(t), \mathscr{B}_{1+\sigma}) \leq C(\|u_0\|_{H^1})e^{-\kappa t},$$

for any $u_0 \in H^1$ *and* $f \in B_{L^\infty H^{1+\sigma}}(R)$.

Idea: Duhamel formula: ($S_a(t)$ =semigroup generated by $i\partial_x^2 - a(x)$)

$$u(t) = S_a(t)u_0 - i\int_0^t S_a(t-s)(|u|^{p-1}u(s) + f(s))\, ds.$$

- Linear spectral gap (Rosier–Zhang 2009): $\|S_a(t)u_0\|_{H^s} \leq Ce^{-\beta t}\|u_0\|_{H^s}$. ✓
- Source term: $\| \int_0^t S_a(t-s)f(s)\|_{H^{1+\sigma}} \leq CR\int_0^t e^{-\beta(t-s)}\, ds \leq CR$. ✓
- Nonlinear effect: $\int_0^t S_a(t-s)(|u|^{p-1}u)\, ds$?

Bourgain space / restricted norm space

$$\|u\|^2_{X^{s,b}} := \|S(-t)u(t)\|_{H^b_t H^s_x} = \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \langle \tau + k^2\rangle^{2b} |\hat{u}(\tau,k)|^2 \, d\tau.$$

Heuristically, $\| \int_0^t S_a(t-s)(|u|^{p-1}u)\, ds\|_{X^{1+\sigma,b}} \leq C\||u|^{p-1}u\|_{X^{1+\sigma,b-1}}$.

Question: Can we trade off spatial and temporal regularity

$$\||u|^{p-1}u\|_{X^{1+\sigma,b-1}} \leq C\|u\|^p_{X^{1,b}} \ ?$$

Bourgain space / restricted norm space

$$\|u\|^2_{X^{s,b}} := \|S(-t)u(t)\|_{H^b_t H^s_x} = \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \langle \tau + k^2\rangle^{2b} |\hat{u}(\tau,k)|^2 \, d\tau.$$

Heuristically, $\| \int_0^t S_a(t-s)(|u|^{p-1}u)\,ds\|_{X^{1+\sigma,b}} \le C\||u|^{p-1}u\|_{X^{1+\sigma,b-1}}$.

Question: Can we trade off spatial and temporal regularity

$$\||u|^{p-1}u\|_{X^{1+\sigma,b-1}} \le C\|u\|^p_{X^{1,b}} \text{ ?}$$

Answer: No. Consider $\mathcal{F}(|u|^2u)(k) = \sum_{k=k_1-k_2+k_3} \hat{u}(k_1)\overline{\hat{u}(k_2)}\hat{u}(k_3)$.

Resonant terms $k=k_1$ or $k=k_3$ sum up to $\|u\|^2_{L^2}u$.

Nonlinear smoothing

Luckily, $\mathrm{NR}(u) :=$ (Non-resonant terms in $|u|^{p-1}u$) behaves better:

Lemma (Erdoğan–Gürel 2013, McConnell 2022, Chen–X.–Zhang–Zhao 2025)

Let $\sigma \in (0, 1/2)$, $b > 1/2$, *then for any* $u \in X^{1,b}$,

$$\| \mathrm{NR}(u) \|_{X^{1+\sigma, b-1}} \leq C \|u\|^p_{X^{1,b}}.$$

Key: Uniform bound on frequencies

$$\sup_k \left[\sum_{k=k_1-k_2+\cdots+k_p,\ \mathrm{NR}} \frac{\langle k \rangle^{2(1+\sigma)}}{\langle k^2 + \sum_{l=1}^p (-1)^l k_l^2 \rangle^{2(1-b)} \prod_{l=1}^p \langle k_l \rangle^2} \right] < \infty$$

Nonlinear smoothing

Luckily, $\mathrm{NR}(u) :=$ (Non-resonant terms in $|u|^{p-1}u$) behaves better:

Lemma (Erdoğan–Gürel 2013, McConnell 2022, Chen–X.–Zhang–Zhao 2025)

Let $\sigma \in (0, 1/2)$, $b > 1/2$, *then for any* $u \in X^{1,b}$,

$$\|\mathrm{NR}(u)\|_{X^{1+\sigma,b-1}} \leq C\|u\|^p_{X^{1,b}}.$$

Key: Uniform bound on frequencies

$$\sup_k \left[\sum_{k=k_1-k_2+\cdots+k_p,\ \mathrm{NR}} \frac{\langle k\rangle^{2(1+\sigma)}}{\langle k^2+\sum_{l=1}^p(-1)^l k_l^2\rangle^{2(1-b)} \prod_{l=1}^p \langle k_l\rangle^2} \right] < \infty$$

Proposition (Nonlinear smoothing, Chen–X.–Zhang–Zhao 2025)

Let $\sigma \in (0, 1/2)$, *then for any* $u_0 \in H^1$ *and* $t \geq t(\|u_0\|_{H^1})$,

$$\|u(t) - e^{i\theta(t)} S_a(t) u_0\|_{H^{1+\sigma}} \leq C(R), \quad \theta(t) \in \mathbb{R}.$$

Idea: Resonance is removed by $e^{-i\theta(t)}u(t,x)$, with appropriate choice of θ.

Part 2: Stability property

$$\boxed{iu_t + u_{xx} + ia(x)u = |u|^{p-1}u, \quad u|_{t=0} = u_0 \in H^s}$$

Theorem

(1) *(Le Balc'h–Martin 2024) If $u_0 \in H^1$ and $p = 3$ (cubic NLS), then*

$$\|u(t)\|_{H^1} \leq Ce^{-\beta t}\|u(0)\|_{H^1}.$$

(2) *(Chen–X.–Zhang–Zhao 2025) Conclusion (1) holds for any odd $p > 3$. Moreover, if $s \geq 1$ and $u_0 \in H^s$, then*

$$\|u(t)\|_{H^s} \leq C(\|u_0\|_{H^s})e^{-\beta' t}.$$

Difficulty: nonlinear observability inequality

$$\int_0^T \int_{\mathbb{T}} a(x)(|u|^2 + |u_x|^2 + |u|^{p+1})\, dxdt \geq C_T E_u(t)$$

Method: global Carleman estimate

Part 3. Control property

$$\boxed{\begin{array}{l} i\partial_t \tilde{u} + \Delta \tilde{u} + ia(x)\tilde{u} - |\tilde{u}|^{p-1}\tilde{u} = f(t,x), \quad \tilde{u}|_{t=0} = \tilde{u}_0; \\ i\partial_t u + \Delta u + ia(x)u - |u|^{p-1}u = f(t,x) + \chi \mathcal{P}_N h(t,x), \quad u|_{t=0} = u_0. \end{array}}$$

$\tilde{u}$: reference solution. u: controlled system. $\mathcal{P}_N$: proj. to first N modes.

Theorem (Chen–X.–Zhang–Zhao 2025)

Let $T > 0$, $R > 0$. *Assume* $\|\tilde{u}\|_{L^\infty H^{1+\sigma}} < R$, *then for any* $u_0 \in H^1$, *if* $\|u_0 - \tilde{u}_0\|_{H^1} \ll 1$, *then* $\exists$ *control* $h \in L^2 H^{1+\sigma}$ *s.t.*

$$\|u(T) - \tilde{u}(T)\|_{H^1} \leq \varepsilon \|u_0 - \tilde{u}_0\|_{H^1}.$$

Frequency analysis

$$\boxed{i\partial_t v + \Delta v + ia(x)v = \frac{p+1}{2}|\tilde{u}|^{p-1}v + \frac{p-1}{2}|\tilde{u}|^{p-3}\tilde{u}^2\bar{v} + \chi(x)\mathcal{P}_N h(t,x)}$$

It suffices to consider the linearized system: $\|v(T)\|_{H^1} \leq \varepsilon \|v_0\|_{H^1}$

> *Dehman–Lebeau: "the energy of each scale of the control force depends (almost) only on the energy of the same scale in the states that one wants to control."*

Low-frequency control

Given $m \in \mathbb{N}$, then $\exists N \in \mathbb{N}$ and control $h \in L^2 H^1$ s.t.

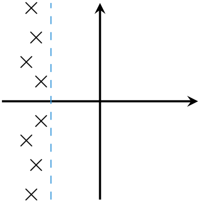
$$P_m v(T) = 0.$$

Idea: Invoke full-frequency null controllability by Laurent (2010).

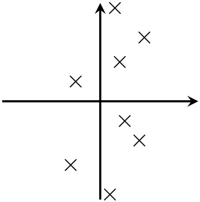
High-frequency dissipation

$$\boxed{i\partial_t v + \Delta v + ia(x)v = \tfrac{p+1}{2}|\tilde{u}|^{p-1}v + \tfrac{p-1}{2}|\tilde{u}|^{p-3}\tilde{u}^2\bar{v}}$$

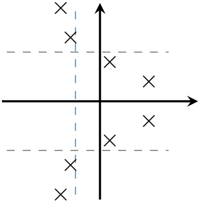
Spectral gap:



$\tilde{u} = 0$ ✓



$\tilde{u} \in L^\infty H^1$ ✗



$\tilde{u} \in L^\infty H^{1+\sigma}$: Hi-freq.

High-frequency dissipation

If $\|\tilde{u}\|_{L^\infty H^{1+\sigma}} \leq R$, then $\exists m \in \mathbb{N}$ s.t.

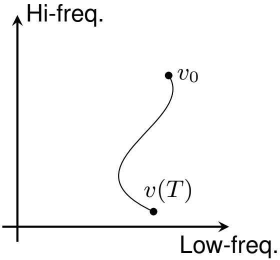
$$\|(I - P_m)v(T)\|_{H^1} \leq \varepsilon \|v_0\|_{H^1}.$$

Idea: exponential asymptotic compactness

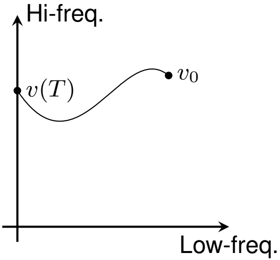
Influence of control on high frequency

How to balance between

- High-frequency dissipation: $h = 0$, $\|(I - P_m)v(T)\|_{H^1} \leq \varepsilon \|v_0\|_{H^1}$
- Low-frequency control: $h \in L^2H^1$, $P_m v(T) = 0$

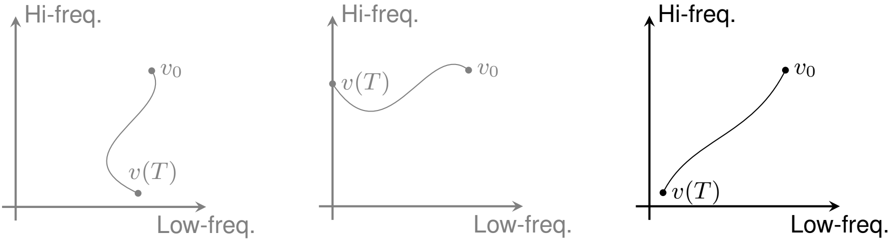


$h = 0$



$h \in L^2H^1$

Difficulty: Control $h \in L^2H^1$ can ruin high-frequency dissipation



$h = 0$ $\quad$ $h \in L^2 H^1$ $\quad$ $h \in L^2 H^{1+\sigma}$ ✓

Theorem (Chen–X.–Zhang–Zhao)

If $\|\tilde{u}\|_{L^\infty H^{1+\sigma}} \leq R$ *, then* $\exists m \in \mathbb{N}$ *and control* $\xi \in L^2 H^{1+\sigma}$*, such that*

$$\|P_m v(T)\|_{H^1} \leq \varepsilon \|v_0\|_{H^1} \quad \text{and} \quad \|(I - P_m) v(T)\|_{H^1} \leq \varepsilon \|v_0\|_{H^1}.$$

A framework

$$\left.\begin{array}{c}\textit{expo. asymptotic compactness}\\ \textit{irreducibility on compact set}\\ \textit{coupling condition on compact set}\end{array}\right\} \Longrightarrow \textit{expo. mixing} + \textit{LDP}$$

Exponential mixing

$$\text{(NLW)} \qquad \partial_{tt}u - \Delta u + a(x)\partial_t u + u^3 = \chi\eta^\omega, \quad x \in D \subset \mathbb{R}^3$$

$$\text{(NLS)} \qquad i\partial_t u + \Delta u + ia(x)u - |u|^{p-1}u = \chi\eta^\omega, \quad x \in \mathbb{T}$$

Features:

- noise can be localized
- damping can be localized, sharp
- general nonlinearity

Thank you!