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Approach to equilibrium in translation-invariant quantum systems

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Zeroth law of thermodynamics

"The zeroth law deals with the observed fact that a large system seems to normally have "states" described by a few macroscopic parameters like a temperature and density, and that any system not in one of these states, left alone, rapidly ***approaches one of these states****. When Boltzmann and Gibbs tried to find a macroscopic basis for thermodynamics, they realized that the* ***approach to equilibrium*** *was the most puzzling and deepest problem in such a formalism."*

from Simon, B.: The Statistical Mechanics of Lattice Gasses.

Heuristic motivation

- Consider an infinite quantum system on a d-dimensional lattice
- At $t = 0$: consider a translation invariant state ρ_0.
  Ex: equilibrium state at inverse temperature β "$\rho_0 = \mathrm{e}^{-\beta H_1}$"
- Evolve it with a distinct dynamics for $t > 0$ "$\tau_t = \mathrm{e}^{\mathrm{i}tH_2} \cdot \mathrm{e}^{-\mathrm{i}tH_2}$"

What happens to the state as $t \to \infty$?

- Equilibrium states for infinite systems are well known: KMS states
- A lot is known when H_2 is a local perturbation: return to equilibrium, non-equilibrium steady states, entropy production...

What happens when the perturbation is infinite, but translation invariant ?

Our program so far

- Formulate the problem
- Develop the structural theory, independent from specific models
- Previously: approach to equilibrium characterized by a strict increase of entropy

  Jaksic, Pillet, T. '24
- Anna's talk: improvement of several conservation laws
- Today's talk: a new characterization of approach to equilibrium, related to constants of motion.
- One outcome: characterization of approach to equilibrium when the system has no constant of motion except the energy (non-integrable models)

Outline

Approach to Equilibrium

Spin algebra, states and entropy

- $\mathcal{F} = \{X \subset \mathbb{Z}^d, |X| < \infty\}$, $\mathcal{U}_X = \mathcal{B}(\mathcal{H}_X)$
- $\mathcal{U} = \overline{\bigcup_{X \in \mathcal{F}} \mathcal{U}_X}$ is the spin C^*-algebra over $\mathbb{Z}^d$
- $\tau_x : \mathcal{U}_X \to \mathcal{U}_{X+x}$ for each $x \in \mathbb{Z}^d$
- $\mathcal{S}_{\mathrm{I}}$ is the set of translation invariant states on $\mathcal{U}$

- Mean specific entropy of $\nu \in \mathcal{S}_{\mathrm{I}}$

$$s(\nu) = -\lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathrm{tr}(\nu_\Lambda \log(\nu_\Lambda))$$

  with $\nu(A) = \mathrm{tr}(\nu_\Lambda A)$ for all $A \in \mathcal{U}_\Lambda$. Always exists, is affine and upper semi-continuous.

- Mean relative entropy of $\nu, \omega \in \mathcal{S}_{\mathrm{I}}$

$$s(\nu|\omega) = -\lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathrm{tr}\big(\nu_\Lambda(\log(\omega_\Lambda) - \log(\nu_\Lambda))\big)$$

  If such a limit exists, then $s(\nu|\omega) \geq 0$.

Interactions, Hamiltonians and Pressure

An interaction is Φ a family $\{\Phi(X)\}_{X\in\mathcal{F}}$ such that $\Phi(X)\in\mathcal{U}_X$ is a self-adjoint. Moreover we assume translation invariance: $\tau_x(\Phi(X))=\Phi(X+x)$.

Big (Banach) space of interactions

$$\mathcal{B}_{\mathrm{b}}=\{\Phi,\ \|\Phi\|_{\mathrm{b}}<\infty\},\qquad \|\Phi\|_{\mathrm{b}}:=\sum_{X\ni 0}\frac{\|\Phi(X)\|}{|X|}.$$

Let $\Phi\in\mathcal{B}_{\mathrm{b}}$

- Local Hamiltonian: $H_\Phi(\Lambda)=\sum_{X\subset\Lambda}\Phi(X)$ (with $\Lambda\in\mathcal{F}$)
- Pressure (Helmholtz free energy) at $\beta>0$:

$$p_\beta(\Phi)=\lim_{\Lambda\to\mathbb{Z}^d}\frac{1}{|\Lambda|}\log(\mathrm{tr}(\mathrm{e}^{-\beta H_\Phi(\Lambda)}))<\infty$$

Equilibrium states

For $\nu \in \mathcal{S}_{\mathrm{I}}$ and $\Phi \in \mathcal{B}_{\mathrm{b}}$, the mean energy reads

$$\lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \nu(H_\Phi(\Lambda)) = \nu(E_\Phi), \qquad E_\Phi = \sum_{X \ni 0} \frac{\Phi(X)}{|X|}$$

Gibbs variational principle

For any $\Phi \in \mathcal{B}_{\mathrm{b}}$ one has

$$p_\beta(\Phi) = \sup_{\nu \in \mathcal{S}_{\mathrm{I}}} \big(s(\nu) - \beta \nu(E_\Phi) \big)$$

Equilibrium states are

$$\mathcal{S}_{\mathrm{eq}}(\beta, \Phi) = \{ \nu \in \mathcal{S}_{\mathrm{I}} \,|\, p_\beta(\Phi) = s(\nu) - \beta \nu(E_\Phi) \}$$

Rk (Dual variational principle): for any $\nu \in \mathcal{S}_{\mathrm{I}}$ one has $s(\nu) = \inf_{\Phi \in \mathcal{B}_{\mathrm{b}}} \big(p_\beta(\Phi) + \beta \nu(E_\Phi) \big)$

Surface energies

Small space of interactions $\mathcal{B}_{\mathrm{s}} \subset \mathcal{B}_{\mathrm{b}}$

$$\mathcal{B}_{\mathrm{s}} = \{\Phi,\, \|\Phi\|_{\mathrm{s}} < \infty\}, \qquad \|\Phi\|_{\mathrm{s}} := \sum_{X \ni 0} \|\Phi(X)\|\,.$$

Proposition

For $\Phi \in \mathcal{B}_{\mathrm{s}}$ and $\Lambda \in \mathcal{F}$ the surface energies

$$W_\Phi(\Lambda) = \lim_{\Lambda' \to \mathbb{Z}^d} (H_\Phi(\Lambda') - H_\Phi(\Lambda) - H_\Phi(\Lambda' \setminus \Lambda)) = \sum_{\substack{X \cap \Lambda \neq \emptyset \\ X \cap \Lambda^c \neq \emptyset}} \Phi(X)$$

exist and satisfy $\lim_{\Lambda \to \mathbb{Z}^d} \dfrac{W_\Phi(\Lambda)}{|\Lambda|} = 0.$

Recall $\mathcal{U}_{\mathrm{loc}} = \bigcup_{X \in \mathcal{F}} \mathcal{U}_X$. For $\Phi \in \mathcal{B}_{\mathrm{s}}$, consider the $*$-derivation $\delta_\Phi : \mathcal{U}_{\mathrm{loc}} \to \mathcal{U}$ defined by

$$\delta_\Phi(A) = \sum_{X \in \mathcal{F}} [\Phi(X), A], \qquad (A \in \mathcal{U}_{\mathrm{loc}}),$$

and denote its closure by $\overline{\delta_\Phi}$.

Theorem

$\overline{\delta_\Phi}$ generates a C^*-dynamics α^t_Φ on $\mathcal{U}$ if and only if $(\mathrm{i} \pm \delta_\Phi)\mathcal{U}_{\mathrm{loc}}$ is dense in $\mathcal{U}$. In that case

$$\alpha^t_\Phi(A) = \lim_{\Lambda \to \mathbb{Z}^d} \mathrm{e}^{\mathrm{i}tH_\Phi(\Lambda)} A \mathrm{e}^{-\mathrm{i}tH_\Phi(\Lambda)}, \qquad A \in \mathcal{U},$$

where the limit is uniform for t compacts.

Definition

$$\mathcal{B}_{\mathrm{sd}} = \{\Phi \in \mathcal{B}_{\mathrm{s}}, \quad \overline{\delta_\Phi} \text{ generates a } C^*\text{-dynamics } \alpha^t_\Phi \text{ on } \mathcal{U}\}$$

$\Phi \in \mathcal{B}_{\mathrm{sd}}$ guarantees the existence of surface energies $W_\Phi(\Lambda)$ and C^*-dynamics α^t_Φ. It contains the usual spaces: $\mathcal{B}_{\mathrm{f}} \subset \mathcal{B}_r \subset \mathcal{B}_{\mathrm{sd}}$ with

$$\mathcal{B}_{\mathrm{f}} = \{\Phi \in \mathcal{B}, \exists r > 0, |X| > r \Rightarrow \Phi(X) = 0\} \qquad \text{(finite range)}$$

and

$$\mathcal{B}_r = \{\Phi \in \mathcal{B}, \|\Phi\|_r < \infty\}, \qquad \|\Phi\|_r = \sum_{X \ni 0} \mathrm{e}^{r(|X|-1)} \|\Phi(X)\| \qquad \text{(short range)}.$$

Prop: For $\Phi \in \mathcal{B}_{\mathrm{sd}}$, $\omega \in \mathcal{S}_{\mathrm{eq}}(\beta, \Phi)$ iff ω is a (β, α_ϕ)-KMS state.

Equilibrium Steady States (ESS)

Let $\omega \in \mathcal{S}_{\mathrm{I}}$ and $\Phi \in \mathcal{B}_{\mathrm{sd}}$. For $T > 0$ define

$$\overline{\omega}_T = \frac{1}{T} \int_0^T \omega \circ \alpha_t^{\Phi} \mathrm{d}t$$

and consider the set of Equilibrium Steady States (ESS) (in contrast to NESS):

$$\mathcal{S}_+(\omega, \Phi) = \{\mathrm{weak} * - \lim(\overline{\omega}_T)_{T>0},\ T \to \infty\}.$$

For $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$ one has $\omega_+ \in \mathcal{S}_{\mathrm{I}}$ and $\omega_+ \circ \alpha_t^{\Phi} = \omega_+$.

Conservation laws

Proposition

For $\Phi \in \mathcal{B}_r$ with $r > 0$ (Jaksic, Pillet, Tauber '24) or $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$ (Anna's talk) one has

$$s(\omega \circ \alpha_\Phi^t) = s(\omega), \qquad \omega \circ \alpha_\Phi^t(E_\Phi) = \omega(E_\phi)$$

for all $t \in \mathbb{R}$ and all $\omega \in \mathcal{S}_{\mathrm{I}}$.

Consequently for $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$, $s(\omega) \leq s(\omega_+)$ and $\omega_+(E_\Phi) = \omega(E_\Phi)$.

E_Φ is a constant of motion of the dynamics.

Approach to Thermal Equilibrium

Let $\omega \in \mathcal{S}_{\mathrm{I}}$ and $\Phi \in \mathcal{B}_{\mathrm{sd}}$, with $e_0 = \omega(E_\Phi)$.

Set, if it exists, β_* such that for $\nu_{\mathrm{eq}} \in \mathcal{S}_{\mathrm{eq}}(\beta_*, \Phi)$ one has $\nu_{\mathrm{eq}}(E_\Phi) = e_0$.

Definition

The pair (ω, Φ) has the property of Approach to Thermal Equilibrium if

- $\mathcal{S}_+(\omega, \Phi) = \{\omega_+\}$
- $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\beta_*, \Phi)$

Main result

Admissible states

Consider $\mathcal{U}_{\mathrm{real}} \subset \mathcal{U}$ the real vector space of self-adjoint elements. $A \in \mathcal{U}_{\mathrm{real}}$ is a constant of motion for $\Phi \in \mathcal{B}_{\mathrm{sd}}$ if

$$\nu \circ \alpha_\Phi^t(A) = \nu(A), \qquad \forall \nu \in \mathcal{S}_{\mathrm{I}}$$

We denote by $C(\alpha_\Phi)$ the set of constants of motion (more structure later).

We say that ω is admissible for $\nu_{\mathrm{eq}} \in \mathcal{S}_{\mathrm{eq}}(\beta_*, \Phi)$ if

$$\omega(A) = \nu_{\mathrm{eq}}(A), \qquad \forall A \in C(\alpha_\Phi).$$

Notice that for E_Φ we always have $\omega(E_\Phi) = \nu_{\mathrm{eq}}(E_\Phi)$ by the choice of β_*.

Regularity

Definition

A pair $(\omega, \Phi) \in \mathcal{S}_{\mathrm{I}} \times \mathcal{B}_{\mathrm{sd}}$ is called regular if the relative entropy $s(\nu|\omega)$ exists for all $\nu \in \mathcal{S}_{\mathrm{I}}$ and satisfies the entropy balance equation:

$$s(\nu|\omega) = -s(\nu) + \nu(E_\Phi) + p(\Phi).$$

- The regular property holds when $\omega \in \mathcal{S}_{\mathrm{eq}}(\Phi)$ with ω weak-Gibbs.
- For $\Phi \in \mathcal{B}_r$ with $\|\Phi\|_r < r$ (or $\Phi \in \mathcal{B}_{\mathrm{f}}$ and $d = 1$) any $\omega \in \mathcal{S}_{\mathrm{eq}}(\Phi)$ is weak-Gibbs, hence regular.

  Jaksic, Pillet, T. '24
- Anna's talk: weak-Gibbs property is preserved along the trajectories
- Entropy balance equation appears everywhere in our proofs.

Main result

Theorem **Jaksic, Pillet, Szczepanek, T. '25**

Let $\omega \in \mathcal{S}_{\mathrm{I}}$ and $\Phi \in \mathcal{B}_{\mathrm{sd}}$. Suppose that $\mathcal{S}_{\mathrm{eq}}(\beta_*, \Phi) = \{\nu_{\mathrm{eq}}\}$ with $(\nu_{\mathrm{eq}}, \Phi)$ regular, $E_\phi \in C(\alpha_\phi)$, and moreover that ω is admissible for ν_{eq}. For $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$, the following statements are equivalent:

1. $\omega_+ = \nu_{\mathrm{eq}}$
2. $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(1, \Psi_+)$ with $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$ such that (ω_+, Ψ_+) is regular, and $E_{\Psi_+} \in C(\alpha_\Phi)$.

Approach to thermal equilibrium is characterized by the existence of some abstract Ψ_+ and its decay properties.

Proof

(2) $\Rightarrow$ (1) Since (ω_+, Ψ_+) is regular one has

$$\begin{aligned} s(\nu_{\mathrm{eq}}|\omega_+) &= -s(\nu_{\mathrm{eq}}) + \nu_{\mathrm{eq}}(E_{\Psi_+}) + P(\Psi_+) && \text{(regularity)} \\ &= -s(\nu_{\mathrm{eq}}) + \omega(E_{\Psi_+}) + P(\Psi_+) && \text{(admissible states)} \\ &= -s(\nu_{\mathrm{eq}}) + \omega_+(E_{\Psi_+}) + P(\Psi_+) && (E_{\Psi_+} \in C(\alpha_\Phi)) \\ &= s(\omega_+) - s(\nu_{\mathrm{eq}}) && (\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\Psi_+)) \end{aligned}$$

from which we infer $s(\omega_+) \geq s(\nu_{\mathrm{eq}})$.

A similar argument shows the converse inequality using the regularity of $(\nu_{\mathrm{eq}}, \Phi)$ and conservation of E_Φ.

From the equalities $s(\omega_+) = s(\nu_{\mathrm{eq}})$ and $\omega_+(E_\Phi) = \nu_{\mathrm{eq}}(E_\phi)$ we infer $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\beta_*, \Phi)$ from Gibbs variational principle.

Dynamical vs structural assumptions

- $\mathcal{S}_{\mathrm{eq}}(\beta_*, \Phi) = \{\nu_{\mathrm{eq}}\}$ with $(\nu_{\mathrm{eq}}, \Phi)$ regular, and $E_\Phi \in C(\alpha_\Phi)$ are easy to satisfy by taking Φ nice and small.
- The admissibility of ω with respect to ν_{eq} is a physical constraint.
- $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(1, \Psi_+)$ with $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$ has to be established for specific models (dynamical problem).
  If Ψ_+ does not exist, approach to thermal equilibrium is not possible
- (ω_+, Ψ_+) regular relies either on a specific model (dynamical) or on general properties of quantum spin on a lattice (structural)
  Can be dropped if the system has only one constant of motion (see below)
- $E_{\Psi_+} \in C(\alpha_\Phi)$ can be formulated based on some structural results (see below)

Structural theory

Commuting dynamics

$$\omega_+ \in \mathcal{S}_+(\omega, \Phi) \cap \mathcal{S}_{\mathrm{eq}}(1, \Psi_+)$$

$\omega_+ \in \mathcal{S}_{\mathrm{eq}}(1, \Psi_+)$ with $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$ implies that ω_+ is a KMS state for dynamics $t \mapsto \alpha^t_{\Psi_+}$

Moreover $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$ is stationary with respect to α_Φ, so it is also a KMS state for the dynamics $t \mapsto \alpha^s_\Phi \circ \alpha^t_{\Psi_+} \circ \alpha^{-s}_\Phi$ for each fixed $s \in \mathbb{R}$. Consequently

$$\forall t, s \in \mathbb{R}, \qquad \alpha^s_\Phi \circ \alpha^t_{\Psi_+} \circ \alpha^{-s}_\Phi = \alpha^t_{\Psi_+}$$

The dynamics α_{Ψ_+} is preserved by α_Φ. Does this imply that $E_{\Psi_+} \in C(\alpha_\Phi)$?

Surface energies of dressed Hamiltonian

Fix $s \in \mathbb{R}$ and consider for each $\Lambda \in \mathcal{F}$ (also appeared in Anna's talk with Ψ_0 instead of Ψ_+)

$$H_s(\Lambda) := \mathrm{e}^{\mathrm{i}sH_\Phi(\Lambda)} H_{\Psi_+}(\Lambda) \mathrm{e}^{-\mathrm{i}sH_\Phi(\Lambda)}.$$

- Well-defined pressure $p_s = p(\Psi_+)$
- Mean energy $\lim_{\Lambda \to \mathbb{Z}^d} |\Lambda|^{-1} \nu(H_s(\Lambda)) = \nu(\alpha_\Phi^s(E_{\Psi_+}))$ for any $\nu \in \mathcal{S}_{\mathrm{I}}$
- Generates dynamics $t \mapsto \alpha_\Phi^s \circ \alpha_{\Psi_+}^t \circ \alpha_\Phi^{-s} = \alpha_{\Psi_+}^t$
- Surface energies? $W_s(\Lambda) := \lim_{\Lambda' \to \mathbb{Z}^d} (H_s(\Lambda') - H_s(\Lambda) - H_s(\Lambda' \setminus \Lambda))$

Theorem **Jaksic, Pillet, Szczepanek, T. '25**

For $\Phi, \Psi_+ \in \mathcal{B}_{\mathrm{sd}}$, assume that it exists $\epsilon > 0$ such that $W_s(\Lambda)$ exists for all $|s| < \epsilon$ and $\Lambda \in \mathcal{F}$ and satisfy $\lim_{\Lambda \to \mathbb{Z}^d} \frac{W_s(\Lambda)}{|\Lambda|} = 0$. Then $E_{\Psi_+} \in C(\alpha_\Phi)$.

Prop: Surface energy assumption holds for $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}(\gamma > 2d)$ and $\Psi_+ \in \mathcal{B}_\gamma^{\mathrm{diam}}(\gamma > d)$ (Anna's talk)

Conservation of relative entropy

Proposition

For $\Phi, \Psi_+ \in \mathcal{B}_{\mathrm{sd}}$, assume that $s(\nu \circ \alpha_\Phi^t) = s(\nu)$ for all $\nu \in \mathcal{S}_{\mathrm{I}}$ and (ω_+, Ψ_+) regular. Then the following statements are equivalent

1. $s(\nu \circ \alpha_\Phi^t | \omega_+) = s(\nu | \omega_+)$ for all $t \in \mathbb{R}$ and $\nu \in \mathcal{S}_{\mathrm{I}}$
2. $E_{\Psi_+} \in C(\alpha_\Phi)$

Prop: Conservation of relative entropy holds for $\Phi \in \mathcal{B}_{\mathrm{f}}$ and $\Psi_+ \in \mathcal{B}_{3r}$ with $\|\Psi_+\|_r < r$ (or $\Psi_+ \in \mathcal{B}_{\mathrm{f}}$ if $d = 1$) (Anna's talk)

Back to constants of motion

Let $\Phi \in \mathcal{B}_{\mathrm{sd}}$ and let $A \in C(\alpha_\Phi)$. Consider

$$\mathrm{const} = \{c\mathbb{1} | c \in \mathbb{R}\}, \qquad \mathcal{I} = \overline{\mathrm{span}_{\mathbb{R}}\{B - \tau^x(B) | B \in \mathcal{U}_{\mathrm{real}}, x \in \mathbb{Z}^d\}}.$$

$B \in \mathcal{I}$ iff $\nu(B) = 0$ for all $\nu \in \mathcal{S}_{\mathrm{I}}$. Consequently, for $A \in C(\alpha_\Phi)$ and $B \in \mathcal{I}$ one has

$$\nu \circ \alpha_\Phi^t(\lambda A + B + c\mathbb{1}) = \nu(\lambda A + B + c\mathbb{1})$$

Definition

The set of distinct constants of motion is $\mathcal{C}(\alpha_\Phi)$

$$\mathcal{C}(\alpha_\Phi) = \{[A], A \in C(\alpha_\Phi)\} \subset \mathcal{U}_{\mathrm{real}}/(\mathcal{I} + \mathrm{const})$$

Theorem **Jaksic, Pillet, Szczepanek, T. '25**

Let $\omega \in \mathcal{S}_{\mathrm{I}}$ and $\Phi \in \mathcal{B}_{\mathrm{sd}}$. Suppose that $\mathcal{S}_{\mathrm{eq}}(\beta_*, \Phi) = \{\nu_{\mathrm{eq}}\}$ with $(\nu_{\mathrm{eq}}, \Phi)$ regular, and moreover that ω is admissible for ν_{eq}. Let $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$ and assume $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(1, \Psi_+)$ with $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$. Moreover assume that

$$\mathcal{C}(\alpha_\Phi) = \{\lambda[E_\Phi], \lambda \in \mathbb{R}\}$$

Then $E_{\Psi_+} \in C(\alpha_\Phi)$ implies $\omega_+ = \nu_{\mathrm{eq}}$.

The regularity of (ω_+, Ψ_+) is not necessary in that case. The proof relies on the concept of physical equivalence from Ruelle.

Recent results on proving the non-integrability (= no other constant of motion) of a large class of 1D-models, also extended to higher D.

Shirashi '19, Yamaguchi, Chiba, Shiraishi '24, Shirashi, Tasaki '25,...

Conclusion

Theorem **Jaksic, Pillet, Szczepanek, T. '25**

Let $\omega \in \mathcal{S}_{\mathrm{I}}$ and $\Phi \in \mathcal{B}_{\mathrm{sd}}$. Suppose that $\mathcal{S}_{\mathrm{eq}}(\beta_*, \Phi) = \{\nu_{\mathrm{eq}}\}$ with $(\nu_{\mathrm{eq}}, \Phi)$ regular, and moreover that ω is admissible for ν_{eq}. For $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$, the following statements are equivalent:

1. $\omega_+ = \nu_{\mathrm{eq}}$
2. $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(1, \Psi_+)$ with $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$ such that (ω_+, Ψ_+) is regular, and $E_{\Psi_+} \in C(\alpha_\Phi)$.

- Approach to Thermal Equilibrium (ATE) requires the existence of Ψ_+ such that $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(1, \Psi_+)$
- Its validity relies only on the properties of Ψ_+. Proving ATE it is a trade-off between:
  - Establishing decay properties of Ψ_+ in specific models.
    (Ex: ATE holds if $d = 1$ and $\Psi_+ \in \mathcal{B}_{\mathrm{f}}$)
  - Establishing structural results for quantum spin systems.
    (Ex: Show that all pairs (ω_+, Ψ_+) are regular for $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$)
- Conservation laws and constants of motion play a central role
- ATE also holds for non-integrable models with a single constant of motion, provided that $E_{\Psi_+} \in C(\alpha_\phi)$.

Perspectives

The problem of approach to equilibrium opens many directions for our program:

- Approach to (constrained) equilibrium in presence of several constants of motion
- General study of constant of motion
- ATE in explicit models

Thank you for your attention!

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- Jaksic, V., Pillet, C. A., Szczepanek, A. & Tauber, C. Approach to equilibrium in translation-invariant quantum systems: Some structural results II, in preparation