Dynamical Conservation Laws in Quantum Spin Systems

(useful to study Approach to Equilibrium)

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- V. Jakšić, C.-A. Pillet, C. Tauber: *Approach to equilibrium in translation-invariant quantum systems: Some structural results.* Annales Henri Poincaré 25, 2024
- V. Jakšić, C.-A. Pillet, C. Tauber: *A note on adiabatic time evolution and quasi-static processes in translation-invariant quantum systems.* Annales Henri Poincaré 25, 2024
- V. Jakšić, C.-A. Pillet, AS, C. Tauber: *Dynamical conservation laws in quantum spin systems*, in preparation
- V. Jakšić, C.-A. Pillet, AS, C. Tauber: *Approach to equilibrium in translation-invariant quantum systems: Some structural results II*, in preparation

Quantum lattice spin systems

- Lattice $\mathbb{Z}^d$ with a family of translations τ_x for $x \in \mathbb{Z}^d$
- $\mathcal{F}$:= finite subsets of $\mathbb{Z}^d$
- Fixed Hilbert space $\mathcal{H}_0 := \mathbb{C}^N$
- $\mathcal{H}_x := \mathcal{H}_0$ for every $x \in \mathbb{Z}^d$
- $\mathcal{H}_X := \otimes_{x \in X} \mathcal{H}_x$ for $X \in \mathcal{F}$, and $\mathcal{U}_X$:= bounded operators on $\mathcal{H}_X$
- $\mathcal{U}_X \ni A \mapsto A \otimes \mathbf{1}_{\tilde{X} \setminus X} \in \mathcal{U}_{\tilde{X}}$ for $X \subset \tilde{X} \in \mathcal{F}$
- Local observables $\mathcal{U}_{\mathrm{loc}} := \bigcup_{X \in \mathcal{F}} \mathcal{U}_X$
- Spin C*-algebra $\mathcal{U}$:= norm-completion of $\mathcal{U}_{\mathrm{loc}}$
- Translation invariant states:
  $\mathcal{S}_I = \{\rho \colon \mathcal{U} \to \mathbb{C} \mid \rho \text{ positive, linear, } \rho(\mathbf{1}) = \mathbf{1} \text{ and } \rho \circ \tau_x = \rho \;\; \forall x \in \mathbb{Z}^d\}$

Interactions, local Hamiltonians, dynamics

- **Interaction**: a family $\{\Phi(X)\}_{X\in\mathcal{F}}$ such that $\Phi(X)\in\mathcal{U}_X$ is self-adjoint.
  We assume translation invariance: $\forall_{x\in\mathbb{Z}^d, X\in\mathcal{F}}\quad \tau_x(\Phi(X)) = \Phi(X+x)$
- $H_\Lambda(\Phi) = \sum_{X\subset\Lambda}\Phi(X)$ is the **local Hamiltonian** on $\Lambda\in\mathcal{F}$
- Corresponding **local dynamics** on Λ:
$$\alpha^t_{\Phi,\Lambda}(A) = \mathrm{e}^{\mathrm{i}tH_\Lambda(\Phi)}A\mathrm{e}^{-\mathrm{i}tH_\Lambda(\Phi)},\quad A\in\mathcal{U}_\Lambda$$
- We say the **(global) dynamics exists** if for $A\in\mathcal{U}$ the limit
$$\alpha^t_\Phi(A) := \lim_{\Lambda\uparrow\mathbb{Z}^d}\alpha^t_{\Phi,\Lambda}(A)$$
  exists and is uniform for t in compact sets.
- **Time evolution of a state**: $\rho_t := \rho\circ\alpha^t_\Phi$

$\Lambda\uparrow\mathbb{Z}^d$ is the limit over an increasing and exhaustive family of cubes in $\mathbb{Z}^d$ centred at 0

Spaces of interactions

- **Big space** of interactions $\mathcal{B}_{\mathrm{b}} = \{\Phi \mid \|\Phi\|_{\mathrm{b}} < \infty\}$ where

$$\|\Phi\|_{\mathrm{b}} = \sum_{X \ni 0} \frac{\|\Phi(X)\|}{|X|}$$

- **Exponentially decaying** interactions $\mathcal{B}^r = \{\Phi \mid \|\Phi\|_r < \infty\}$, $r > 0$

$$\|\Phi\|_r = \sum_{X \ni 0} \mathrm{e}^{r(|X|-1)} \|\Phi(X)\|$$

  $\mathcal{B}^r \subset \mathcal{B}_{\mathrm{b}}$, both are Banach spaces, and $\mathcal{B}_{\mathrm{f}}$ is a dense subset of each

  - α_Φ may not exist for $\Phi \in \mathcal{B}_b$
  - α_Φ does exist for $\Phi \in \mathcal{B}^r$

- **Finite range** interactions:

$$\mathcal{B}_{\mathrm{f}} = \{\Phi \mid \exists_{R \in \mathbb{N}} \; \mathrm{diam}(X) > R \;\Rightarrow\; \Phi(X) = 0\}$$

Conserved quantities

Let $\Phi \in \mathcal{B}_b$ and $\rho, \omega \in \mathcal{S}_I$. Notation: $\rho_\Lambda \in \mathcal{U}_\Lambda$ is s.t. $\rho(A) = \mathrm{tr}(\rho_\Lambda A)$ for all $A \in \mathcal{U}_\Lambda$

- **Specific entropy**

  $$s(\rho) := -\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathrm{tr}(\rho_\Lambda \log(\rho_\Lambda)) \in [0, \log N]$$

  Always exists, is affine and upper semi-continuous.

- **Specific energy**

  $$E_\Phi := \sum_{X \ni 0} \frac{1}{|X|} \Phi(X)$$

  Satisfies $\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \rho(H_\Lambda(\Phi)) = \rho(E_\Phi)$ for $\rho \in \mathcal{S}_I$.

- **Specific relative entropy** (assuming the limit exists)

  $$s(\rho|\omega) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathrm{tr}(\rho_\Lambda(\log \rho_\Lambda - \log \omega_\Lambda)) \geq 0$$

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CL for specific entropy and energy. For $\Phi \in \mathcal{B}^r$, $\rho \in \mathcal{S}_\mathrm{I}$, $t \in \mathbb{R}$:

- $s(\rho \circ \alpha_\Phi^t) = s(\rho)$ [Lanford-Robinson'68]
- $(\rho \circ \alpha_\Phi^t)(E_\Phi) = \rho(E_\Phi)$ [Jakšić-Pillet-Tauber'24]

diam-spaces & Lieb–Robinson bound

Set $\gamma > 0$ and consider $\mathcal{B}_\gamma^{\mathrm{diam}} = \{\Phi \mid \|\Phi\|_{\gamma,\mathrm{diam}} < \infty\}$ with

$$\|\Phi\|_{\gamma,\mathrm{diam}} = \sum_{X \ni 0} |X| [\mathrm{diam}(X)]^\gamma \|\Phi(X)\|$$

- $\mathcal{B}_\gamma^{\mathrm{diam}}$ is a Banach space, $\mathcal{B}_{\mathrm{f}}$ is its dense subset, and $\mathcal{B}_\gamma^{\mathrm{diam}} \subset \mathcal{B}_b$
- $\mathcal{B}_\gamma^{\mathrm{diam}}$ and $\mathcal{B}^r$ are incomparable

Thm [Nachtergaele-Sims-Young'19] Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > d$, and $0 < \epsilon < \gamma - d$

- **Dynamics α_Φ exists**
- **Lieb-Robinson bound:** Let $t \in \mathbb{R}$, $\Lambda_0 \subset \Lambda$, and $A \in \mathcal{U}_{\Lambda_0}$.

$$\exists c_t > 0 \quad \|\alpha_\Phi^t(A) - \alpha_{\Phi,\Lambda}^t(A)\| \le c_t \|A\| |\Lambda_0| (1 + \mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda))^{-(\gamma - d - \epsilon)}$$

CL for specific entropy and energy [Jakšić-Pillet-S-Tauber'25]
Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$, $\rho \in \mathcal{S}_I$, $t \in \mathbb{R}$

$$s(\rho \circ \alpha_\Phi^t) = s(\rho) \quad \text{and} \quad (\rho \circ \alpha_\Phi^t)(E_\Phi) = \rho(E_\Phi).$$

CLs for diam-spaces: energy $1/3$

- For $\mathcal{B}^r$, the proofs of CL for energy and entropy are completely different
- For $\mathcal{B}_\gamma^{\mathrm{diam}}$, the **Lieb–Robinson bound is the main tool** for both CLs
  This proof strategy was pointed out by Wreszinski:
  - W. F. Wreszinski: *Irreversibility, the time arrow and a dynamical proof of the second law of thermodynamics.* Quantum Stud.: Math. Found. 7 (2020)
  - W. F. Wreszinski: *The second law of thermodynamics as a deterministic theorem for quantum spin systems.* Rev. Math. Phys. 35 (2023)

CL for specific energy: sketch of the proof

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CL for specific energy: sketch of the proof

For $\Lambda \in \mathcal{F}$:

$$(\rho \circ \alpha^t_{\Phi,\Lambda})(H_\Lambda(\Phi)) = \rho(\mathrm{e}^{\mathrm{i}tH_\Lambda(\Phi)} H_\Lambda(\Phi) \mathrm{e}^{-\mathrm{i}tH_\Lambda(\Phi)}) = \rho(H_\Lambda(\Phi)),$$

which gives

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} (\rho \circ \alpha^t_{\Phi,\Lambda})(H_\Lambda(\Phi)) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \rho(H_\Lambda(\Phi)) = \rho(E_\Phi).$$

On the other hand,

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} (\rho \circ \alpha^t_{\Phi})(H_\Lambda(\Phi)) = (\rho \circ \alpha^t_{\Phi})(E_\Phi).$$

CLs for diam-spaces: energy 2/3

Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$, $\rho \in \mathcal{S}_I$, $t \in \mathbb{R}$. Then

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} |(\rho \circ \alpha_\Phi^t)(H_\Lambda(\Phi)) - (\rho \circ \alpha_{\Phi,\Lambda}^t)(H_\Lambda(\Phi))| = 0.$$

$$\begin{aligned} |(\rho \circ \alpha_\Phi^t)(H_\Lambda(\Phi)) - (\rho \circ \alpha_{\Phi,\Lambda}^t)(H_\Lambda(\Phi))| &= \left| \rho\Big(\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi)) \Big) \right| \\ &\leq \| \alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi)) \| \end{aligned}$$

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$$\begin{aligned}|(\rho \circ \alpha_\Phi^t)(H_\Lambda(\Phi)) - (\rho \circ \alpha_{\Phi,\Lambda}^t)(H_\Lambda(\Phi))| &= \left|\rho\Big(\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\Big)\right| \\ &\leq \|\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\|\end{aligned}$$

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$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \|\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\| = 0.$$

Let Λ be a cube and Λ_0 a sub-cube. Define

$$\tilde{H}_{\Lambda,\Lambda_0}(\Psi) := H_\Lambda(\Phi) - H_{\Lambda_0}(\Phi).$$

$$\begin{aligned}
&\|\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\| \\
&\quad \le \quad \|\alpha_\Phi^t(H_{\Lambda_0}(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_{\Lambda_0}(\Phi))\| + \|\alpha_\Phi^t(\tilde{H}_{\Lambda,\Lambda_0}(\Psi))\| + \|\alpha_{\Phi,\Lambda}^t(\tilde{H}_{\Lambda,\Lambda_0}(\Phi))\|
\end{aligned}$$

CLs for diam-spaces: energy 2/3

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$$\|\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\|$$

$$\leq \quad \|\alpha_\Phi^t(H_{\Lambda_0}(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_{\Lambda_0}(\Phi))\| + 2\,\|\tilde{H}_{\Lambda,\Lambda_0}(\Phi)\|$$

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&\qquad \leq \quad \|H_{\Lambda_0}(\Phi)\| \sup_{\substack{A \in \mathcal{U}_{\Lambda_0} \\ \|A\| \leq 1}} \|\alpha_\Phi^t(A) - \alpha_{\Phi,\Lambda}^t(A)\| + 2\,\|\tilde{H}_{\Lambda,\Lambda_0}(\Phi)\|
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&\quad \leq \ c_1|\Lambda_0| \cdot \left[c_2|\Lambda_0|(1 + \mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda))^{-\gamma+d+\epsilon}\right] + c_3|\Lambda \setminus \Lambda_0|
\end{aligned}$$

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&\qquad \leq \quad \|H_{\Lambda_0}(\Phi)\| \sup_{\substack{A \in \mathcal{U}_{\Lambda_0} \\ \|A\| \leq 1}} \|\alpha_\Phi^t(A) - \alpha_{\Phi,\Lambda}^t(A)\| + \ 2\ \|\tilde{H}_{\Lambda,\Lambda_0}(\Phi)\| \\
&\qquad \leq \ c_1|\Lambda_0| \cdot \left[c_2|\Lambda_0|(1 + \mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda))^{-\gamma+d+\epsilon}\right] + c_3|\Lambda \setminus \Lambda_0|
\end{aligned}$$

(because $\|H_{\Lambda_0}(\Phi)\| \leq \sum_{X \cap \Lambda_0 \neq \emptyset} \|\Phi(X)\| \leq |\Lambda_0| \sum_{X \ni 0} \|\Phi(X)\| \leq |\Lambda_0| c_1$)

CLs for diam-spaces: energy 2/3

Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$, $\rho \in \mathcal{S}_I$, $t \in \mathbb{R}$. Then

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \| \alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi)) \| = 0.$$

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$$\frac{1}{|\Lambda|} \| \alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi)) \|$$

$$\leq \frac{1}{|\Lambda|} \| \alpha_\Phi^t(H_{\Lambda_0}(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_{\Lambda_0}(\Phi)) \| + \frac{2}{|\Lambda|} \| \tilde{H}_{\Lambda,\Lambda_0}(\Phi) \|$$

$$\leq \frac{1}{|\Lambda|} \| H_{\Lambda_0}(\Phi) \| \sup_{\substack{A \in \mathcal{U}_{\Lambda_0} \\ \|A\| \leq 1}} \| \alpha_\Phi^t(A) - \alpha_{\Phi,\Lambda}^t(A) \| + \frac{2}{|\Lambda|} \| \tilde{H}_{\Lambda,\Lambda_0}(\Phi) \|$$

$$\leq c_1 \frac{|\Lambda_0|}{|\Lambda|} \cdot \left[c_2 |\Lambda_0| (1 + \mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda))^{-\gamma + d + \epsilon} \right] + c_3 \frac{|\Lambda \setminus \Lambda_0|}{|\Lambda|}$$

CLs for diam-spaces: energy 2/3

Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$, $\rho \in \mathcal{S}_I$, $t \in \mathbb{R}$. Then

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&\qquad \leq \frac{1}{|\Lambda|} \|H_{\Lambda_0}(\Phi)\| \sup_{\substack{A \in \mathcal{U}_{\Lambda_0} \\ \|A\| \leq 1}} \|\alpha_\Phi^t(A) - \alpha_{\Phi,\Lambda}^t(A)\| + \frac{2}{|\Lambda|} \|\tilde{H}_{\Lambda,\Lambda_0}(\Phi)\| \\
&\qquad \leq c_1 \frac{|\Lambda_0|}{|\Lambda|} \cdot \left[c_2 |\Lambda_0| (1 + \mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda))^{-\gamma + d + \epsilon} \right] + c_3 \frac{|\Lambda \setminus \Lambda_0|}{|\Lambda|}
\end{aligned}$$

Can we construct families Λ, Λ_0 such that $\lim_{\Lambda \uparrow \mathbb{Z}^d} \ldots = 0$?

CLs for diam-spaces: energy 3/3

Lemma. Assume $\gamma > 2d$. Let Λ be a family of cubes such that $\Lambda \uparrow \mathbb{Z}^d$ and denote the side of Λ by L_Λ. There exists $p \in (0,1)$ s.t. the sub-cubes $\Lambda_0 = (1 - L_\Lambda^{-p})\Lambda$ satisfy

(i) $\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{|\Lambda \setminus \Lambda_0|}{|\Lambda|} = 0$,

(ii) $\lim_{\Lambda \uparrow \mathbb{Z}^d} |\Lambda_0|(1 + \mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda))^{-\gamma+d+\epsilon} = 0$ for any $0 < \epsilon < \gamma - 2d$.

(ii) Since $|\Lambda_0| = \left(1 - L_\Lambda^{-p}\right)^d L_\Lambda^d$ and $\mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda) = L_\Lambda^{1-p}$, we obtain

$$|\Lambda_0|(1 + \mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda))^{-\gamma+d+\epsilon} \leq L_\Lambda^{d-(1-p)(\gamma-d-\epsilon)}.$$

It follows that $d < (1-p)(\gamma - d - \epsilon)$ if $0 < p < p_0$ with $p_0 := \frac{\gamma-2d-\epsilon}{\gamma-d-\epsilon}$, so (ii) holds

$\Longrightarrow$ **CL for specific energy holds for $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$**

Conserved quantities

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  $$s(\rho|\omega) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathrm{tr}(\rho_\Lambda(\log \rho_\Lambda - \log \omega_\Lambda)) \geq 0$$

CL for specific entropy and energy. Let $\Phi \in \mathcal{B}^r$ with $r > 0$ or $\mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$

- $s(\rho \circ \alpha_\Phi^t) = s(\rho)$
- $(\rho \circ \alpha_\Phi^t)(E_\Phi) = \rho(E_\Phi)$

CL for relative entropy via Regularity

Let $\Psi \in \mathcal{B}_b$. **Pressure** $P_\Psi := \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log(\mathrm{tr}(\mathrm{e}^{-H_\Lambda(\Psi)})) < \infty$

- Gibbs variational principle: $P_\Psi = \sup_{\rho \in \mathcal{S}_I} (s(\rho) - \rho(E_\Psi))$
- Equilibrium states: $\mathcal{S}_{\mathrm{eq}}(\Psi) = \{\rho \in \mathcal{S}_I \mid P_\Psi = s(\rho) - \rho(E_\Psi)\}$

Definition. Let $\omega \in \mathcal{S}_{\mathrm{I}}$, $\Psi \in \mathcal{B}_b$. We call (ω, Ψ) a **regular pair** if

$$\forall \nu \in \mathcal{S}_{\mathrm{I}}(\mathcal{U}) \quad s(\nu|\omega) \text{ exists and } s(\nu|\omega) = -s(\nu) + \nu(E_\Psi) + P_\Psi$$

- (ω, Ψ) is regular $\Longrightarrow \omega \in \mathcal{S}_{\mathrm{eq}}(\Psi)$
- **R-Conjecture:** (ω, Ψ) is regular for every reasonable Ψ and $\omega \in \mathcal{S}_{\mathrm{eq}}(\Psi)$

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CL for regularity: if (ω, Ψ) regular, then $(\omega_t = \omega \circ \alpha^t_\Phi, \Psi_t)$ regular for some Ψ_t

Obs. CL for regularity, $P_{\Psi_t} = P_{\Psi_0}$, and $\forall_\nu\ \nu_t(E_{\Psi_t}) = \nu(E_{\Psi_0}) \Rightarrow$ **CL for rel. entropy**

$$\begin{aligned} s(\nu|\omega) &= -s(\nu) + \nu(E_{\Psi_0}) + P_{\Psi_0} \\ &= -s(\nu_t) + \nu_t(E_{\Psi_t}) + P_{\Psi_t} = s(\nu_t|\omega_t) \end{aligned}$$

CL for relative entropy via weak Gibbsianity

Def. $\omega \in \mathcal{S}_I$ is **weak Gibbs** for $\Psi \in \mathcal{B}_\mathrm{b}$ if there exist constants $C_\Lambda > 0$ s.t.

$$C_\Lambda^{-1} \frac{\mathrm{e}^{-H_\Lambda(\Psi)}}{\mathrm{tr}\, \mathrm{e}^{-H_\Lambda(\Psi)}} \leq \omega_\Lambda \leq C_\Lambda \frac{\mathrm{e}^{-H_\Lambda(\Psi)}}{\mathrm{tr}\, \mathrm{e}^{-H_\Lambda(\Psi)}} \quad \text{and} \quad \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{\log C_\Lambda}{|\Lambda|} = 0$$

- $\mathcal{S}_\mathrm{wg}(\Psi) :=$ set of weak Gibbs states for Ψ
- $\omega \in \mathcal{S}_\mathrm{wg}(\Psi) \Longrightarrow (\omega, \Psi)$ is regular $\Longrightarrow \omega \in \mathcal{S}_\mathrm{eq}(\Psi)$
- **Theorem:** $\mathcal{S}_\mathrm{wg}(\Psi) = \mathcal{S}_\mathrm{eq}(\Psi)$ if $\Psi \in \mathcal{B}^r$ and $\|\Psi\|_r < r$, or $\Psi \in \mathcal{B}_\mathrm{f}$ and $d = 1$

[Jakšić-Pillet-Tauber'24]

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CL for weak Gibbsianity: if $\omega \in \mathcal{S}_{\mathrm{wg}}(\Psi_0)$, then $\omega_t \in \mathcal{S}_{\mathrm{wg}}(\Psi_t)$ for some Ψ_t

Obs. CL for weak Gibbsianity, $P_{\Psi_t} = P_{\Psi_0}$, $\nu_t(E_{\Psi_t}) = \nu(E_{\Psi_0}) \Longrightarrow$

CL for relative entropy

CL for weak Gibbsianity: dressed Hamiltonians

Let $\Phi, \Psi_0 \in \mathcal{B}_b$. Fix $t \in \mathbb{R}$ and define

$$H_\Lambda(t) := \alpha^{-t}_{\Phi,\Lambda}(H_\Lambda(\Psi_0)) = \mathrm{e}^{-\mathrm{i}tH_\Lambda(\Phi)} H_\Lambda(\Psi_0) \mathrm{e}^{\mathrm{i}tH_\Lambda(\Phi)}$$

The corresponding translation-invariant interaction is uniquely defined as:

$$\Psi_t(X) = \sum\nolimits_{Y \subset X} (-1)^{|X|-|Y|} H_Y(t)$$

- Ψ_t need not even be in $\mathcal{B}_b$!

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- Pressure and specific energy of Ψ_t exist, $P_{\Psi_t} = P_{\Psi_0}$, and $E_{\Psi_t} = \alpha_\Phi^{-t}(E_{\Psi_0})$. Thus $\nu_t(E_{\Psi_t}) = (\nu \circ \alpha_\Phi^t)(E_{\Psi_t}) = (\nu \circ \alpha_\Phi^t)(\alpha_\Phi^{-t}(E_{\Psi_0})) = \nu(E_{\Psi_0})$
- If Φ, Ψ_0 generate dynamics, $\{H_\Lambda\}_\Lambda$ generates dynamics $\alpha^s = \alpha_\Phi^{-t} \circ \alpha_{\Psi_0}^s \circ \alpha_\Phi^t$

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CL for weak Gibbsianity: Assume that $\Phi \in \mathcal{B}_{\mathrm{f}}$ and either

(a) $d \geq 1$ and $\Psi_0 \in \mathcal{B}^{3r}$ for some $r > 0$ and such that $\|\Psi_0\|_r < r$, or

(b) $d = 1$ and $\Psi_0 \in \mathcal{B}_{\mathrm{f}}$.

Then $\omega \in \mathcal{S}_{\mathrm{wg}}(\Psi_0) \Longrightarrow \omega_t := \omega \circ \alpha_\Phi^t \in \mathcal{S}_{\mathrm{wg}}(\Psi_t)$ for $|t|$ small enough.

In dim $d = 1$ for all $t \in \mathbb{R}$.

CL for weak Gibbsianity: proof overview

Characterization of weak Gibbsianity [Jakšić–Pillet–Tauber'24]:

$\omega_t \in \mathcal{S}_{\mathrm{wg}}(\Psi_t)$ iff

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|} \log \inf_{\substack{A\in\mathcal{U}_\Lambda \\ A>0}} \frac{\omega_t(A)}{(\omega_t)_{-W_\Lambda(t)}(A)} = \lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|} \log \sup_{\substack{A\in\mathcal{U}_\Lambda \\ A>0}} \frac{\omega_t(A)}{(\omega_t)_{-W_\Lambda(t)}(A)} = 0$$

$(\omega_t)_{-W_\Lambda(t)}$ is the local perturbation of ω_t by the surface energies:

$$W_\Lambda(t) = \lim_{\Lambda'\uparrow\mathbb{Z}^d} W_{\Lambda,\Lambda'}(t), \quad \text{where } W_{\Lambda,\Lambda'}(t) = H_{\Lambda'}(t) - H_\Lambda(t) - H_{\Lambda'\setminus\Lambda}(t)$$

CL for surface energies [Jakšić–Pillet–S–Tauber'25]

$$\forall_{t\in\mathbb{R}} \quad W_\Lambda(t) \text{ exists} \quad \text{and} \quad \lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|}\|W_\Lambda(t)\| = 0$$

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Key bound [Lenci–Rey-Bellet'05] For any $A \in \mathcal{U}_\Lambda$ such that $A > 0$

$$\mathrm{e}^{-\|W_\Lambda(t)\| - \|\alpha^{\mathrm{i}/2}_{-W_\Lambda(t)}(W_\Lambda(t))\|} \le \frac{\omega_t(A)}{(\omega_t)_{-W_\Lambda(t)}(A)} \le \mathrm{e}^{\|W_\Lambda(t)\| + \|\alpha^{\mathrm{i}/2}(W_\Lambda(t))\|}$$

CL for weak Gibbsianity: Ruelle's bound

We need bounds for $\|\alpha^{\mathrm{i}/2}(W_\Lambda(t))\|$ and $\|\alpha^{\mathrm{i}/2}_{-W_\Lambda(t)}(W_\Lambda(t))\|$.

Ruelle's bound ('69) Assume $\Psi \in \mathcal{B}^r$. The map

$$\mathbb{R} \ni s \mapsto \alpha^s_\Psi(A) \in \mathcal{U}$$

has an analytic extension to the strip $|\operatorname{Im} z| < \frac{r}{2\|\Psi\|_r}$. For any z in this strip

$$\|\alpha^z_\Psi(A)\| \leq \|A\| \mathrm{e}^{r|\mathrm{supp} A|} C_{z,\Psi} \quad \text{with} \quad C_{z,\Psi} = (1 - \tfrac{2}{r}\|\Psi\|_r |\operatorname{Im} z|)^{-1}.$$

Since $\|\alpha^s(W_\Lambda(t))\| = \|\alpha^s_{\Psi_0} \circ \alpha^t_\Phi(W_\Lambda(t))\|$ we simply use Ruelle's bound twice.

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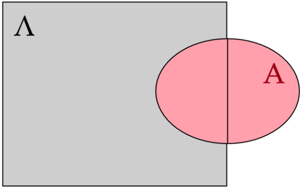
The perturbed dynamics is more problematic. Since

$$\alpha^s_{-W_\Lambda(t)} = \alpha^s_{\mathbb{Z}^d \setminus \Lambda} \circ \alpha^s_\Lambda$$

we need an analogous bound for restricted dynamics

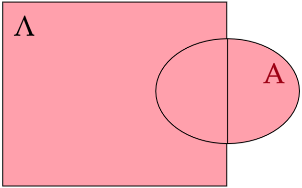
$$\alpha^s_K = \alpha^{-t}_{\Phi|_K} \circ \alpha^s_{\Psi_0|_K} \circ \alpha^t_{\Phi|_K}, \quad K \subset \mathbb{Z}^d.$$

CL for weak Gibbsianity: Ruelle's bound



$\alpha_{\Lambda}^{s}(A)$

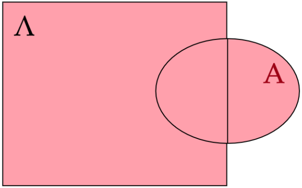
CL for weak Gibbsianity: Ruelle's bound



$$\alpha^s_\Lambda(A)$$

$$\mathrm{supp}\Big(\alpha^s_\Lambda(A)\Big) \subset \mathrm{A} \cup \Lambda$$

CL for weak Gibbsianity: Ruelle's bound



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$$\alpha^s_{\mathbb{Z}^d\setminus\Lambda}\Big(\alpha^s_\Lambda(A)\Big)$$

$$\|\alpha^s_{\mathbb{Z}^d\setminus\Lambda}\Big(\alpha^s_\Lambda(A)\Big)\| \ \leq \|A\|\, \mathrm{e}^{|\mathrm{A}\cup\Lambda|}$$

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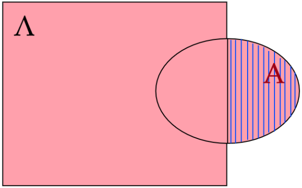

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$$\|\alpha^s_{\mathbb{Z}^d\setminus\Lambda}\left(\alpha^s_\Lambda(A)\right)\| \leq \|A\|\, \mathrm{e}^{|\mathrm{A}\cup\Lambda|}$$

What we should see on RHS:

$$\mathrm{e}^{|(\mathrm{A}\cup\Lambda)\cap\left(\mathbb{Z}^d\setminus\Lambda\right)|} = \mathrm{e}^{|\mathrm{A}\cap\left(\mathbb{Z}^d\setminus\Lambda\right)|} \leq \mathrm{e}^{|\mathrm{A}|}$$

CL for weak Gibbsianity: Ruelle's bound

Let $K \subset \mathbb{Z}^d$ and $A \in \mathcal{U}_{\rm loc}$. Recall that $\alpha_K^s = \alpha_{\Phi|_K}^{-t} \circ \alpha_{\Psi_0|_K}^{s} \circ \alpha_{\Phi|_K}^{t}$.

Ruelle's bound generalized. Assume $\Psi \in \mathcal{B}^r$. The map

$$\mathbb{R} \ni s \mapsto \alpha_{\Psi|_K}^s(A) \in \mathcal{U}$$

has an analytic extension to the strip $|\operatorname{Im} z| < \frac{r}{2\|\Psi\|_r}$. For any z in this strip

$$\|\alpha_{\Psi|_K}^z(A)\| \leq \|A\| \mathrm{e}^{r|\mathrm{supp} A \cap K|} C_{z,\Psi} \quad \text{with} \quad C_{z,\Psi} = (1 - \tfrac{2}{r}\|\Psi\|_r |\operatorname{Im} z|)^{-1}.$$

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Bound for composite dynamics. Assume $\Psi_0 \in \mathcal{B}^{3r}$ such that $\|\Psi_0\|_r < r$ and $\Phi \in \mathcal{B}_{\mathrm{f}}$. Set $R = \operatorname{range} \Phi$ and $T_0 = \left(\frac{2}{r}\|\Phi\|_r C_{z,\Psi_0} \mathrm{e}^{rR^d}\right)^{-1}$.
For all $|t| < T_0$ the map

$$\mathbb{R} \ni s \mapsto \alpha_K^s(A)$$

has an analytic extension to the strip $|\operatorname{Im} z| < \frac{r}{2\|\Psi_0\|_r}$. For any z in this strip

$$\|\alpha_K^z(A)\| \leq \|A\| \mathrm{e}^{2r|\mathrm{supp} A \cap K|} \frac{C_{z,\Psi_0}}{1 - |t|/T_0}.$$

CL for weak Gibbsianity: Araki's bound in dim $= 1$

Let $\Psi \in \mathcal{B}_{\mathrm{f}}$ and $F_n(x) := \exp\big((n-R+1)x + 2\sum_{r=1}^{R} \frac{\exp(rx)-1}{r}\big)$ with $R := \operatorname{range}\Psi$

Araki's bound ('69): For every $A \in \mathcal{U}_{\mathrm{loc}}$, the map

$$\mathbb{R} \ni s \mapsto \alpha_\Psi^s(A)$$

has an analytic extension to the whole complex plane, and for any $z \in \mathbb{C}$

$$\|\alpha_\Psi^z(A)\| \leq F_n(C_\Psi |z|)\|A\|,$$

where $n = \max\{\mathrm{diam}(\mathrm{supp} A), R-1\}$ and $C_\Psi = 2(R+1)\|\Psi\|_{\mathrm{s}}$

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Araki's bound generalized: Additionally, let $K \subset \mathbb{Z}$. For every $A \in \mathcal{U}_{\mathrm{loc}}$,

$$\mathbb{R} \ni s \mapsto \alpha_{\Psi|_K}^s(A)$$

has an analytic extension to the whole complex plane, and for any $z \in \mathbb{C}$

$$\|\alpha_{\Psi|_K}^z(A)\| \leq F_n(C_\Psi|z|)\|A\|,$$

where $n = \max\{\mathrm{diam}(\mathrm{supp} A \cap K), R-1\}$ and $C_\Psi = 2(R+1)\|\Psi\|_{\mathrm{s}}$

CL for weak Gibbsianity: proof recap

Recall for any $A \in \mathcal{U}_\Lambda$ such that $A > 0$ we have

$$\mathrm{e}^{-\|W_\Lambda(t)\| - \|\alpha^{\mathrm{i}/2}_{-W_\Lambda(t)}(W_\Lambda(t))\|} \leq \frac{\omega_t(A)}{(\omega_t)_{-W_\Lambda(t)}(A)} \leq \mathrm{e}^{\|W_\Lambda(t)\| + \|\alpha^{\mathrm{i}/2}(W_\Lambda(t))\|} \tag{1}$$

- Using the Lieb-Robinson bound: $\lim\limits_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|}\|W_\Lambda(t)\| = 0$,
- Using Ruelle/Araki generalized bounds:

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|}\|\alpha^{\mathrm{i}/2}(W_\Lambda(t))\| = 0 \quad \text{and} \quad \lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|}\|\alpha^{\mathrm{i}/2}_{-W_\Lambda(t)}(W_\Lambda(t))\| = 0.$$

- $\frac{1}{|\Lambda|}\log()$ of each bound in (1) goes to zero as $\Lambda \uparrow \mathbb{Z}^d$
- So $\omega_t \in \mathcal{S}_{\mathrm{wg}}(\Psi_t)$, i.e., **CL for weak Gibbsianity holds**

CL for weak Gibbsianity/regularity & relative entropy

CL for weak Gibbsianity: Assume that $\Phi \in \mathcal{B}_{\mathrm{f}}$ and either

(a) $d \geq 1$ and $\Psi_0 \in \mathcal{B}^{3r}$ for some $r > 0$ and such that $\|\Psi_0\|_r < r$, or

(b) $d = 1$ and $\Psi_0 \in \mathcal{B}_{\mathrm{f}}$.

Then $\omega \in \mathcal{S}_{\mathrm{wg}}(\Psi_0) \Longrightarrow \omega_t \in \mathcal{S}_{\mathrm{wg}}(\Psi_t)$ for $|t| < T_0$.

In dim $d = 1$ one can take $T_0 = \infty$.

CL for weak Gibbsianity, $P_{\Psi_t} = P_{\Psi_0}$ and $\nu_t(E_{\Psi_t}) = \nu(E_{\Psi_0})$

$\Longrightarrow$ **CL for relative entropy**

Recall that under (a) or (b) we have

$$\omega \in \mathcal{S}_{\mathrm{wg}}(\Psi_0) \iff (\omega, \Psi_0) \text{ regular} \iff \omega \in \mathcal{S}_{\mathrm{eq}}(\Psi_0)$$

In consequence: **CL for regularity**

Summary of new CLs. Let $\nu \in \mathcal{S}_I$ and $t \in \mathbb{R}$

CL for specific entropy and energy. Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$. Then

- $s(\nu_t) = s(\nu)$
- $\nu_t(E_\Phi) = \nu(E_\Phi)$

CL for specific relative entropy. Assume that $\Phi \in \mathcal{B}_{\mathrm{f}}$ and either

(a) $d \geq 1$ and $\Psi_0 \in \mathcal{B}^{3r}$ such that $\|\Psi_0\|_r < r$, or

(b) $d = 1$ and $\Psi_0 \in \mathcal{B}_{\mathrm{f}}$.

If (ω, Ψ_0) is regular, then $s(\nu_t|\omega_t)$ exists and $s(\nu_t|\omega_t) = s(\nu|\omega)$ if $|t| < T_0$.

In dim $d = 1$ we can take $T_0 = \infty$.

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In dim $d = 1$ we can take $T_0 = \infty$.

CL for weak Gibbsianity/regularity. Under the same assumptions:

there exists Ψ_t such that $\omega_t \in \mathcal{S}_{\mathrm{wg}}(\Psi_t)$, i.e., (ω_t, Ψ_t) is regular

CL for surface energies Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$ and $\Psi_0 \in \mathcal{B}_{\gamma'}^{\mathrm{diam}}$ with $\gamma' \geq 0$.
Then $W_\Lambda(t)$ exists for every $t \in \mathbb{R}$, and $\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|}\|W_\Lambda(t)\| = 0$

Thank you!

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- D. Ruelle: *Statistical Mechanics: Rigorous Results* (1969)
- M. Lenci, L. Rey-Bellet: *Large deviations in quantum lattice systems: one-phase region.* J. Stat. Phys. 119 (2005)
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Bonus: Hiai–Petz type bound

vanishing of surface energies $\implies$ if exists, specific relative entropy
$\lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|}\|W_\Lambda(t)\| = 0$ does not increase along the trajectory

Thm. Assume $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$ and $\Psi_0 \in \mathcal{B}^r$ with any $r > 0$.
Let $\omega \in \mathcal{S}_{\mathrm{eq}}(\Psi_0)$ and $\nu \in \mathcal{S}_I$. Then

$$\limsup_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|} S((\nu_t)_\Lambda|(\omega_t)_\Lambda) \leq -s(\nu) + \nu(E_{\Psi_0}) + P_{\Psi_0}.$$

In particular, if (Ψ_0, ω) is regular, then

$$\limsup_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|} S((\nu_t)_\Lambda|(\omega_t)_\Lambda) \leq s(\nu|\omega),$$

and if $s(\nu_t|\omega_t)$ exists, then

$$s(\nu_t|\omega_t) \leq s(\nu|\omega).$$