Gibbs measures as local equilibrium Kubo-Martin-Schwinger states for focusing nonlinear Schrödinger equations

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General setup

For a phase space $\mathcal{S}$ and vector field $X : \mathcal{S} \to \mathcal{S}$, we consider the initial value problem

$$\dot{u}(t) = X(u(t)), \qquad u(0) = u_0 \in \mathcal{S}.$$

One can study the initial value problem in two ways.

(1) Deterministically. Well-posedness in a suitable function space (existence, uniqueness, stability,...)

(2) Probabilistically. Study the dynamics of ensembles of initial data rather than a single point in phase space.
One obtains the *Liouville equation* for $\mu \in \mathscr{P}(\mathcal{S})$ – a probability measure on $\mathcal{S}$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{S}} F(u)\, \mu_t(\mathrm{d}u) = \int_{\mathcal{S}} \langle \nabla F(u), X(u) \rangle\, \mu_t(\mathrm{d}u).$$

$F \in \mathscr{C}(\mathcal{S})$
Φ_t– the flow map of the IVP
$\mu_t = (\Phi_t)_\sharp \mu \in \mathscr{P}(\mathcal{S})$; recall $(\Phi_t)_\sharp \mu(A) \equiv \mu(\Phi_t^{-1}(A))$
$\langle \cdot, \cdot \rangle$ – inner product on $\mathcal{S}$.

General setup

A solution $\mu_t \in \mathscr{P}(\mathcal{S})$ to the Liouville equation is called stationary if for all F

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\mathcal{S}} F(u)\, \mu_t(\mathrm{d}u) = \int_{\mathcal{S}} \langle \nabla F(u), X(u) \rangle\, \mu_t(\mathrm{d}u) = 0\,.$$

Assume $\mathcal{S}$ has a Poisson bracket : a map $\{\cdot,\cdot\} : \mathscr{C}(\mathcal{S}) \times \mathscr{C}(\mathcal{S}) \to \mathscr{C}(\mathcal{S})$ satisfying

- Antisymmetry : $\{F, G\} = -\{G, F\}$.
- Distributivity : $\{F + G, H\} = \{F, H\} + \{G, H\}$.
- Leibniz rule: $\{FG, H\} = \{F, H\}G + F\{G, H\}$.
- Jacobi identity : $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$.

Take $F = 1$ in the Leibniz rule and deduce that

$$\{G, H\} = \{1, H\}G + \{G, H\}\,.$$

Hence $\{1, H\} = \{H, 1\} = 0$.

KMS states

- Let $\beta > 0$ be given.
  We say that $\mu \in \mathscr{P}(\mathcal{S})$ is a (classical) (β, X) – Kubo–Martin–Schwinger (KMS) state if for all test functions F, G we have

$$\int_{\mathcal{S}} \{F, G\}\, \mu(\mathrm{d}u) = \beta \int_{\mathcal{S}} \langle \nabla F(u), X(u) \rangle\, G(u)\, \mu(\mathrm{d}u)\,.$$

- Taking $G \equiv 1$, we get that $\mu_t = \mu$ is a stationary solution to the Liouville equation since

$$\int_{\mathcal{S}} \langle \nabla F(u), X(u) \rangle\, \mu(\mathrm{d}u) = 0$$

  for all test functions F.
- This concept was first introduced in the infinite-dimensional setting by Gallavotti and Verboven (1975) (in statistical mechanics).
  Further work by Aizenman-Goldstein-Gruber-Lebowitz-Martin (1977), Arsen'ev (1983), Chueshov (1986).

The nonlinear Schrödinger equation

Consider the spatial domain $\Lambda = \mathbb{T}^d$ for $d = 1, 2, 3$.

- Study the ***nonlinear Schrödinger equation (NLS)***.

$$\begin{cases} \mathrm{i}\partial_t u_t(x) = \big(- \Delta + 1 \big) u_t(x) + \big(\int V(x-y)\, |u_t(y)|^2 \,\mathrm{d}y \big)\, u_t(x) \\ u_0(x) = \Psi(x) \in H^s(\Lambda)\,. \end{cases}$$

- Interaction: $V \in L^1(\Lambda)$ positive or $V = \delta$.
- Conserved energy (Hamiltonian)

$$h(u) = \frac{1}{2} \int \bar{u}(x)(1-\Delta)u(x)\,\mathrm{d}x + \frac{1}{4} \int \int |u(x)|^2\, V(x-y)\, |u(y)|^2 \,\mathrm{d}x\,\mathrm{d}y\,.$$

- Infinite-dimensional Hamiltonian system, $\mathcal{S} = H^s(\Lambda)$

$$\frac{\mathrm{d}}{\mathrm{d}t} F(u_t) = \{F, h\}(u_t)\,, \qquad F \in \mathscr{C}^\infty(\mathcal{S})$$

and

$$\{u, v\} \equiv \mathrm{Im} \int u(x)\overline{v(x)}\,\mathrm{d}x\,.$$

Gibbs measures for the NLS

- The ***Gibbs measure*** μ associated with h is the probability measure on the space of fields $u : \Lambda \to \mathbb{C}$

$$\mu(\mathrm{d}u) := \frac{1}{z}\,\mathrm{e}^{-h(u)}\,\mathrm{d}u\,, \qquad z := \int \mathrm{e}^{-h(u)}\,\mathrm{d}u\,.$$

$\mathrm{d}u =$ (formally-defined) Lebesgue measure.

- Formally, μ is invariant under the flow of the NLS:

$$(\Phi_t)_\sharp \mu = \mu\,,$$

where $\Phi_t :=$ flow map of NLS.

Gibbs measures for the NLS: known results

- **Rigorous construction of Gibbs measure:** CQFT literature in the 1970-s (Nelson, Glimm-Jaffe, Simon), also Lebowitz-Rose-Speer (1988).
- **Proof of invariance:** Bourgain (1990s).
- **Application to nonlinear dispersive PDE:** *Obtain low-regularity solutions μ-almost surely*.
  Bourgain-Bulut, Burq-Tzvetkov, Bringmann, Burq-Thomann-Tzvetkov, Cacciafesta- de Suzzoni, Deng-Nahmod-Yue, Dinh-Rougerie, Dinh-Rougerie-Tolomeo-Wang, Fan-Ou-Staffilani-Wang, Genovese-Lucà-Valeri, Genovese-Lucà-Tzvetkov, Krieger-Lührmann-Staffilani, Lührmann-Mendelson, Nahmod-Oh-Rey-Bellet-Staffilani, Nahmod-Rey-Bellet-Sheffield-Staffilani, Oh-Pocovnicu, Oh-Sosoe-Tolomeo, . . .
- **Mean-field limits of quantum many-body Gibbs states**
  Lewin-Nam-Rougerie, Fröhlich-Knowles-Schlein-S., Ammari-Ratsimanetrimanana, Ammari-Farhat-Petrat, Rout-S., Dinh-Rougerie, Nam-Zhu-Zhu.

Gibbs measures for the NLS

- Let $h_0(u) := \frac{1}{2}\int \mathrm{d}x\,(|\nabla u(x)|^2 + |u(x)|^2)$. Define the ***Wiener measure*** μ_0

$$\mu_0(\mathrm{d}u) := \frac{1}{z_0}\,\mathrm{e}^{-h_0(u)}\,\mathrm{d}u\,, \qquad z_0 := \int \mathrm{e}^{-h_0(u)}\,\mathrm{d}u\,.$$

  Typical elements of $\operatorname{supp}\mu_0$ can be written as

$$\sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{(|n|^2+1)^{1/2}}\,\mathrm{e}^{2\pi i n\cdot x}\,, \;\; (g_n) = \text{i.i.d. complex Gaussians.}$$

  Series converges almost surely in $H^{1-\frac{d}{2}-\varepsilon}(\Lambda)$.

- $d = 1$ and $V \in L^\infty, V \geqslant 0$ pointwise, almost surely

$$h^I := \frac{1}{4}\int\int |u(x)|^2\, V(x-y)\,|u(y)|^2\,\mathrm{d}x\,\mathrm{d}y \in [0,\infty)\,.$$

- $d = 2,3$ and $V \in L^\infty, \hat{V} \geqslant 0$ pointwise; *renormalize* by ***Wick ordering***.

$$h^{I,\mathrm{Wick}} := \frac{1}{4}\int\int \big(|u(x)|^2 - \infty\big)\,V(x-y)\,\big(|u(y)|^2-\infty\big)\,\mathrm{d}x\,\mathrm{d}y \in [0,\infty)\,.$$

  Here $\infty = \mathbb{E}_{\mu_0}(|u(\cdot)|^2)$. Obtain $\mu \ll \mu_0$.

(Informal) Statement of result for defocusing problems

- **General question:** Can we compare Gibbs measures and classical KMS states?
  Answer/Theorem: They coincide (under appropriate assumptions)!
- **Structure of proof:**
  - Establish the Gibbs-KMS equivalence in general nonlinear infinite-dimensional systems (under appropriate assumptions).
  - Verify the assumptions for the relevant examples from *defocusing* nonlinear dispersive PDEs.
- In order to interpret our result in the PDE context, we reintroduce the (inverse temperature) β.

$$\mu_\beta(\mathrm{d}u) := \frac{1}{z_\beta}\,\mathrm{e}^{-\beta h(u)}\,\mathrm{d}u\,, \qquad z_\beta := \int \mathrm{e}^{-\beta h(u)}\,\mathrm{d}u\,.$$

$$\mu_{\beta,0}(\mathrm{d}u) := \frac{1}{z_{\beta,0}}\,\mathrm{e}^{-\beta h_0(u)}\,\mathrm{d}u\,, \qquad z_{\beta,0} := \int \mathrm{e}^{-\beta h_0(u)}\,\mathrm{d}u\,.$$

Outline

Outline of the rest of the talk

1. Obtain Gibbs-KMS equivalence in finite-dimensional dynamical systems.
2. Generalize to infinite-dimensional linear dynamical systems.
3. Obtain Gibbs-KMS equivalence in nonlinear infinite-dimensional dynamical systems.
   Study ***defocusing*** nonlinear dispersive PDEs.
   $\rightarrow$ Use ***Malliavin derivatives*** and ***Gross-Sobolev spaces***.
4. Study the ***focusing*** nonlinear Schrödinger/Hartree equation on $\mathbb{T}^d$ with $d = 1, 2, 3$.
   $\rightarrow$ Prove a ***local*** Gibbs-KMS equivalence.

- 1–3: Based on Ammari-S. (Revista Matemática Iberoamericana 2023).
- 4: Based on Ammari-Rout-S. (Preprint 2024).

Finite-dimensional systems

The linear algebra setup.

- E – Hermitian space with inner product $\langle \cdot, \cdot \rangle$ such that $\dim_{\mathbb{C}}(E) = n$.
  $\{e_1, \ldots, e_n\}$ – orthonormal basis of E.
- View E as a **real** vector space $E_{\mathbb{R}}$ with inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}} := \mathrm{Re}\langle \cdot, \cdot \rangle$.
  Let $f_j := \mathrm{i} e_j, j = 1, \ldots, n$.
  $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ – orthonormal basis of $E_{\mathbb{R}}$.
- For $F, G \in \mathscr{C}^{\infty}(E)$,

$$\nabla F(u) := \sum_{j=1}^{n} \frac{\partial F}{\partial e_j}(u)\, e_j + \sum_{j=1}^{n} \frac{\partial F}{\partial f_j}(u)\, f_j$$

$$\{F, G\}(u) := \sum_{j=1}^{n} \left(\frac{\partial F}{\partial e_j}(u) \frac{\partial G}{\partial f_j}(u) - \frac{\partial G}{\partial e_j}(u) \frac{\partial F}{\partial f_j}(u) \right).$$

- Hamiltonian $h \in \mathscr{C}^1(E)$. Define $X := -\mathrm{i}\nabla h$.

Finite-dimensional systems

- **Hamiltonian system:**
  $h \in \mathscr{C}^1(E), \quad X = -\mathrm{i}\nabla h : E \to E$.
  $$\dot{u}(t) = X(u(t))\,.$$
  $\rightarrow$ Assume that it has a global flow.
- **Gibbs measure:**
  $\mathrm{d}L$ – Lebesgue measure on E, $\beta > 0$.
  Assume that
  $$z_\beta := \int_E \mathrm{e}^{-\beta h(u)}\,\mathrm{d}L < \infty\,.$$
  Define
  $$\mu_\beta := \frac{\mathrm{e}^{-\beta h(\cdot)}\,\mathrm{d}L}{\int_E \mathrm{e}^{-\beta h(u)}\,\mathrm{d}L} \equiv \frac{1}{z_\beta}\,\mathrm{e}^{-\beta h(\cdot)}\,\mathrm{d}L\,.$$
  $\rightarrow$ Invariant under the flow.
- **KMS condition:**
  $\mu \in \mathscr{P}(E)$ is a (β, X)-KMS state if for all $F, G \in \mathscr{C}_c^\infty(E)$ we have
  $$\int_E \{F, G\}(u)\,\mathrm{d}\mu = \beta \int_E \mathrm{Re}\,\langle \nabla F(u), X(u)\rangle\, G(u)\,\mathrm{d}\mu\,.$$

Finite-dimensional systems

$\mathrm{Re}\langle \nabla F(u), X(u)\rangle = \{F,h\}(u) \Rightarrow (\beta, X)$- KMS condition is equivalent to

$$\int_E \{F,G\}(u)\,\mathrm{d}\mu = \beta \int_E \{F,h\}(u)\,G(u)\,\mathrm{d}\mu$$

Theorem 1: Ammari-S. (2023).

Let $\mu \in \mathscr{P}(E)$. Then μ is a (β, X)-KMS state if and only if $\mu = \mu_\beta$.

- μ_β: Stationary solution of the Liouville equation, i.e.

$$\int_E \mathrm{Re}\,\langle \nabla F(u), X(u)\rangle\,\mathrm{d}\mu_\beta(u) = 0$$

  for all $F \in \mathscr{C}_c^\infty(E)$.
- KMS states are stationary solutions to the Liouville equation.
- By Theorem 1, the two notions (Gibbs and KMS) coincide.

Finite-dimensional systems

Theorem 1: Ammari-S. Rev. Matemática Iberoam. (2023).

Let $\mu \in \mathscr{P}(E)$. Then μ is a (β, X)-KMS state if and only if $\mu = \mu_\beta$.

$\Leftarrow$: Let $\mu = \mu_\beta$. We integrate by parts to write

$$\int_E \frac{\partial F}{\partial e_j}(u)\,\frac{\partial G}{\partial f_j}(u)\,\mathrm{d}\mu_\beta = -\frac{1}{z_\beta}\int G(u)\,\frac{\partial}{\partial f_j}\left(\frac{\partial F}{\partial e_j}(u)\,\mathrm{e}^{-\beta h(u)}\right)\mathrm{d}L$$
$$= -\int_E G(u)\,\frac{\partial^2 F}{\partial f_j \partial e_j}(u)\,\mathrm{d}\mu_\beta + \beta\int_E G(u)\,\frac{\partial F}{\partial e_j}\,\frac{\partial h}{\partial f_j}\,\mathrm{d}\mu_\beta\,.$$

Similarly

$$\int_E \frac{\partial G}{\partial e_j}(u)\,\frac{\partial F}{\partial f_j}(u)\,\mathrm{d}\mu_\beta = -\int_E G(u)\,\frac{\partial^2 F}{\partial e_j \partial f_j}(u)\,\mathrm{d}\mu_\beta + \beta\int_E G(u)\,\frac{\partial F}{\partial f_j}\,\frac{\partial h}{\partial e_j}\,\mathrm{d}\mu_\beta\,.$$

Hence

$$\int_E \{F,G\}(u)\,\mathrm{d}\mu_\beta = \beta\int_E \{F,h\}(u)\,G(u)\,\mathrm{d}\mu_\beta \Rightarrow \mu_\beta \text{ is a } (\beta, X)\text{-KMS state.}$$

Finite-dimensional systems

$\Rightarrow$: Let $\mu \in \mathscr{P}(E)$ be a (β, X)-KMS state.
We adapt an argument from Aizenman-Goldstein-Gruber-Lebowitz-Martin (1977) to show that $\mu = \mu_\beta$.

- By the Leibniz rule for $\{\cdot, \cdot\}$, for all $F, G \in \mathscr{C}_c^\infty(E)$

$$\{F, G\, \mathrm{e}^{-\beta h(u)}\} = \{F, G\}\, \mathrm{e}^{-\beta h(u)} - \beta\{F, h\}\, G(u)\, \mathrm{e}^{-\beta h(u)}\,.$$

- Since μ is a (β, X)-KMS state, we get

$$\int_E \{F, G\, \mathrm{e}^{-\beta h(u)}\}\, \mathrm{e}^{\beta h(u)}\, \mathrm{d}\mu = 0\,.$$

- Define $\nu := \mathrm{e}^{\beta h(u)}\, \mu$. Then for all $F \in \mathscr{C}_c^\infty(E), G \in \mathscr{C}_c^1(E)$, we have

$$\nu(\{F, G\}) \equiv \int_E \{F, G\}(u)\, \mathrm{d}\nu = 0\,.$$

$\Longrightarrow$ $\nu = cL$. Since $\mu \in \mathscr{P}(E)$, we deduce $\mu = \mu_\beta$. $\square$

Infinite-dimensional linear systems

H - a separable complex Hilbert space.

- **Hamiltonian system:**
  Consider a linear operator A densely defined on H.
  - $\exists c > 0$ s.t. $A \geqslant c\mathbf{1}$.
  - $\exists$ o.n.b. of H consisting of eigenvectors (e_j) with eigenvalues λ_j.
  - $\exists s \geqslant 0$ such that $\sum_{j=1}^{\infty} \frac{1}{\lambda_j^{1+s}} = \mathrm{tr}(A^{-1-s}) < \infty$.

  The Hamiltonian is $h_0 : D(A^{1/2}) \to \mathbb{R}\,, \quad h_0(u) := \frac{1}{2}\langle u, Au\rangle\,.$
  $\langle u, v\rangle_{H^r} := \langle A^{r/2}u, A^{r/2}v\rangle$. H^r-associated Hilbert space.
  We have the embedding $H^s \subseteq H \subseteq H^{-s}$.
  Example: For $d \leqslant 3$, consider $H = L^2(\mathbb{T}^d), \quad A = -\Delta + 1, \quad s > \frac{d}{2} - 1$.
  $h_0(u) = \frac{1}{2}\int \bar{u}(x)(1-\Delta)u(x)\,\mathrm{d}x$.
- Think of H as a real Hilbert space with

$$\langle \cdot, \cdot \rangle_{L^2, \mathbb{R}} := \mathrm{Re}\langle \cdot, \cdot \rangle_{L^2}$$

  and o.n.b. $\{e_j, f_j\,, j \in \mathbb{N}\}, \quad f_j = \mathrm{i}e_j\,.$
- **Notation:** Fix $s > \frac{d}{2} - 1$ henceforth.

Cylindrical smooth functions

- Define $\pi_n : H^{-s} \to \mathbb{R}^{2n}$ by

$$\pi_n(u) := \big(\langle u, e_1\rangle_{L^2,\mathbb{R}}, \ldots, \langle u, e_n\rangle_{L^2,\mathbb{R}}; \langle u, f_1\rangle_{L^2,\mathbb{R}}, \ldots, \langle u, f_n\rangle_{L^2,\mathbb{R}}\big)\,.$$

- We say $F \in \mathscr{C}^\infty_{c,cyl}(H^{-s})$ if $\exists n \in \mathbb{N}\ \exists \varphi \in \mathscr{C}^\infty_c(\mathbb{R}^{2n})$ such that $F = \varphi \circ \pi_n$

$$F(u) = \varphi\big(\langle u, e_1\rangle_{L^2,\mathbb{R}}, \ldots, \langle u, e_n\rangle_{L^2,\mathbb{R}}; \langle u, f_1\rangle_{L^2,\mathbb{R}}, \ldots, \langle u, f_n\rangle_{L^2,\mathbb{R}}\big)\,.$$

$\to$ **Cylindrical smooth functions.**

- For $F = \varphi \circ \pi_n \in \mathscr{C}^\infty_{c,cyl}(H^{-s})$

$$\nabla F(u) := \sum_{j=1}^{n} \Big[\partial_j \varphi(\pi_n(u))\, e_j + \partial_{n+j}\varphi(\pi_n(u))\, f_j\Big]\,.$$

- For $F = \varphi \circ \pi_n\,, \quad G = \psi \circ \pi_m \in \mathscr{C}^\infty_{c,cyl}(H^{-s})$, we define

$$\{F, G\}(u) :=$$

$$\sum_{j=1}^{\min(n,m)} \Big[\partial_j \varphi(\pi_n(u))\, \partial_{m+j}\psi(\pi_m(u)) - \partial_j \psi(\pi_m(u))\, \partial_{n+j}\varphi(\pi_n(u))\Big]\,.$$

Infinite-dimensional linear systems

- **Gibbs measure:**
  Formally $\mu_{\beta,0} = \frac{\mathrm{e}^{-\beta h(\cdot)}\,\mathrm{d}u}{\int \mathrm{e}^{-\beta h(\cdot)}\,\mathrm{d}u}$.
- Rigorously, consider $\mu_{\beta,0}$ to be the unique centered Gaussian measure on H^{-s} with covariance $\beta^{-1}\,A^{-1-s}$, i.e. for all $f, g \in H^{-s}$

$$\frac{1}{\beta}\,\langle f, A^{-1-s} g\rangle_{H^{-s},\mathbb{R}} = \int_{H^{-s}} \langle f, u\rangle_{H^{-s},\mathbb{R}}\,\langle u, g\rangle_{H^{-s},\mathbb{R}}\,\mathrm{d}\mu_{\beta,0}\,.$$

- Define the orthogonal projection $P_n : H^{-s} \to E_n \equiv \mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_n\}$.
  Fact:

$$(P_n)_\sharp \mu_{\beta,0} = \frac{\mathrm{e}^{-\frac{\beta}{2}\langle\cdot, A\cdot\rangle}\,\mathrm{d}L_{2n}}{\int_{E_n} \mathrm{e}^{-\frac{\beta}{2}\langle\cdot, A\cdot\rangle}\,\mathrm{d}L_{2n}}\,,$$

  where $\mathrm{d}L_{2n} =$ Lebesgue measure on E_n.

Infinite-dimensional linear systems

We consider $X_0 = -\mathrm{i}A$.
$\mu \in \mathscr{P}(H^{-s})$ is a (β, X_0)-KMS state if for all $F, G \in \mathscr{C}^\infty_{c,cyl}(H^{-s})$ we have

$$\int_{H^{-s}} \{F,G\}(u)\,\mathrm{d}\mu = \beta \int_{H^{-s}} \mathrm{Re}\,\langle \nabla F(u), X_0(u)\rangle_{L^2}\, G(u)\,\mathrm{d}\mu\,.$$

Theorem 2: Ammari-S. Rev. Matemática Iberoam.(2023).

$\mu \in \mathscr{P}(H^{-s})$ is a (β, X_0)-KMS state if and only if $\mu = \mu_{\beta,0}$.

Sketch of proof of $\Rightarrow$: Let $\tilde{\mu}_n := (P_n)_\sharp \mu \in \mathscr{P}(E_n)$.
One shows that for $F = \varphi \circ \pi_n, G = \psi \circ \pi_n \in \mathscr{C}^\infty_{c,cyl}(H^{-s})$

$$\int_{E_n} \{F,G\}(u)\,\mathrm{d}\tilde{\mu}_n = \beta \int_{E_n} \mathrm{Re}\,\langle \nabla F(u), X_0(u)\rangle_{E_n}\, G(u)\,\mathrm{d}\tilde{\mu}_n\,.$$

Hence $\tilde{\mu}_n$ is a KMS state. Theorem 1 implies that for all n

$$\tilde{\mu}_n \equiv (P_n)_\sharp \mu = \frac{\mathrm{e}^{-\frac{\beta}{2}\langle\cdot,A\cdot\rangle}\,\mathrm{d}L_{2n}}{\int_{E_n} \mathrm{e}^{-\frac{\beta}{2}\langle\cdot,A\cdot\rangle}\,\mathrm{d}L_{2n}} = (P_n)_\sharp \mu_{\beta,0}\,.$$

The claim follows by taking $n \to \infty$. $\square$

Infinite-dimensional nonlinear systems

- **The Malliavin derivative:**
  Recall that for $F = \varphi \circ \pi_n \in \mathscr{C}^\infty_{c,cyl}(H^{-s})$,

$$\nabla F(u) = \sum_{j=1}^{n} \Big[\partial_j \varphi(\pi_n(u))\, e_j + \partial_{n+j} \varphi(\pi_n(u))\, f_j \Big]\,.$$

  Fact: For $p \in [1, \infty)$, the operator

$$\nabla : \mathscr{C}^\infty_{c,cyl}(H^{-s}) \subseteq L^p(\mu_{\beta,0}) \to L^p(\mu_{\beta,0}; H^{-s})$$

  is closable.
  $\rightarrow$ *The Malliavin derivative*.
- **Gross-Sobolev spaces:**
  $\mathbb{D}^{1,p}(\mu_{\beta,0})$- closure domain of ∇.

$$\|F\|_{\mathbb{D}^{1,p}(\mu_{\beta,0})} \sim \|F\|_{L^p(\mu_{\beta,0})} + \|\nabla F\|_{L^p(\mu_{\beta,0}; H^{-s})}$$

  $\rightarrow$ a Banach space; a Hilbert space for $p = 2$.

Infinite-dimensional nonlinear systems: Results

Consider $h^I : H^{-s} \to \mathbb{R}$, Borel measurable such that

$$\mathrm{e}^{-\beta h^I} \in L^1(\mu_{\beta,0}), \qquad h^I \in \mathbb{D}^{1,2}(\mu_{\beta,0}).$$

Define

$$X = -\mathrm{i}A - \mathrm{i}\nabla h^I .$$

We say that $\mu \in \mathscr{P}(H^{-s})$ is a (β, X)-KMS state if for all $F, G \in \mathscr{C}^\infty_{c,cyl}(H^{-s})$ we have

$$\int_{H^{-s}} \{F, G\}(u)\,\mathrm{d}\mu = \beta \int_{H^{-s}} \mathrm{Re}\,\langle \nabla F(u), X(u)\rangle_{L^2}\, G(u)\,\mathrm{d}\mu .$$

Theorem 3: Ammari-S. (Rev. Matemática Iberoam. 2023).

The following hold (under additional technical assumptions).

(i) The Gibbs measure

$$\mu_\beta = \frac{\mathrm{e}^{-\beta h^I}\,\mu_{\beta,0}}{\int_{H^{-s}} \mathrm{e}^{-\beta h^I}\,\mathrm{d}\mu_{\beta,0}}$$

is a (β, X)-KMS state.

(ii) Let $\mu \in \mathscr{P}(H^{-s})$ be a (β, X)-KMS state. Then $\mu = \mu_\beta$.

Infinite-dimensional nonlinear systems: Comments

- **Technical assumptions needed:**
  - For (ii), assume that $\varrho := \frac{\mathrm{d}\mu}{\mathrm{d}\mu_{\beta,0}} \in \mathbb{D}^{1,2}(\mu_{\beta,0})$.
  - Obtain that ϱ solves
    $$\nabla\varrho + \beta\varrho\nabla h^I = 0\,.$$
  - Use PDE to get
    $$\nabla(\mathrm{e}^{\beta h^I}\varrho) = 0 \Rightarrow \mathrm{e}^{\beta h^I}\varrho = C \Rightarrow \varrho = \frac{1}{\int \mathrm{e}^{-\beta h^I}\,\mathrm{d}\mu_{\beta,0}}\,\mathrm{e}^{-\beta h^I}\,.$$
- **Method of proof, Integration by parts:**
  For $F \in \mathscr{C}^\infty_{c,cyl}(H^{-s}),\ G \in \mathbb{D}^{1,2}(\mu_{\beta,0}),\ \varphi \in H^1$, we have

$$\int_{H^{-s}} G(u)\,\langle\nabla F(u),\varphi\rangle_{L^2}\,\mathrm{d}\mu_{\beta,0} =$$
$$\int_{H^{-s}} F(u)\left(-\langle\nabla G(u),\varphi\rangle_{L^2} + \beta G(u)\,\langle u, A\varphi\rangle_{L^2}\right)\mathrm{d}\mu_{\beta,0}\,.$$

Applications to defocusing nonlinear dispersive PDEs

Example: Let $d = 2, 3$.
Consider the *Wick-ordered nonlocal NLS (Hartree) equation.*

$$\mathrm{i}\partial_t u + (\Delta - 1)u = \Big[V * (:|u|^2:)\Big]\, u \text{ on } \mathbb{T}^d \times \mathbb{R}\,.$$

$:|u|^2: \, := |u|^2 - \mathbb{E}_{\mu_0}(|u(\cdot)|^2)$ denotes Wick ordering with respect to μ_0.

- Assume

$$\begin{cases} 0 \leqslant \hat{V}(k) \leqslant \frac{C}{\langle k \rangle^{\varepsilon}} & \text{if } \; d = 2 \\ 0 \leqslant \hat{V}(k) \leqslant \frac{C}{\langle k \rangle^{2+\varepsilon}} & \text{if } \; d = 3\,. \end{cases}$$

  $\rightarrow$ Analogous to assumptions in Bourgain (JMPA 1997); recent improvement in $3D$ by Deng-Nahmod-Yue (JMP 2021).

- $\mu_{\beta,0}$-almost surely, we have

$$h^I \equiv \frac{1}{4} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} :|u(x)|^2: V(x-y) :|u(y)|^2: \mathrm{d}x\, \mathrm{d}y \in [0, \infty)\,.$$

$$\Rightarrow \mathrm{e}^{-\beta h^I(\cdot)} \in L^2(\mu_{\beta,0})\,, \qquad h^I \in \mathbb{D}^{1,2}(\mu_{\beta,0})\,.$$

Focusing NLS/Hartree equation

We study the following *focusing* problems.

1. Focusing NLS on $\Lambda = \mathbb{T}^1$:

$$\mathrm{i}\partial_t u + (\Delta - 1) = -|u|^r u\,, \quad r = 3, 5\,.$$

$$h(u) = \frac{1}{2}\int_{\mathbb{T}^1} (|\nabla u|^2 + |u|^2)\, dx - \frac{1}{r+1}\int_{\mathbb{T}^1} |u|^{r+1}\, \mathrm{d}x\,.$$

2. (Wick-ordered) focusing Hartree equation on $\Lambda = \mathbb{T}^d$ for $d = 2, 3$:

$$\mathrm{i}\partial_t u + (\Delta - 1) = \left[V * :|u|:^2\right] u$$

with no sign assumptions on V.

$$h(u) = \frac{1}{2}\int_{\Lambda} (|\nabla u|^2 + |u|^2)\, dx + \frac{1}{4}\int_{\mathbb{T}^d} \left[V * :|u|^2:\right] :|u|^2: \mathrm{d}x\,.$$

Here

$$h^{I,\mathrm{Wick}} = \frac{1}{4}\int_{\mathbb{T}^d} (V * :|u|^2:) :|u|^2: \mathrm{d}x$$

is not necessarily $\geqslant 0$.

- One in general has $z_\beta = \int \mathrm{e}^{-\beta h^I(u)} \mathrm{d}\mu_{\beta,0} = \infty$.

Local Gibbs measures

- Let us fix $\beta = 1$ throughout the sequel.
- We study ***local Gibbs measures***

$$\mu_{\mathrm{Gibbs},R} := \frac{1}{z_R} \mathrm{e}^{-h^I} \, \Xi_R(\mathcal{M}(u)) \, \mathrm{d}\mu_{1,0} \,, \qquad z_R := \int \mathrm{e}^{-h^I} \, \Xi_R(\mathcal{M}(u)) \, \mathrm{d}\mu_{1,0} \,.$$

Here, $R > 0$, $\Xi_R = \mathbf{1}_{(-R,R)}$ and

$$\mathcal{M}(u) := \begin{cases} \|u\|_{L^2}^2 & \text{if } d = 1 \,, \\ :\|u\|_{L^2}^2 : \equiv \|u\|_{L^2}^2 - \mathbb{E}_{\mu_{1,0}}\big[\|u\|_{L^2}^2\big] & \text{if } d = 2, 3 \,. \end{cases}$$

- Lebowitz-Rose-Speer, Bourgain, Brydges-Slade, Thomann-Tzvetkov, Oh-Okamoto-Tolomeo, Oh-Sosoe-Tolomeo, Dinh-Rougerie-Tolomeo-Wang, Carlen-Fröhlich-Lebowitz, Weber-Tolomeo, Robert-Seong-Tolomeo-Wang,. . . .
  $\to$ obtain a well-defined probability measure.
- When $d = 1$ and $p = 5$, one needs to take $R > 0$ sufficiently small; otherwise one takes $R > 0$ arbitrary.
- **Goal:** Relate local Gibbs measures to suitable *(local) KMS states*.

Local KMS states

Let us consider the (Wick-ordered) focusing Hartree equation on $\Lambda = \mathbb{T}^d, d = 2, 3$.

$$\mathrm{i}\partial_t u + (\Delta - 1) = \left[V * : |u|^2 :\right] u$$

$\rightarrow$ Rewrite as $\quad \dot{u} = X(u),\ X(u) = \mathrm{i}(\Delta - 1)u + \mathrm{i}\nabla h^{I,\mathrm{Wick}}(u)\,.$

Define

$$\mathbb{B}_R := \{u \in H^{-s}\,,\ |\mathcal{M}(u)| < R\}\,, \qquad \mathbb{B}_R^c \equiv H^{-s} \setminus \mathbb{B}_R\,.$$

$$\mathbb{D}_R^{1,2}(\mu_{1,0}) := \left\{G \in \mathbb{D}^{1,2}(\mu_{1,0})\,,\ \exists\, R' \in (0,R) \text{ s.t. } G(u) = 0\ \mu_{1,0}\text{-a.s. on } \mathbb{B}_{R'}^c\right\}.$$

Definition (Local KMS condition)

$\mu \in \mathscr{P}(H^{-s})$ is a ***local KMS state*** if

$$\int_{H^{-s}} \{F, G\}(u)\,\mathrm{d}\mu = \int_{H^{-s}} \langle \nabla F(u), X(u)\rangle_{L^2,\mathbb{R}}\, G(u)\,\mathrm{d}\mu\,.$$

for all $G \in \mathbb{D}_R^{1,2}(\mu_{1,0})$ and $F \in \mathscr{C}_{c,cyl}^{\infty}(H^{-s})$.

Statement of results

Theorem 4: Ammari-Rout-S. (Preprint, 2024).

The following results hold

(i) The local Gibbs measure

$$\mu_{\mathrm{Gibbs},R} := \frac{1}{z_R}\,\mathrm{e}^{-h^I}\,\Xi_R(\mathcal{M}(u))\,\mathrm{d}\mu_{1,0}\,, \quad z_R := \int \mathrm{e}^{-h^I}\,\Xi_R(\mathcal{M}(u))\,\mathrm{d}\mu_{1,0}\,.$$

is a local KMS state.

(ii) Let $\mu \in \mathscr{P}(H^{-s})$ be a local KMS state. Suppose that

$$\mathrm{d}\mu = \varrho\,\mathrm{d}\mu_{1,0}$$

with $\varrho \in \mathbb{D}^{1,2}(\mu_{1,0}) \cap L^4(\mu_{1,0})$. Then μ is locally a Gibbs measure: $\exists c_0 \geqslant 0$ such that $\mu_{1,0}$ almost surely on $\mathbb{B}_R$ one has

$$\varrho(u) = c_0 \mathrm{e}^{-h^I(u)}\,.$$

Statement of results

One also notes that local Gibbs measures are stationary solutions to the Liouville equation.
(**Note:** This does not follow from Theorem 4 (i) since we *cannot take* $G = 1$.)

Theorem 5: Ammari-Rout-S. (Preprint, 2024).

The following results hold

(i) The local Gibbs measure $\mu_{\mathrm{Gibbs,R}}$ satisfies

$$\int_{H^{-s}} \langle \nabla F, X(u) \rangle_{L^2,\mathbb{R}} \, \mathrm{d}\mu_{\mathrm{Gibbs,R}} = 0\,.$$

for all $F \in \mathscr{C}^{\infty}_{c,cyl}(H^{-s})$.

Using the results of Ammari-Farhat-S. (AIM 2024), we obtain an alternative proof of Bourgain's almost sure global existence result.

Corollary

The focusing NLS/(Wick-ordered) Hartree equation admits a global solution for $\mu_{\mathrm{Gibbs},R}$ every initial data in H^{-s}.

Elements of the proofs

- The proofs of Theorem 4 (i) and Theorem 5 are more complicated in the local setting, due to the presence of the cut-off.
- An important observation is that $\{h, \mathcal{M}\} = 0$. This allows us to show Theorem 5 in the finite-dimensional setting and then extend to the general setting by considering cylindrical functions and using methods of Malliavin calculus.
- When showing that local KMS states $\mu = \varrho\mu_{1,0}$ are local Gibbs measures, we obtain the differential equation

$$\nabla(\mathrm{e}^{-h^I}\tilde{\Xi}_R(\mathcal{M})\varrho) = 0$$

  on $\mathbb{B}_R \equiv \{u \in H^{-s}\,,\ |\mathcal{M}(u)| < R\}$
  ($\tilde{\Xi}_R$ is a smoothed-out characteristic function of $[-R, R]$.).
- We can deduce that $\mathrm{e}^{-h^I}\tilde{\Xi}_R(\mathcal{M})\varrho$ is constant by using work of Aida (1998) provided we show that $\mathbb{B}_R$ is H^1 connected, i.e. for all $u \in \mathbb{B}_R$, the set

$$\{w \in H^1 \,:\, u + w \in \mathbb{B}_R\}$$

  is connected in H^1.
- This is immediate when $d = 1$, but it requires an additional construction when $d = 2, 3$.

Finite-dimensional calculation

Work on $\mathbb{R}^{2n}$ with $h \in \mathscr{C}^\infty(\mathbb{R}^{2n})$ such that

$$\{h, M\} = 0\,, \quad M(p,q) = \sum_{j=1}^{n} (p_j^2 + q_j^2)\,.$$

Vector field

$$X : \mathbb{R}^{2n} \to \mathbb{R}^{2n}\,, \quad X = \left(\left(\frac{\partial h}{\partial q_j} \right)_{j=1,\ldots,n}, \left(-\frac{\partial h}{\partial p_j} \right)_{j=1,\ldots,n} \right).$$

For $\psi \in \mathscr{C}_c^\infty(\mathbb{R})$, consider the local Gibbs measure

$$d\mu_\psi := \frac{1}{z_\psi}\, \mathrm{e}^{-h(p,q)} \psi(M)\, dp\, dq \in \mathscr{P}(\mathbb{R}^{2n})\,, \quad z_\psi := \int \mathrm{e}^{-h(p,q)}\, \psi(M)\, \mathrm{d}p\, \mathrm{d}q\,.$$

Proposition (Ammari-Rout-S.)

For all $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^{2n})$, we have

$$\int_{\mathbb{R}^{2n}} \mathrm{Re}\, \langle \nabla\varphi, X \rangle\, \mathrm{d}\mu_\psi = 0\,.$$

Finite-dimensional calculation

Vector field $X = \left(\left(\frac{\partial h}{\partial q_j} \right)_{j=1,\ldots,n}, \left(-\frac{\partial h}{\partial p_j} \right)_{j=1,\ldots,n} \right)$.

Local Gibbs measure

$d\mu_\psi := \frac{1}{z_\psi} \mathrm{e}^{-h(p,q)} \psi(M)\, dp\, dq \in \mathscr{P}(\mathbb{R}^{2n})\,, \quad z_\psi := \int \mathrm{e}^{-h(p,q)}\, \psi(M)\, \mathrm{d}p\, \mathrm{d}q\,.$

We compute:

$$\int_{\mathbb{R}^{2n}} \mathrm{Re}\, \langle \nabla\varphi, X \rangle\, \mathrm{d}\mu_\psi = \frac{1}{z_\psi} \int_{\mathbb{R}^{2n}} \mathrm{Re}\, \langle \nabla\varphi, X \rangle\, \mathrm{e}^{-h}\, \psi(M)\, \mathrm{d}p\, \mathrm{d}q$$

$$= \frac{1}{z_\psi} \sum_{j=1}^{n} \int_{\mathbb{R}^{2n}} \left(\frac{\partial\varphi}{\partial p_j} \frac{\partial h}{\partial q_j} - \frac{\partial\varphi}{\partial q_j} \frac{\partial h}{\partial p_j} \right) \mathrm{e}^{-h}\, \psi(M)\, \mathrm{d}p\, \mathrm{d}q\,.$$

Integrate by parts to rewrite as:

$$\frac{1}{z_\psi} \sum_{j=1}^{n} \int_{\mathbb{R}^{2n}} \varphi \Bigg[\left(-\frac{\partial^2 h}{\partial p_j \partial q_j} + \frac{\partial^2 h}{\partial q_j \partial p_j} \right) \mathrm{e}^{-h}\, \psi(M) - \frac{\partial h}{\partial q_j} \frac{\partial}{\partial p_j} (\mathrm{e}^{-h}\, \psi(M))$$

$$+ \frac{\partial h}{\partial p_j} \frac{\partial}{\partial q_j} (\mathrm{e}^{-h}\, \psi(M)) \Bigg]\, \mathrm{d}p\, \mathrm{d}q$$

$$= \frac{1}{z_\psi} \int_{\mathbb{R}^{2n}} \varphi\, \{h, \mathrm{e}^{-h}\, \psi(M)\}\, \mathrm{d}p\, \mathrm{d}q\,.$$

Finite-dimensional calculation

By the Leibniz rule for the Poisson bracket

$$\begin{aligned}\frac{1}{z_\psi}\int_{\mathbb{R}^{2n}} \varphi\,\{h, \mathrm{e}^{-h}\,\psi(M)\}\,\mathrm{d}p\,\mathrm{d}q \\ &= \frac{1}{z_\psi}\int_{\mathbb{R}^{2n}} \big(\varphi\,\{h,\mathrm{e}^{-h}\}\,\psi(M) + \varphi\,\{h,\psi(M)\}\,\mathrm{e}^{-h}\big)\,\mathrm{d}p\,\mathrm{d}q \\ &= \frac{1}{z_\psi}\int_{\mathbb{R}^{2n}} \big(\varphi\,\{h,\mathrm{e}^{-h}\}\,\psi(M) + \varphi\,\psi'(M)\,\{h,M\}\,\mathrm{e}^{-h}\big)\,\mathrm{d}p\,\mathrm{d}q = 0\,.\end{aligned}$$

Above, we used

$$\{h,\mathrm{e}^{-h}\} = \{h,M\} = 0\,.$$

We conclude

$$\int_{\mathbb{R}^{2n}} \mathrm{Re}\,\langle\nabla\varphi, X\rangle\,\mathrm{d}\mu_\psi = 0\,. \quad \square$$

Thank you for your attention!