The large-mass limit of interacting Bose gases in the continuum

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Summary

We consider a gas of spinless bosons in the continuum.

Bosons interact with a radial two-body potential.

We assume the potential is *stable* and Hölder continuous.

We show the *grand-canonical partition function* converges to that of the classical gas when the mass of the bosons is increased to infinity.

We obtain quantitative estimates on the convergence.

We use the Ginibre loop representation of the partition function. This allows us to use Gaussian bounds on the Brownian bridge measures.

Similar work (non exhaustive)

Lewin, Nam, Rougerie
Derivation of nonlinear Gibbs measures from many-body quantum mechanics, 2015
Mean-field large-temperature limit in the continuum.

Fröhlich, Knowles, Schlein, Sohinger
The mean-field limit of quantum Bose gases at positive temperature, 2022
Mean-field large-mass limit in the continuum.

Interacting loop ensembles and Bose gases, 2023
Large-mass limit on a lattice.

Setting

Spatial domain is a torus

$$\Lambda \equiv \Lambda_L := [-L/2, L/2]^d, \quad L \geq 1.$$

Hamiltonian for n bosons is

$$\mathbb{H}_n^\nu := \underbrace{-\frac{\nu}{2}\sum_{i=1}^{n} \Delta_i}_{\text{kinetic part}} + \underbrace{\sum_{1 \leq i < j \leq n} v(x_i - x_j)}_{\textit{interaction}}.$$

where $\nu \equiv 1/m > 0$ is an inverse mass of a boson.

The domain of $\mathbb{H}_n$ are symmetric L^2 functions on Λ^n:

$$\mathcal{H}_n := \{f \in L^2(\Lambda^n) : f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})\ \forall \sigma \in S_n\}.$$

Grand-canonical ensemble

$$\mathbb{H}_n^\nu = -\frac{\nu}{2}\sum_{i=1}^{n}\Delta_i + \sum_{1\le i<j\le n} v(x_i - x_j)$$

$$\mathcal{H}_n = \mathrm{Symmetrised}(L^2(\Lambda^n))$$

Fix an inverse temperature $\beta > 0$ and a chemical potential $\mu < 0$.

Grand-canonical ensemble is a sequence of operators $(\rho_n)_{n\in\mathbb{N}}$:

$$\rho_n^\nu := \frac{1}{\Xi^\nu}\mathrm{e}^{-\beta(\mathbb{H}_n^\nu - \mu n)},$$

$$\Xi^\nu := 1 + \sum_{n=1}^{\infty}\mathrm{Tr}_{\mathcal{H}_n}\,\mathrm{e}^{-\beta(\mathbb{H}_n^\nu - \mu n)}.$$

We are interested in the behaviour of Ξ^ν as $\nu \to 0$.

Tuning μ

$$\rho_n^\nu = \frac{1}{\Xi^\nu} \mathrm{e}^{-\beta(\mathbb{H}_n^\nu - \mu n)}$$

$$\Xi^\nu = 1 + \sum_{n=1}^{\infty} \mathrm{Tr}_{\mathcal{H}_n} \mathrm{e}^{-\beta(\mathbb{H}_n^\nu - \mu n)}$$

If μ is kept fixed as $\nu \to 0$, then Ξ^ν diverges.

Fix a constant $\zeta > 0$ and set

$$\mu = \mu(\zeta, \nu) = \log \zeta + \frac{d}{2} \log \nu.$$

Then $\mathrm{e}^{\mu} \nu^{-d/2} = \zeta$. This will cancel out the singularity of the heat kernel at 0.

Assumptions on v

$$\mathbb{H}_n^\nu = -\frac{\nu}{2}\sum_{i=1}^{n}\Delta_i + \sum_{1\leq i<j\leq n} v(x_i - x_j)$$

Recall $v\colon \Lambda \to \mathbb{R}$ is the potential. Assume:

(i) v is **even**,

(ii) v is ***stable***: $\exists B \geq 0$ such that for all $n \in \mathbb{N}^*$ and all $x_1, \ldots, x_n \in \Lambda$

$$\sum_{1\leq i<j\leq n} v(x_i - x_j) \geq -Bn,$$

(iii) v is **α-Hölder continuous** for some $\alpha \in (0,1]$: there exists $C > 0$ so that for all $x, y \in \Lambda$

$$|v(x) - v(y)| \leq C|x-y|_\Lambda^\alpha$$

where $|u|_\Lambda := \min_{m\in\mathbb{Z}^d}|u - Lm|$ is the periodic Euclidean norm.

Results: Partition function

$$\mathbb{H}_n^\nu = -\frac{\nu}{2}\sum_{i=1}^{n} \Delta_i + \sum_{1\le i<j\le n} v(x_i - x_j)$$

$$\Xi^\nu = 1 + \sum_{n=1}^{\infty} \mathrm{Tr}_{\mathcal{H}_n} \mathrm{e}^{-\beta(\mathbb{H}_n^\nu - \mu n)}$$

Theorem 1 (Garouniatis-S.-Sohinger 2025)

For $v \colon \Lambda \to \mathbb{R}$ even, stable, and α-Hölder continuous we have

$$\left|\Xi^\nu - \Xi^0\right| \lesssim_{d,\zeta,v,\alpha,B,L} \nu^{\alpha/2}$$

where

$$\Xi^0 := 1 + \sum_{n=1}^{\infty} \frac{\zeta^n}{(2\pi)^{nd/2}\, n!} \int_{\Lambda^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \exp\Bigg(-\sum_{1\le i<j\le n} v(x_i - x_j)\Bigg)$$

is the grand-canonical partition function of a classical interacting gas.

Reduced density matrices

$$\mathbb{H}_n^\nu = -\frac{\nu}{2}\sum_{i=1}^{n}\Delta_i + \sum_{1\le i<j\le n} v(x_i - x_j)$$

$$\rho_n^\nu = \frac{1}{\Xi^\nu}\mathrm{e}^{-\beta(\mathbb{H}_n^\nu - \mu n)} \qquad \Xi^\nu = 1 + \sum_{n=1}^{\infty}\mathrm{Tr}_{\mathcal{H}_n}\,\mathrm{e}^{-\beta(\mathbb{H}_n^\nu - \mu n)}$$

Given $p \in \mathbb{N}^*$ define the ***reduced*** p***-particle density matrix*** as

$$\Gamma_p^\nu := \sum_{n=0}^{\infty}\frac{(n+p)!}{n!}\,\mathrm{Tr}_{p+1,\dots,n+p}(\rho_{n+p}^\nu)$$

where $\mathrm{Tr}_{p+1,\dots,n+p}$ is the partial trace over coordinates $x_{p+1},\dots,x_{n+p}$.

For fixed $\boldsymbol{x},\boldsymbol{y} \in \Lambda^p$, let $\Gamma_p^\nu(\boldsymbol{x},\boldsymbol{y})$ be the operator kernel of Γ_p^ν.

Results: Reduced density matrices

$$\rho_n^\nu = \frac{1}{\Xi^\nu} \mathrm{e}^{-\beta(\mathbb{H}_n^\nu - \mu n)}$$

$\Gamma_p^\nu(\boldsymbol{x}, \boldsymbol{y})$ is the operator kernel of $\Gamma_p^\nu = \sum_{n=0}^{\infty} \frac{(n+p)!}{n!} \mathrm{Tr}_{p+1,\ldots,n+p}(\rho_{n+p}^\nu)$

Theorem 2 (Garouniatis-S.-Sohinger 2025)

For any $p \in \mathbb{N}^*$ and any $\boldsymbol{x}, \boldsymbol{y} \in \Lambda^p$ we have

$$\lim_{\nu \to 0} \Gamma_p^\nu(\boldsymbol{x}, \boldsymbol{y}) = \Gamma_p^0(\boldsymbol{x}, \boldsymbol{y})$$

where

$$\Gamma_p^0(\boldsymbol{x}, \boldsymbol{y}) := \frac{\zeta^p}{(2\pi)^{pd/2}} \sum_{\pi \in S_p} \delta_K(\boldsymbol{x} - \pi \boldsymbol{y}) \frac{\Xi^0(\boldsymbol{x})}{\Xi^0}$$

and $\Xi^0(\boldsymbol{x})$ is the interacting partition function:

$$\Xi^0(\boldsymbol{x}) := \sum_{n=0}^{\infty} \frac{\zeta^n}{(2\pi)^{nd/2}\, n!} \int_{\Lambda^n} \mathrm{d}x_{p+1} \cdots \mathrm{d}x_{p+n} \exp\Big(- \sum_{1 \le i < j \le n+p} v(x_i - x_j) \Big).$$

Ginibre representation

Gases of interacting bosons are represented in terms of interacting Brownian paths – ***Ginibre ensemble***.

A ***path*** ω of ***duration*** T is a continuous map $\omega\colon [0,T] \to \Lambda$.

Let $\Omega^T_{y,x}$ be the space of all paths of duration T from $x \in \Lambda$ to $y \in \Lambda$.
Let $\Omega^T := \bigcup_{x,y\in\Lambda} \Omega^T_{y,x}$.

Let $\mathbb{W}^T_x$ be the law of Brownian motion on Ω^T that starts at x.
Let $\mathbb{W}^T_{y,x}(\mathrm{d}\omega) := \mathbb{1}_{\omega(T)=y}\mathbb{W}_x(\mathrm{d}\omega)$ be the ***Brownian bridge measure.***

Interaction of paths

Two paths $\omega \in \Omega^T$, $\tilde{\omega} \in \Omega^{\tilde{T}}$ with $T, \tilde{T} \in \nu\mathbb{N}$ interact with potential

$$\mathcal{V}^\nu(\omega, \tilde{\omega}) := \frac{1}{\nu} \sum_{s \in \nu\mathbb{N}} \mathbb{1}_{s<T} \sum_{\tilde{s} \in \nu\mathbb{N}} \mathbb{1}_{\tilde{s}<\tilde{T}} \int_0^\nu \mathrm{d}t \, v\big(\omega(s+t) - \tilde{\omega}(\tilde{s}+t)\big).$$

Every path interacts with itself as well:

$$\mathcal{V}^\nu(\omega) := \frac{1}{\nu} \sum_{s,s' \in \nu\mathbb{N}} \mathbb{1}_{s<s'<T} \int_0^\nu \mathrm{d}t \, v\big(\omega(s+t) - \omega(s'+t)\big).$$

Given n paths $(\omega_1, \ldots, \omega_n) = \boldsymbol{\omega}$, their total interaction is

$$\mathcal{V}^\nu(\boldsymbol{\omega}) := \sum_{1 \le i < j \le n} \mathcal{V}^\nu(\omega_i, \omega_j) + \sum_{i=1}^n \mathcal{V}^\nu(\omega_i).$$

Interaction of paths

$$\mathcal{V}^\nu(\omega,\tilde\omega) = \frac{1}{\nu}\sum_{s\in\nu\mathbb{N}} \mathbb{1}_{s<T} \sum_{\tilde s\in\nu\mathbb{N}} \mathbb{1}_{\tilde s<\tilde T} \int_0^\nu \mathrm{d}t\, v\big(\omega(s+t)-\tilde\omega(\tilde s+t)\big),$$

$$\mathcal{V}^\nu(\omega) = \frac{1}{\nu}\sum_{s,s'\in\nu\mathbb{N}} \mathbb{1}_{s<s'<T} \int_0^\nu \mathrm{d}t\, v\big(\omega(s+t)-\omega(s'+t)\big).$$

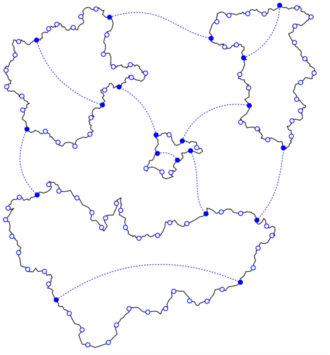

Figure: Interaction of four loops. Source: *Interacting loop ensembles and Bose gases* by Fröhlich, Knowles, Schlein, Sohinger, '23

Partition function

$$\Xi^\nu = 1 + \sum_{n=1}^{\infty} \mathrm{Tr}_{\mathcal{H}_n} \mathrm{e}^{-\beta(\mathbb{H}_n^\nu - \mu n)}$$

Define a new measure on loops:

$$\mathbb{L}^\nu(\mathrm{d}\omega) := \sum_{k=1}^{\infty} \frac{\zeta^k \nu^{kd/2}}{k} \int_\Lambda \mathrm{d}x \int_{\Omega^{k\nu}} \mathbb{W}_{x,x}^{k\nu}(\mathrm{d}\omega).$$

The partiton function in the Ginibre representation is

$$\Xi^\nu = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathbb{L}^\nu(\mathrm{d}\omega_1) \cdots \mathbb{L}^\nu(\mathrm{d}\omega_n) \exp(-\mathcal{V}^\nu(\boldsymbol{\omega})).$$

This follows from the ***Feynman-Kac formula***.

Recall theorem for partition function

$$\Xi^\nu = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathbb{L}^\nu(\mathrm{d}\omega_1) \cdots \mathbb{L}^\nu(\mathrm{d}\omega_n) \exp(-\mathcal{V}^\nu(\boldsymbol{\omega}))$$

$$\Xi^0 = 1 + \sum_{n=1}^{\infty} \frac{\zeta^n}{(2\pi)^{nd/2}\, n!} \int_{\Lambda^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \exp\Big(-\sum_{1\leq i<j\leq n} v(x_i - x_j)\Big)$$

Theorem 1 (Garouniatis-S.-Sohinger 2025)

For $v : \Lambda \to \mathbb{R}$ even, stable, and α-Hölder continuous we have

$$\left|\Xi^\nu - \Xi^0\right| \lesssim_{d,\zeta,v,\alpha,B,L} \nu^{\alpha/2}.$$

Proof: Overview

$$\Xi^{\nu} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathbb{L}^{\nu}(\mathrm{d}\omega_1) \cdots \mathbb{L}^{\nu}(\mathrm{d}\omega_n) \exp(-\mathcal{V}^{\nu}(\boldsymbol{\omega}))$$

$$\Xi^{0} = 1 + \sum_{n=1}^{\infty} \frac{\zeta^n}{(2\pi)^{nd/2}\, n!} \int_{\Lambda^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \exp\Big(-\sum_{1\le i<j\le n} v(x_i - x_j)\Big)$$

Step 1: Derive a lower bound on $\mathcal{V}(\boldsymbol{\omega})$. This follows from the stability assumption on the potential v.

Step 2: Loops of duration longer than ν are negligible when $\nu \to 0$.

Step 3: Loops of duration ν collapse to points as $\nu \to 0$.

Step 1: Lower bound on $\mathcal{V}(\boldsymbol{\omega})$

$$\mathcal{V}^{\nu}(\omega, \tilde{\omega}) := \frac{1}{\nu} \sum_{s \in \nu\mathbb{N}} \mathbb{1}_{s<T} \sum_{\tilde{s} \in \nu\mathbb{N}} \mathbb{1}_{\tilde{s}<\tilde{T}} \int_0^{\nu} \mathrm{d}t \, v(\omega(s+t) - \tilde{\omega}(\tilde{s}+t))$$

Recall the stability assumption: There exists $B \geq 0$ such that for all $n \in \mathbb{N}^*$ and all $x_1, \ldots, x_n \in \Lambda$

$$\sum_{1 \leq i < j \leq n} v(x_i - x_j) \geq -Bn.$$

Stability of v implies

$$\mathcal{V}^{\nu}(\boldsymbol{\omega}) \geq -\frac{B}{\nu} \sum_{i=1}^{n} T_i$$

where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$, and T_i is the duration of ω_i.

Step 2: Only loops of duration ν are relevant

$$\mathbb{L}^\nu(\mathrm{d}\omega) = \sum_{k=1}^{\infty} \frac{\zeta^k \nu^{kd/2}}{k} \int_\Lambda \mathrm{d}x \int_{\Omega^{k\nu}} \mathbb{W}^{k\nu}_{x,x}(\mathrm{d}\omega)$$

$$\Xi^\nu = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathbb{L}^\nu(\mathrm{d}\omega_1) \cdots \mathbb{L}^\nu(\mathrm{d}\omega_n) \exp(-\mathcal{V}^\nu(\boldsymbol{\omega})) \tag{1}$$

(1) can be interpreted as a formal Riemann sum of mesh size ν. It does not correspond to a convergent integral. The main contribution should come from the $k = 1$ terms.

Define

$$\widehat{\mathbb{L}}^\nu(\mathrm{d}\omega) := \zeta \nu^{d/2} \int_\Lambda \mathrm{d}x \int_{\Omega^\nu} \mathbb{W}^\nu_{x,x}(\mathrm{d}\omega) \quad \text{Loops of duration } \nu.$$

and

$$\widehat{\Xi}^\nu := 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \widehat{\mathbb{L}}^\nu(\mathrm{d}\omega_1) \cdots \widehat{\mathbb{L}}^\nu(\mathrm{d}\omega_n) \exp(-\mathcal{V}^\nu(\boldsymbol{\omega})).$$

Then

$$|\Xi^\nu - \widehat{\Xi}^\nu| \lesssim_{d,\zeta,B,L} \nu^{d/2}.$$

Step 2: Only loops of duration ν are relevant

$$\mathbb{L}^\nu(\mathrm{d}\omega) = \sum_{k=1}^\infty \frac{\zeta^k \nu^{kd/2}}{k} \int_\Lambda \mathrm{d}x \int_{\Omega^{k\nu}} \mathbb{W}^{k\nu}_{x,x}(\mathrm{d}\omega)$$

$$\Xi^\nu = 1 + \sum_{n=1}^\infty \frac{1}{n!} \int \mathbb{L}^\nu(\mathrm{d}\omega_1)\cdots\mathbb{L}^\nu(\mathrm{d}\omega_n) \exp\big(-\mathcal{V}^\nu(\boldsymbol{\omega})\big)$$

$$\Xi^\nu - \widehat{\Xi}^\nu = \sum_{n=1}^\infty \frac{n}{n!} \int (\mathbb{L}^\nu - \widehat{\mathbb{L}}^\nu)(\mathrm{d}\omega_1)\,\mathbb{L}^\nu(\mathrm{d}\omega_2)\cdots\mathbb{L}^\nu(\mathrm{d}\omega_n) \exp\big(-\mathcal{V}^\nu(\boldsymbol{\omega})\big)$$

From $\mathcal{V}^\nu(\boldsymbol{\omega}) \geq -\frac{B}{\nu}\sum_{i=1}^n T_i$ we get

$$|\Xi^\nu - \widehat{\Xi}^\nu| \leq \sum_{n=1}^\infty \frac{n}{n!} \int (\mathbb{L}^\nu - \widehat{\mathbb{L}}^\nu)(\mathrm{d}\omega_1)\,\mathbb{L}^\nu(\mathrm{d}\omega_2)\cdots\mathbb{L}^\nu(\mathrm{d}\omega_n) \exp\left(B\sum_{i=1}^n k_i\right).$$

From Gaussian bounds on $\mathbb{W}^\nu_{x,x}$ we get

$$\int \mathrm{e}^{Bk_1}\,(\mathbb{L}^\nu - \widehat{\mathbb{L}}^\nu)(\mathrm{d}\omega_1) \lesssim_{d,\zeta,B,L} \nu^{d/2},$$

$$\int \mathrm{e}^{Bk_i}\,\mathbb{L}^\nu(\mathrm{d}\omega_i) \lesssim_{d,\zeta,B,L} (\nu^{d/2}+1) \quad \text{for } i = 2,\ldots,n.$$

Step 3: Remaining loops collapse to points

$$\widehat{\mathbb{L}}^{\nu}(\mathrm{d}\omega) := \zeta \nu^{d/2} \int_{\Lambda} \mathrm{d}x \int_{\Omega^{\nu}} \mathbb{W}^{\nu}_{x,x}(\mathrm{d}\omega) \quad \text{Loops of duration } \nu.$$

$$\widehat{\Xi}^{\nu} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \widehat{\mathbb{L}}^{\nu}(\mathrm{d}\omega_1) \cdots \widehat{\mathbb{L}}^{\nu}(\mathrm{d}\omega_n) \exp\big(-\mathcal{V}^{\nu}(\boldsymbol{\omega})\big)$$

$$\Xi^{0} = 1 + \sum_{n=1}^{\infty} \frac{\zeta^n}{(2\pi)^{nd/2}\, n!} \int_{\Lambda^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \exp\Big(-\sum_{1 \le i < j \le n} v(x_i - x_j)\Big)$$

$$\widetilde{\Xi}^{\nu} := 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \widehat{\mathbb{L}}^{\nu}(\mathrm{d}\omega_1) \cdots \widehat{\mathbb{L}}^{\nu}(\mathrm{d}\omega_n) \exp\Big(-\sum_{1 \le i < j \le n} v(x_i - x_j)\Big)$$

By the mean value theorem, stability and α-Hölder continuity of v, and Gaussian bounds:

$$\big|\widehat{\Xi}^{\nu} - \widetilde{\Xi}^{\nu}\big| \lesssim_{d,\zeta,v,\alpha,B,L} \nu^{\alpha/2}.$$

By Gaussian bounds:

$$\big|\widetilde{\Xi}^{\nu} - \Xi^{0}\big| \lesssim_{d,\zeta,v,\alpha,B,L} \nu^{d/2}.$$