Semiclassical concentration of eigenfunctions of Toeplitz operators in phase space

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15 september 2025

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1. Eigenfunctions concentration

2. A framework on $\mathbb{R}^3$

3. Toeplitz quantisation on manifolds

Notion of concentration

Context: Semiclassical operator $P_\hbar$ (depends on $\hbar$, $\hbar \to 0$) on manifold $M/\mathbb{R}^d/\mathbb{C}^d$.

$P_\hbar u_\hbar = \lambda_\hbar u_\hbar$, $\|u_\hbar\|_{L^2} = 1$, behaviour of $u_\hbar$ as $\hbar \to 0$. Typical behaviours:

- $u_\hbar$ independent of $\hbar$ then $\|u_\hbar\|_{L^p}$ independent of $\hbar$.
- $u_\hbar \sim \hbar^{-\beta}\chi_{\Omega_\hbar}$ with $|\Omega_\hbar| = \hbar^\alpha$ then $\|u_\hbar\|_{L^p} \sim \hbar^{\frac{\alpha}{p}-\beta}$.
- Sum of previous behaviours.

Example

$$-\hbar^2\Delta u_\hbar + x^2 u_\hbar = \lambda_\hbar u_\hbar.$$

- Concentration: $U \subset \left\{x \in \mathbb{R} / \ |x| > \sqrt{\lambda_\hbar}\right\}$ open, then $\|u_\hbar\|_{L^2(U)} = O(e^{-\frac{c}{\hbar}})$. (linked to wavefront set)
- How much: depends where.

Hermite functions on $\mathbb{R}$

Eigenvalue: $E = \hbar(2k+1)$, for $k \in \mathbb{N}$, $V = x^2$.

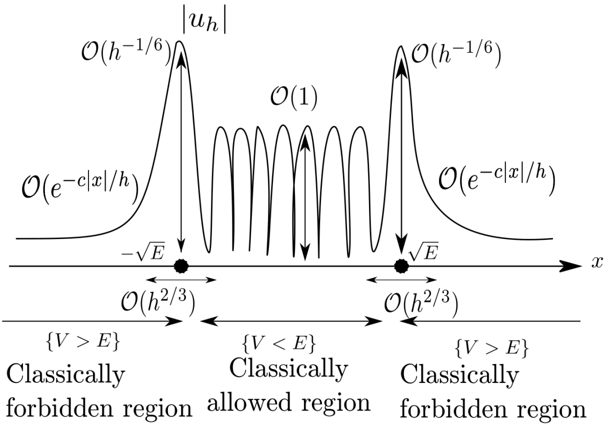


Figure: Ngoc Nhi Nguyen. "Fermionic semiclassical L^p estimates". (2024)

Known results

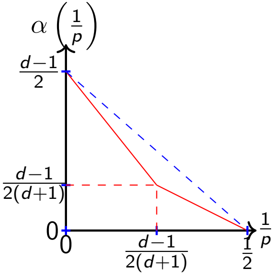
(M, g) Riemann manifold dimension $d \geq 2$,

$$-\hbar^2 \Delta_g u_\hbar = \lambda_\hbar u_\hbar,$$

$$\|u_\hbar\|_{L^p(M)} = O\left(\hbar^{-\alpha\left(\frac{1}{p}\right)}\right).$$

Saturation on the the d sphere.[a]

[a] Christopher Sogge. "Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds". (1988).



Other known frameworks: $-\hbar^2\Delta_g + V$ for V with some hypothesis[1], Δ on M without boundary[2], second order elliptic operators[3].

[1] Herbert Koch and Daniel Tataru. "L^p eigenfunction bounds for the Hermite operator". (2005), Herbert Koch, Daniel Tataru and Maciej Zworski. "Semiclassical L^p Estimates". (2007).

[2] Yaiza Canzani and Jeffrey Galkowski. *Geodesic Beams in Eigenfunction Analysis*. 2023.

[3] Christopher Sogge. *Problems related to the concentration of eigenfunctions*. 2015.

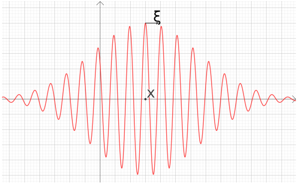
Concentration on phase space

Gaussian wave packets for $\hbar > 0$ and $y \in \mathbb{R}^3$

$$\Phi^{\hbar}_{x_0,\xi_0}(y) = (\pi h)^{-d/4} e^{-\frac{(y-x_0)^2}{2\hbar}} e^{i\frac{\xi_0\cdot(x_0-y)}{\hbar}}$$

such that

$$|\langle \Phi^{\hbar}_{x_0,\xi_0}, \Phi^{\hbar}_{x_1,\xi_1}\rangle| \leq e^{-\frac{(x_0-x_1)^2+(\xi_0-\xi_1)^2}{4\hbar}}$$



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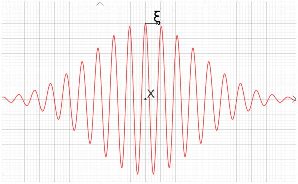

Denote: $Bu(x,\xi) = (2\pi h)^{-n/2}(u, \Phi_{x,\xi}(\cdot))_{L^2(\mathbb{R}^3)}$, $\forall u \in L^2(\mathbb{R}^3)$.

Lemma

$B : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^6)$ is an isometry, $B^*B = id_{L^2(\mathbb{R}^3)}$.

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Example

$$-\hbar^2\Delta u_\hbar + x^2 u_\hbar = \lambda_\hbar u_\hbar.$$

- Concentration: $V \subset \left\{(x,\xi) \in \mathbb{R}^2 /\ x^2 + \xi^2 \neq \lambda_\hbar\right\}$ open, then $\|Bu_\hbar\|_{L^2(V)} = O(\hbar^\infty)$. (definition of wavefront set)
- How much: depends where.

New state space

Definition

Bargmann space,

$$\mathbf{F}_\hbar = \mathrm{Im}(B) = L^2(\mathbb{C}^d) \cap \left\{ e^{-\frac{|z|^2}{2\hbar}} f \ / \ f \in \mathcal{H}(\mathbb{C}^d) \right\}$$

where $z = x - i\xi$. With Hilbert basis: $\left(e_\alpha = C_{\alpha,\hbar} e^{-\frac{|z|^2}{2\hbar}} z^\alpha \right)_{\alpha \in \mathbb{N}^d}$.

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Let $f \in C^0(\mathbb{C}^n)$ grow at most polynomially, the Berezin-Toeplitz operator associated is:

$$\begin{aligned} T_\hbar(f) : D &\to \mathbf{F}_\hbar \\ g &\mapsto \Pi_{\mathbf{F}_\hbar}(fg) \end{aligned}$$

with domain $D \subset \mathbf{F}_\hbar$.

Examples

- If f is holomorphic then $T_\hbar(f) = f \times id_{\mathbf{F}_N}$.
- For all $\alpha, \beta \in \mathbb{N}^d$,

$$T_\hbar(z^\alpha \overline{z}^\beta)u = e^{-\frac{|z|^2}{2\hbar}} (\hbar\partial)^\beta \left(e^{\frac{|z|^2}{2\hbar}} z^\alpha u \right)$$

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No concentration result for Berezin-Toeplitz operators!

Harmonic oscillator : $T_\hbar(|z|^2) = \sum_{1\le j\le d} T_\hbar(z_j\overline{z_j})$

Theorem

The spectrum of $T_\hbar(|z|^2)$ is purely discrete equal to $\{(k+d)\hbar \ / \ k \in \mathbb{N}\}$. The eigenspace of eigenvalue $(k+d)\hbar$ is span $\{e_\alpha / \ |\alpha| = k\}$.

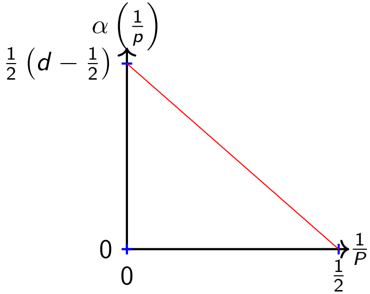
Theorem

Consider the eigenvalue $(|\nu| + d)\hbar$, then for all $p \in [2, +\infty]$ there exists $C > 0$ such that

$$\|e_\nu\|_{L^p(\mathbb{C}^n)} \leq C\hbar^{\left(\frac{1}{2}-d\right)\left(\frac{1}{2}-\frac{1}{p}\right)}.$$

$$\|e_\nu\|_{L^p(\mathbb{C}^n)} \leq C\hbar^{-\alpha\left(\frac{1}{p}\right)}$$

Optimal: becomes an equality for $\nu = (|\nu|, 0, \cdots, 0)$.



Projective space

$\mathbb{CP}^d = \left(\mathbb{C}^{d+1} \backslash \{0\}\right) / R$ with local coordinates

$$\left\{[z_0, \cdots, z_n] \in \mathbb{CP}^d / \; z_0 \neq 0\right\} \to \mathbb{C}^d$$

$$[z_0, \cdots, z_n] \mapsto \left(1, \frac{z_1}{z_0}, \cdots, \frac{z_n}{z_0}\right) = (1, w_1, \cdots, w_n).$$

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Denote $\tilde{w} = (1, \overline{w_1},\cdots,\overline{w_n})$ and $\langle w\rangle = \sqrt{1+|w|^2}$.
Here $N \in \mathbb{N}$ and $\frac{1}{\hbar} = N \to +\infty$. Consider functions of the form

$$\frac{f(w_1,\cdots,w_n)\tilde{w}^{\otimes N}}{\langle w\rangle^{2N}}$$

with f holomorphic.

Lemma

$$\mathcal{H}_N = \left\{\frac{Q(w_1,\cdots,w_n)\tilde{w}^{\otimes N}}{\langle w\rangle^{2N}} /\ Q \in \mathbb{C}_{\leq N}[X_1,\cdots,X_d]\right\}.$$

with orthonormal basis $\left(e_a = C_{n,N,\alpha}\frac{w^a\tilde{w}^{\otimes N}}{\langle w\rangle^{2N}}\right)_{|a|\leq N}$.

Theorem

On M a compact Kähler manifold or $\mathbb{C}^d$.
$T_\hbar(f) = \Pi_\hbar(f_0 + \hbar f_1 + \cdots)$ *with principal symbol* f_0.
$E \in \mathbb{R} / f_0(x) = E \Rightarrow df_0(x) \neq 0$, *sequence* $\lambda_\hbar \xrightarrow[\hbar \to +\infty]{} E$, $e_\hbar$ *such that*

$$\begin{aligned} \|e_\hbar\|_{L^2(M)} =& 1 \\ T_\hbar(f) e_\hbar =& \lambda_\hbar e_\hbar + O(\hbar^\infty). \end{aligned}$$

Then, for all $\hbar \in \mathbb{N}$ *and* $p \in [2, +\infty]$

$$\|e_\hbar\|_{L^p(M)} = O\left(\hbar^{\left(d-\frac{1}{2}\right)\left(\frac{1}{p}-\frac{1}{2}\right)}\right)$$

Corollary

The same result applies on $\mathbb{C}^d$ *for* $d \in \mathbb{N}^*$.

Saturation on $\mathbb{CP}$

Consider $H \in C^0(\mathbb{CP})$ such that

$$H([z_0, z_1]) = \frac{|z_0|^2}{|z_0|^2 + |z_1|^2} = \frac{|w|^2}{|w|^2 + 1}.$$

For all $a \in \mathbb{N}$ such that $a \leq N$,

$$T_N(H)(e_a) = \frac{a+1}{N+2} e_a.$$

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For all $a \in \mathbb{N}$ such that $a \leq N$,

$$T_N(H)(e_a) = \frac{a+1}{N+2} e_a.$$

Theorem

Consider a sequence $a_N \sim \frac{N}{4}$ and $p \in [2, +\infty]$, then there exists $C > 0$, independent of p, such that

$$\|e_{a_N}\|_{L^p(\mathbb{CP}^n)} \sim C N^{\frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right)}.$$

Ideas of proof

Lagrangian states (possibly time dependent)

Stable by Toeplitz operators

Stable by time derivative

Solution to $(-i\hbar\partial_t + T_\hbar(f))\psi = 0$ as a Lagrangian state.
Quantum propagator $U_{\hbar,t}$ as a Lagrangian state.

Operators as Lagrangian states

$$\begin{aligned} &\rho(N(E - T_\hbar(f)))(x,x) \\ &= \hbar^{\frac{1}{2}}\mathcal{F}_N^{-1}(\hat{\rho}(t)U_{\hbar,t}(x,x))(E) \\ &= \sum_\lambda \rho(N(E-\lambda))|e_{\lambda_N}(x)|^2 \end{aligned}$$

$$\rho(\hbar^{-1}(E - T_\hbar(f)))(x,x) = \hbar^{\frac{1}{2}-d} e^{-\frac{|E-f_0(x)|}{\hbar}}\hat{\rho}(0)a_0(x) + O(1)$$

- N R. *Semiclassical concentration estimates for Berezin-Toeplitz quasimodes for regular energies.* 2025
- N R. *Eigenvalues of non self-adjoint Toeplitz operators near an elliptic critical value with analytic regularity.* (soon)
- PhD defense in june 2026

Thank you

[1] Pierre Bérard. "On the wave equation on a compact Riemannian manifold without conjugate points". (1977).

[2] Yaiza Canzani **and** Jeffrey Galkowski. *Geodesic Beams in Eigenfunction Analysis*. 2023.

[3] Laurent Charles **and** Yohann Le Floch. *Quantum propagation for Berezin-Toeplitz operators*. 2021.

[4] Herbert Koch **and** Daniel Tataru. "L^p eigenfunction bounds for the Hermite operator". (2005).

[5] Herbert Koch, Daniel Tataru **and** Maciej Zworski. "Semiclassical L^p Estimates". (2007).

[6] Ngoc Nhi Nguyen. "Fermionic semiclassical L^p estimates". (2024).

[7] N R. *Eigenvalues of non self-adjoint Toeplitz operators near an elliptic critical value with analytic regularity*. (soon).

[8] N R. *Semiclassical concentration estimates for Berezin-Toeplitz quasimodes for regular energies*. 2025.

[9] Christopher Sogge. "Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds". (1988).

[10] Christopher Sogge. *Problems related to the concentration of eigenfunctions*. 2015.

Toeplitz quantisation on manifolds

- Complex (symplectic) compact differential manifold M.
- Holomorphic functions are constants $\to$ sections of a holomorphic bundle $\pi : L \to M$.

For a suitable bundle : quantum space $\mathcal{H}_N = H^0(M, L^{\otimes N})$ with $N = \frac{1}{\hbar} \to +\infty$.

Definition

$\Pi_N : L^2(M, L^{\otimes N}) \to \mathcal{H}_N$ the orthogonal projector, for $f \in C^0(M)$,

$$T_N(f) = \Pi_N f : \mathcal{H}_N \to \mathcal{H}_N$$

- They are bounded: $\|T_N(f)\| \leq \|f\|_{L^\infty(M)}$.
- $T_N(f)^* = T_N(\overline{f})$, they are self-adjoint if and only if f is real-valued.