Stochastically Generated Matrix Product States:

Correlation Decay in the Thermodynamic Limit

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Joint (ongoing) work with Albert H Werner

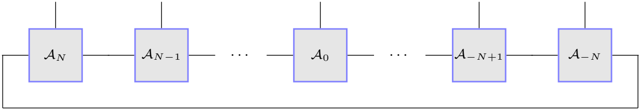
Deterministic MPS I

Fix a bi-infinite sequence $(\mathcal{A}_n)_{n\in\mathbb{Z}}$ of d-tuples of complex $D \times D$ matrices with

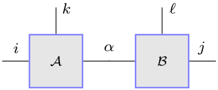
$$\mathcal{A}_n = (A_1^{(n)}, \ldots, A_d^{(n)}) \in \mathbb{C}^d \otimes (\mathbb{C}^D)^{\otimes 2}, \qquad A_j^{(n)} \in \mathbb{M}_D(\mathbb{C}) \cong (\mathbb{C}^D)^{\otimes 2}.$$

Here we denote, $(\mathcal{A}_n)_{\text{phys}=j,\ \text{bond}=(i,k)} = (A_j^{(n)})_{i,k}$.

On the 1-dimensional lattice, place $\mathcal{A}_n$ at site n. For a symmetric window $[-N, N]$ we obtain the (generally unnormalized) MPS on $2N+1$ sites:

$|\Psi(N)\rangle =$ 

Here, contraction of two rank-3 tensors along a virtual index α:



$\sum_\alpha (A_k)_{i,\alpha}(B_\ell)_{\alpha,j}$

Deterministic MPS II

Therefore, we have that

$$|\Psi(N)\rangle = \sum_{j_{-N},\dots,j_N=1}^{d} \text{Tr}\left[A_{j_{-N}}^{(-N)} \cdots A_{j_N}^{(N)}\right] |j_{-N},\dots,j_N\rangle \,.$$

Normalization is *not* assumed a priori, i.e. $\langle\Psi(N)|\Psi(N)\rangle$ need not equal 1.

1. Matrix product states (MPS) form a subset of many-body quantum states.
2. They are good approximations of several 'physically relevant' states, such as ground states of gapped local Hamiltonians on 1-dimensional systems.
3. Random (TI) MPS typically have correlations between observables measured on distinct sites that decay ***fast*** (with the distance separating the sites).

Generating a Random MPS

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A random MPS can be obtained in different ways, such as (but not limited to)

1. **Random homogeneous (TI):** draw a single tensor $\mathcal{A} \sim \lambda$ and set $\mathcal{A}_n = \mathcal{A}$ for all n.
2. **IID local tensors:** $(\mathcal{A}_n)_{n\in\mathbb{Z}}$ independent with common law λ.
3. **Stochastically Stationary + Stochastically Correlated:** $(\mathcal{A}_n)$ jointly distributed with identical marginals λ, but non-trivial spatial stochastic correlations.

Example

Let $X = (X_n)_{n\in\mathbb{Z}}$ be an irreducible, aperiodic Markov chain on the state space $S = \{1, \dots, m\}$ ($m \geq 2$), started in the stationary distribution π. For each $i \in S$, let $(\mathcal{B}_n^{(i)})_{n\in\mathbb{Z}}$ be i.i.d. with one-site law λ_{B_i} on the tensor space $\mathbb{C}^d \otimes (\mathbb{C}^D)^{\otimes 2}$, independent across i and independent of X. Set

$$\mathcal{A}_n := \mathcal{B}_n^{(X_n)}, \qquad n \in \mathbb{Z},$$

and place $(\mathcal{A}_n)$ as local tensors to generate the random MPS that is strictly stationary (but *not* IID) with one-site law

$$\mathrm{Law}(\mathcal{A}_0) = \sum_{i=1}^{m} \pi_i \, \lambda_{B_i} := \lambda.$$

4. **Non-stationary + inhomogeneous:** Take $\mathcal{A}_n$ with law λ_n.

Random MPS considered here

Let λ be a probability distribution on $\mathbb{C}^d \otimes (\mathbb{C}^D)^{\otimes 2}$. Assume that $(\mathcal{A}_n)_{n\in\mathbb{Z}}$ is a bi-infinite sequence of random tensors such that

$$\mathrm{Law}(\mathcal{A}_n) = \lambda, \qquad \forall\, n \in \mathbb{Z}.$$

This setting encompasses: the IID case, the random TI (homogeneous) case, and deterministic TI as the degenerate law.

Such a sequence can be assumed to be defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ is determined by the marginal law λ.

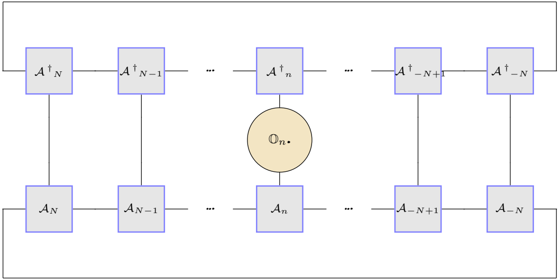
For any random object f on Ω we denote its realization at $\omega \in \Omega$ by f_ω.

Expectation of a Local Observable

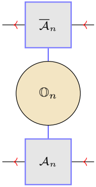
Define the (random) transfer operator at site n by $\phi_n(\cdot) = \sum_{i=1}^{d} A_i^{(n)}(\cdot)(A_i^{(n)})^\dagger$, we see that for a local observable $\mathbb{O}_n$ at site n, the expectation of $\mathbb{O}_n$ in state $|\Psi(N)\rangle$ (for $N > |n|$) is given by

$$\frac{\langle \Psi(N)| \mathbb{O}_n |\Psi(N)\rangle}{\langle \Psi(N)|\Psi(N)\rangle} = \frac{\mathrm{Tr}\left\{\phi_N \circ \ldots \circ \phi_{n+1} \circ \widehat{\mathcal{O}_n} \circ \phi_{n-1} \circ \ldots \circ \phi_{-N}\right\}}{\mathrm{Tr}\{\phi_N \circ \ldots \phi_{-N}\}}.$$

With $\mathrm{Tr}\{\cdot\}$ the Liouville (superoperator) trace:
$\phi : x \to \sum_{i=1}^{d} A_i x B_i^\dagger \implies M_\phi = \sum_{i=1}^{d} A_i \otimes \overline{B_i}$.



(a) Expectation of a local observable $\mathbb{O}_n$ at site n in state $|\Psi(N)\rangle$.



(b) $\widehat{\mathcal{O}}_n$

Two-Point Function and Thermodynamic Limit

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For local observables $\mathbb{O}_m, \mathbb{O}_n$ supported at sites m, n (with $N > \max\{|m|, |n|\}$), define

$$f_N(m,n) := \left| \frac{\langle \Psi(N)| \, \mathbb{O}_m \mathbb{O}_n \, |\Psi(N)\rangle}{\langle \Psi(N)|\Psi(N)\rangle} - \frac{\langle \Psi(N)| \, \mathbb{O}_m \, |\Psi(N)\rangle}{\langle \Psi(N)|\Psi(N)\rangle} \frac{\langle \Psi(N)| \, \mathbb{O}_n \, |\Psi(N)\rangle}{\langle \Psi(N)|\Psi(N)\rangle} \right|.$$

When the limit exists, set

$$f_\infty(m,n) := \lim_{N\to\infty} f_N(\mathbb{O}_m, \mathbb{O}_n).$$

This quantity is random (via the marginal laws λ of the local tensors).

Two results I

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Theorem (Movassagh & Schenker - '21)

If the sequence of transfer operators ϕ_n associated to the sequence of local tensors A_n satisfies

1. *Stationary and* ***ergodic*** *(for example: an IID sequence).*
2. *There is some $N_0 \in \mathbb{N}$ so that $\phi_{N_0-1} \circ \ldots \circ \phi_0$ is strictly positive with positive probability.*
3. *With probability 1, ϕ_0 and ${\phi_0}^\dagger$ have no quantum states in the kernels.*

then, there is a $\mu \in (0,1)$ such that for any $x \in \mathbb{Z}$ and for any two local observables $\mathbb{O}_m$, $\mathbb{O}_n$ at sites m, n (resp.)

$$f_\infty(m,n) \leq C_{\lambda,\mu,x,\mathbb{O}_m,\mathbb{O}_n} \mu^{n-m}$$

for all $m < n$ with $m < x < n$, where $\Omega \ni \omega \mapsto C_{\lambda,\mu,x,\mathbb{O}_m,\mathbb{O}_n}(\omega)$ is a ***(random)*** *constant that is almost surely finite.*

Two results II

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Theorem (Lancien & Perez-Garcia - '21)

If a random TI MPS is constructed by sampling $\mathcal{A} \in \mathbb{C}^d \otimes (\mathbb{C}^D)^{\otimes 2} \cong \mathbb{C}^{dD^2}$ *w.r.t. the Gaussian distribution on* $\mathbb{C}^{dD^2}$ *with mean* 0 *and covariance* $[1/(dD)]\mathbb{I}_{dD^2}$*,*

(i.e. by sampling $\{(g_j)_{i,k})\}_{1 \leq j \leq d, 1 \leq i,k \leq D}$ *independent w.r.t. to the complex Gaussian distribution with mean* 0 *and variance* $1/(dD)$*)*

then, there are absolute constants c_1, c_2, c_3, c_4 *such that, for any local observables* $\mathbb{O}_m$ *and* $\mathbb{O}_n$ *at sites* m, n *with* $|n-m| \leq N - c_1 \log D / \log d$*,*

$$f_N(m,n) \leq c_4 \left(\frac{c_3}{\sqrt{d}} \right)^{|n-m|} \|\mathbb{O}_n\| \|\mathbb{O}_m\|$$

with probability at least

$$1 - e^{-c_2 \min\{D, d^{1/3}\}}.$$

Common Structure

Both frameworks have:

1. Identical marginals for the local tensors (hence similarly distributed transfer maps).
2. Almost surely no states in the kernels of ϕ_n and $\phi_n^\dagger$.
3. *Finite-time strict positivity* of products: a random time $\boldsymbol{N}_*(\omega)$ with $\phi_{\boldsymbol{N}_*(\omega)-1} \circ \cdots \circ \phi_0$ strictly positive a.s.

   (In Gaussian TI case, $\boldsymbol{N}_*$ can be chosen deterministically: $\boldsymbol{N}_* = 2\lceil \log D / \log d \rceil$, a.s.)

Assumptions

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The similarly distributed sequence of local tensors $(\mathcal{A}_n)_{n\in\mathbb{Z}}$ defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfy the following assumptions:

Standing assumptions

(A1) $\mathbb{P}\Big\{\ker(\phi_0) \cap \mathbb{S}_D = \emptyset,\ \ \ker(\phi_0^\dagger) \cap \mathbb{S}_D = \emptyset\Big\} = 1,$
where $\mathbb{S}_D := \{\rho \in \mathbb{M}_D :\ \ \rho \geq 0,\ \ \mathrm{Tr}\,[\rho] = 1\}$.

(A2) Finite-time strict positivity:
$\mathbb{P}$-a.s. there exists $\boldsymbol{N}_*(\omega) \in \mathbb{N}$ with $\phi_{\boldsymbol{N}_*(\omega)-1;\omega} \circ \cdots \circ \phi_{0;\omega}$ strictly positive;

Absolutely continuous laws on $\mathbb{C}^d \otimes (\mathbb{C}^D)^{\otimes 2}$ satisfy (A1) and (A2) in both TI and IID settings, with $\tau = 2\lceil \log D / \log d \rceil$.

Any (deterministic) primitive quantum channel satisfies both (A1) and (A2).

We denote

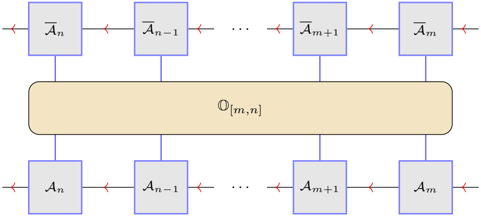
$$\tau(\omega) = \inf\{n : \phi_{n-1;\omega} \circ \ldots \circ \phi_{0;\omega} \text{ is strictly positive}\}$$

There are other examples where $(A2)$ happens with $\mathbb{P}\{\omega : \tau(\omega) > n\} \searrow 0$ as $n \to \infty$, with various rates in n.

Implications Under the Assumptions

Under (A1)-(A2), products $\Phi^{(n)} := \phi_{n-1} \circ \cdots \circ \phi_0$ are eventually irreducible and contracting a.s. Hence there exist full-rank density matrices $(Z_n)_{n\in\mathbb{Z}}$ and $(Z'_n)_{n\in\mathbb{Z}}$ such that for any local observable $\mathbb{O}_{[m,n]}$,

$$\lim_{N\to\infty} \frac{\langle \Psi(N)|\ \mathbb{O}_{[m,n]}\ |\Psi(N)\rangle}{\langle \Psi(N)|\Psi(N)\rangle} = \frac{\left\langle Z'_{n+1}|\widehat{\mathcal{O}}_{[m,n]}|Z_{m-1}\right\rangle}{\left\langle Z'_{n+1}\middle|\left(\phi_n \circ \cdots \circ \phi_m\right)\middle|Z_{m-1}\right\rangle} \quad \mathbb{P}\text{-almost surely.}$$



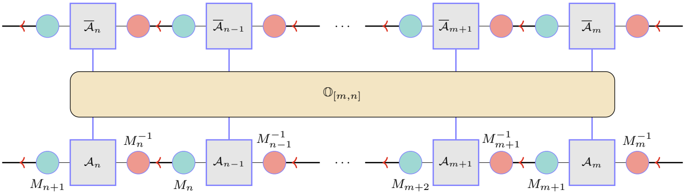
Gauge Fixing I

Define the gauge transform (dynamic normalization) by

$$B_i^{(n)} := \frac{1}{\sqrt{\mathrm{Tr}\left[\phi_n^\dagger(Z'_{n+1})\right]}} (Z'_{n+1})^{1/2} A_i^{(n)} (Z'_n)^{-1/2}, \qquad \widetilde{\phi}_n(X) = \sum_{i=1}^{d} B_i^{(n)}\, X\, (B_i^{(n)})^\dagger.$$

One checks $\widetilde{\phi}_n$ is CPTP and that, for any local observable, the thermodynamic limit of expectations is invariant under this gauge.

The resulting MPS / Expectation is the following:



Here $M_n \approx \sqrt{Z'_n}$.

Result 1: Almost-sure exponential decay

Theorem (Pathirana-Werner '25+)

Under (A1)-(A2) there exists a random $\mu(\omega) \in (0,1)$ *such that for any* $x \in \mathbb{Z}$ *there is an a.s. finite* $g_x(\omega)$ *with the property that for all* $m < x < n$ *with* $n - m \geq 2$,

$$f_\infty^\omega(n,m) \;\leq\; \|\mathbb{O}_m\|\,\|\mathbb{O}_n\|\,g_x(\omega)\,\mu(\omega)^{\,n-m} \quad \text{a.s.}$$

In the IID (or more generally, ergodic) case, μ *is a (deterministic) constant.*

In the random TI (homogeneous) case, g_x *can be replaced by a site-independent almost surely finite* $g(\omega)$.

Random TI Case

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Theorem (Pathirana-Werner '25+)

Assume (A1) and (A2). If $\mathcal{A}_n \equiv \mathcal{A}$ with $\mathcal{A} \sim \lambda$ (random TI), then for each error margin $0 < \epsilon < 1$ there exist deterministic constants $K = K(\epsilon) > 0$ and $\delta = \delta(\epsilon) \in (0,1)$, such that the same bound holds for all $2 \leq |n-m|$:

$$\mathbb{P}\{f_\infty(n,m) \leq K\|\mathbb{O}_n\|\|\mathbb{O}_n\|\delta^{|n-m|}\} \geq 1-\epsilon.$$

IID Case

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Theorem (Pathirana-Werner '25+)

If $(\mathcal{A}_n)$ *are IID and satisfy (A1)-(A2), then there are deterministic constant* $C_{\mathrm{pr}}, \beta > 0$ *such that*

$$\mathbb{P}\Big\{ f_\infty(n,m) \ \leq \ C_{\mathrm{pr}} \, \|\mathbb{O}_m\| \, \|\mathbb{O}_n\| \, e^{-\beta|n-m|} \Big\} \ \geq \ 1 - e^{-\beta|n-m|}.$$

for all $|m-n| \geq 2$.

Remark

If one does not assume independence between local tensors, but assumes asymptotic stochastic decorrelation (in "some" sense: such as mixing coefficients $\rho_n, \psi_n, \phi_n \to 0$*), we can obtain that for each* $k \in \mathbb{N}$ *there exists* $C_{\mathrm{poly}(k)}$ *with*

$$\mathbb{P}\Big\{ f_\infty(n,m) \ \leq \ C_{\mathrm{poly}(k)} \, \|\mathbb{O}_m\| \, \|\mathbb{O}_n\| \, |n-m|^{-k} \Big\} \ \geq \ 1 - |n-m|^{-k}.$$

for all $|n-m| \geq 2$.

If one assumes certain rates of convergence for the mixing coefficients ρ_n, ψ_n, ϕ_n*, then the above rates can be improved to stretched exponentials.*

Maximum Separation Window

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Our last method is to consider the case where sites $m, n \in \mathbb{Z}$ are separated by a maximum size L, i.e. $2 \leq |m-n| \leq L$. For such cases, we obtain:

Theorem (Pathirana-Werner '25+)

Assume (A1) and (A2). For any maximum separation of length $L > 2$ and $0 < \epsilon < 1$ there exist deterministic constants $K = K(\epsilon, L) > 0$ and $\delta = \delta(\epsilon, L) \in (0,1)$ such that for all $2 \leq |n-m| \leq L$,

$$\mathbb{P}\Big\{ f_\infty(n,m) \;\leq\; K \, \|\mathbb{O}_m\| \, \|\mathbb{O}_n\| \, \delta^{\,|n-m|} \Big\} \;\geq 1-\epsilon.$$

There is no meaningful way to generalize this result for $L \to \infty$ (or for $\epsilon \to 0$) since there is no guarantee that δ^L does not converge to 1 as $L \to \infty$ (or as $\epsilon \to 0$).

However, there are techniques that utilize rates of decay for $\mathbb{P}\{\tau > n\}$ and rates (tails) of ***contraction strength*** so that given $m, n \in \mathbb{Z}$ with $2 \leq |n-m| = \ell$ we may produce $K := K(\ell)$, $\delta := \delta(\ell)$ and $\epsilon(\ell)$ so that

$$\mathbb{P}\Big\{ f_\infty(n,m) \;\leq\; K \, \|\mathbb{O}_m\| \, \|\mathbb{O}_n\| \, \delta^{\,|n-m|} \Big\} \;\geq 1-\epsilon(\ell).$$

with $K(\ell) \to K'$ (for some finite K') , $\delta^\ell, \epsilon(\ell) \;\to 0$ as $\ell \to \infty$.

Thank You!

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Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and let* $\mathcal{A}$*,* $\mathcal{B}$ *be two sub* σ*-algebras. Then we define the following measures of the correlation between* $\mathcal{A}$ *and* $\mathcal{B}$*:*

$$\begin{aligned}
\rho(\mathcal{A},\mathcal{B}) &= \sup\left\{|\mathit{corr}(X,Y)| \;:\; Y \in L^2(\mathcal{A}),\; X \in L^2(\mathcal{B}),\; X,Y \neq 0\right\}, \\
\psi(\mathcal{A},\mathcal{B}) &= \sup\left\{\left|1 - \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)\mathbb{P}(B)}\right| : A \in \mathcal{A},\; B \in \mathcal{B},\; \mathbb{P}(A),\mathbb{P}(B) \neq 0\right\} \\
\varphi(\mathcal{A},\mathcal{B}) &= \sup\left\{|\mathbb{P}(B|A) - \mathbb{P}(B)| \;:\; A \in \mathcal{A},\; B \in \mathcal{B},\; \mathbb{P}(A) > 0\right\}.
\end{aligned}$$

Definition

Let (X_n) *be a sequence of random variables and let* $\mathcal{F}_k = \sigma(X_n : n \leq k)$ *and* $\mathcal{F}^k = \sigma(X_n : n \geq k)$*. Then we define,*

$$\rho_n := \sup_{k\in\mathbb{N}} \rho(\mathcal{F}_k, \mathcal{F}^{n+k}), \quad \psi_n := \sup_{k\in\mathbb{N}} \psi(\mathcal{F}_k, \mathcal{F}^{n+k}), \quad \varphi_n := \sup_{k\in\mathbb{N}} \varphi(\mathcal{F}_k, \mathcal{F}^{n+k}).$$

Other Rates & Probabilities

Let $\tau(\omega) = \inf\{n \in \mathbb{N} : \ \phi^{\omega}_{n-1} \circ \cdots \circ \phi^{\omega}_0$ is strictly positive$\}$, $f(b) = \mathbb{P}\{\tau > b\}$, and, for $b \in \mathbb{N}$ and $u \geq 2$, $\zeta_b(u) := \sum_{t=1}^{b} \mathbb{P}\Big(\{\tau = t\} \cap \{\mathsf{c}(\phi^{\omega}_{t-1} \circ \cdots \circ \phi^{\omega}_0) > 1 - 1/u\}\Big)$. Given $m, n \in \mathbb{Z}$ with $2 \leq |n - m| = \ell$, there exist $K(\ell) > 0$, $\delta(\ell) \in (0,1)$, and $\epsilon(\ell) \in (0,1)$ such that

$$\mathbb{P}\Big\{ f_{\infty}(\mathbb{O}_m, \mathbb{O}_n) \ \leq \ K(\ell)\, \|\mathcal{O}_m\|\, \|\mathcal{O}_n\|\, \delta(\ell)^{\,\ell} \Big\} \ \geq \ 1 - \epsilon(\ell).$$

Assumptions on f, ζ	Rates for $\epsilon(\ell)$, $\delta(\ell)^{\ell}$, $K(\ell)$
(A) Exp-Poly: $f(b) \leq C_1 e^{-\gamma b}$, $\quad \zeta_b(u) \leq C_2 u^{-\beta}$	$\epsilon(\ell) = \Theta\Big(1/\log(2+\ell)\Big)$ $\delta(\ell)^{\ell} = \exp\Big(- \Theta\Big(\ell^{\,1-1/\beta}/\log(2+\ell)\Big)\Big)$ $K(\ell) = C_1 \exp\Big\{\Theta\ \Big(2 + l)^{-1/\beta}\Big)\Big\}$
(B) Poly-Poly: $f(b) \leq C_1 b^{-\alpha}$, $\quad \zeta_b(u) \leq C_2 u^{-\beta}$ *Pick exponents* ρ, σ *with* $\dfrac{1}{\alpha+1} < \rho < 1, \quad \dfrac{1-\rho}{\beta} < \sigma < 1 - \rho$	$\epsilon(\ell) = \Theta\Big((2+\ell)^{-\eta}\Big)$ for any $\eta \in \Big(0,\ \min\{(\alpha+1)\rho - 1,\ \beta\sigma + \rho - 1\}$ $\delta(\ell)^{\ell} = \exp\Big(-\ \Theta(l^{\,1-\rho-\sigma})\Big)$ $K(\ell) = C_2\ \exp\Big(\Theta\Big(l^{-\sigma}\Big)\Big)$
(C) Exp & cutoff: $f(b) \leq C_1 e^{-\gamma b}$, $\quad \zeta_b(u) = 0$ for all $u \geq u_0$	$\epsilon(\ell) = \Theta\Big(1/\log(2+\ell)\Big)$ $\delta(\ell)^{\ell} = \exp\left(-\Theta(l/\log(2+l))\right)$ $K(\ell) = C_3$