Renormalizing Generalized Spin–Boson Models

Joint work with Benjamin Alvarez, Davide Lonigro and Javier Martín

Sascha Lill
sali@math.ku.dk

University of Copenhagen

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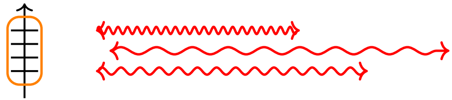
Generalized Spin–Boson Models: The Hilbert Space

- (Generalized) Spin–boson models are standard in quantum optics to describe **emission and absorption of light**.



- "**Generalized spin**" means D-level quantum system with Hilbert space $\mathbb{C}^D$. (E.g., for $N \in \mathbb{N}$ qubits we have $D = 2^N$)
- "**Boson**" field described by symmetric Fock space

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^d)^{\otimes_s n} \, .$$

- Total Hilbert space is $\boxed{\mathfrak{H} = \mathbb{C}^D \otimes \mathcal{F}}$

Generalized Spin–Boson Models: The Hamiltonian

- Hamiltonians from the physics literature look like this:

$$H^{\mathsf{bare}} = K \otimes I + I \otimes \mathrm{d}\Gamma(\omega) + \underbrace{\sum_{j=1}^{N} B_j \otimes a^*(f_j)}_{=:A^*} + \underbrace{\sum_{j=1}^{N} B_j^* \otimes a(f_j)}_{=:A} \ .$$

- $K \in \mathbb{C}^{D \times D}$, symmetric
- $\omega \in L^2_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}_+)$ is the dispersion relation, with second quantization $(\mathrm{d}\Gamma(\omega)\psi)^{(n)}(k_1, \dots, k_n) = \sum_{\ell=1}^{n} \omega(k_\ell)\psi^{(n)}(k_1, \dots, k_n)$.
  Typical examples: $\omega(k) = |k|^2 + m$ or $\omega(k) = |k|$
- $N \in \mathbb{N}$ is the number of interacting spins with $B_j \in \mathbb{C}^{D \times D}$
- $a^*(f_j)$, $a(f_j)$ are creation/annihilation operators with $f_j \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$
  CCR: $[a(f), a^*(g)] = \langle f, g \rangle$, $[a(f), a(g)] = [a^*(f), a^*(g)] = 0$ for $f, g \in L^2(\mathbb{R}^d)$
- Challenge: "**non-perturbative renormalization**", i.e., interpreting H^{bare} as self-adjoint operator $H : \mathfrak{H} \supseteq \mathrm{dom}(H) \to \mathfrak{H}$

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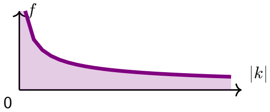
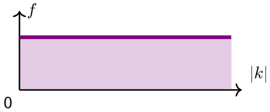
UV Divergences

- Physically interesting: $f_j(k) = |k|^{-1/2}$ or $f_j(k) = 1$, so $f_j \notin L^2$.
  (At $|k| \to \infty$, the decay would have to be faster than $|k|^{-d/2}$.)





- A is OK for $\psi \in \mathcal{F}$ with fast decay:

$$(a(f_j)\psi)^{(n)}(k_1, \dots, k_n) = \sqrt{n+1} \int \overline{f_j(k)} \psi^{(n+1)}(k_1, \dots, k_n, k) \mathrm{d}k \,.$$

- A^* is problematic for any $\psi \in \mathcal{F}$:

$$(a^*(f_j)\psi)^{(n+1)} = \sqrt{n+1} f_j \otimes_{\mathrm{s}} \psi^{(n)}$$

$$\Rightarrow \quad \|(a^*(f_j)\psi)^{(n+1)}\|_{L^2} = \sqrt{n+1} \underbrace{\|f_j\|_{L^2}}_{=\infty} \|\psi^{(n)}\|_{L^2} = \infty \,.$$

So on $\Psi = v \otimes \psi \in \mathfrak{H}$ we have

$$A^* \Psi = \sum_j (B_j v) \otimes (a^*(f_j)\psi) \notin \mathfrak{H} \,.$$

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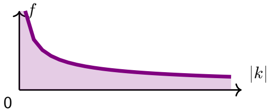

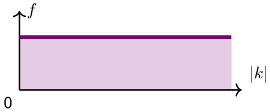

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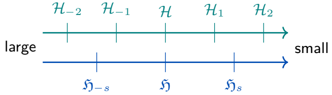
- Way out: **Hilbert space riggings**. Assume $\omega(k) \geqslant m > 0$.
  For $s \in [0, \infty)$, we set
  $$\mathcal{H}_s := \operatorname{dom}(\omega^{s/2}) = \{f \in L^2(\mathbb{R}^d) \ : \ \omega^{s/2} f \in L^2(\mathbb{R}^d)\} \ .$$
  This is a Hilbert space with $\|f\|_s := \|\omega^{s/2} f\|_{L^2}$. (so $\mathcal{H} := \mathcal{H}_0 = L^2$)
- Dual space is $\mathcal{H}_{-s} := \mathcal{H}_s'$ with distribution pairing
  $$\langle f, g \rangle = \langle \omega^{s/2} f, \omega^{-s/2} g \rangle_{L^2}.$$
  So $\ldots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2} \subset \ldots$



- Similarly, one can define $\mathfrak{H}_s \subset \mathfrak{H} \subset \mathfrak{H}_{-s}$ with
  $\|\Psi\|_{\mathfrak{H}_s} := \|(\mathrm{d}\Gamma(\omega) + 1)^{s/2} \Psi\|_{\mathfrak{H}}$. So $\operatorname{dom}(\mathrm{d}\Gamma(\omega)) = \mathfrak{H}_2$.
- So, $\mathrm{d}\Gamma(\omega) \in \mathcal{B}(\mathfrak{H}_2, \mathfrak{H})$, and further,
  for $f_j \in \mathcal{H}_{-s}$ one can show $A^* \in \mathcal{B}(\mathfrak{H}, \mathfrak{H}_{-s})$ and $A \in \mathcal{B}(\mathfrak{H}_s, \mathfrak{H})$.
  $\Rightarrow$ Then, at least, $H^{\text{bare}} \in \mathcal{B}(\mathfrak{H}_s, \mathfrak{H}_{-s})$ **is well-defined**.

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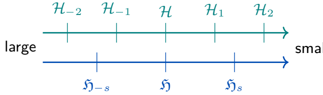

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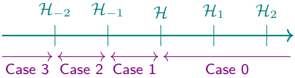
- Q: Can we turn H^{bare} into self-adjoint $H : \mathrm{dom}(H) \to \mathfrak{H}$?
  We distinguish 4 cases:



- **Case 0**: $f_j \in \mathcal{H}_0$ is **almost trivial**.
  $H : \mathfrak{H}_2 \to \mathfrak{H}$ is self-adjoint
  by Kato–Rellich theorem on $\mathrm{dom}(H) = \mathfrak{H}_2$. Key estimates:
  $\|a(f)\psi\| \leqslant \left\|\frac{f}{\omega^{1/2}}\right\| \|\mathrm{d}\Gamma(\omega)^{1/2}\psi\|$
  $\|a^*(f)\psi\|^2 = \|a(f)\psi\|^2 + \|f\|^2\|\psi\|^2$

- **Case 1**: $f_j \in \mathcal{H}_{-1} \backslash \mathcal{H}_0$ is **still easy**. H is self-adjoint by KLMN theorem, but we lose information on $\mathrm{dom}(H)$.

- **Case 2**: $f_j \in \mathcal{H}_{-2} \backslash \mathcal{H}_{-1}$ is **not so easy**.

- **Case 3**: $f_j \notin \mathcal{H}_{-2}$ is **really hard**.

Main Results: Case 1

- First result; application of abstract theorem by [Posilicano 2020].
- Assumptions:
  (i) B_j is normal, i.e., $B_j^* B_j = B_j B_j^*$,
  (ii) $[B_j, B_{j'}] = 0$, $\forall j, j' = 1, \dots, N$, and
  (iii) $\bigcap_{j=1}^{N} \mathrm{Ker}(B_j) = \{0\}$.
  (iv) f_j are $\mathcal{H}$-independent, i.e., $\sum_j c_j f_j \in \mathcal{H}$ implies $c_j = 0\ \forall j$.

Theorem (L., Lonigro 2023)

*Let $\omega \geqslant m > 0$, $f_j \in \mathcal{H}_{-1} \backslash \mathcal{H}_0$ and (i)–(iv) hold. Then H is **self-adjoint** on*

$$\boxed{\mathrm{dom}(H) = \left\{\Psi \in \mathfrak{H} \ : \ \left(1 + \left(K + \mathrm{d}\Gamma(\omega) - z_0\right)^{-1} A^*\right) \Psi \in \mathfrak{H}_2\right\}},$$

*for any $z_0 \in \mathbb{R} \cap \rho(K + \mathrm{d}\Gamma(\omega))$. Further, there exist $(f_{j,n})_{n\in\mathbb{N}} \subset \mathcal{H}$ with $\|f_j - f_{j,n}\|_{\mathcal{H}_{-1}} \to 0$ such that in **norm resolvent sense**:*

$$\boxed{H_n := \left(K + \mathrm{d}\Gamma(\omega) + \sum_{j=1}^{N} \left(B_j a^*(f_{j,n}) + B_j^* a(f_{j,n})\right)\right) \xrightarrow{n\to\infty} H}.$$

Main Results: Case 2

- Second result uses “completion of the square” argument.
- We assume only

  (ii) $[B_j, B_{j'}] = 0,\ \forall j, j' = 1, \ldots, N$

Theorem (Alvarez, L., Lonigro, Martín, 2025)

*Let $\omega \geqslant m > 0$, $f_j \in \mathcal{H}_{-2}$ and (ii) hold. Then, we construct a densely defined H, which is bounded below. Thus, it allows for a **self-adjoint Friedrichs extension**, also called H.*

Further, for any $(f_{j,n})_{n\in\mathbb{N}} \subset \mathcal{H}$ with $\|f_j - f_{j,n}\|_{\mathcal{H}_{-2}} \to 0$, we have

$$\boxed{H_n := \left(K + \mathrm{d}\Gamma(\omega) + \sum_{j=1}^{N} (B_j a^*(f_{j,n}) + B_j^* a(f_{j,n})) - E_n \right) \xrightarrow{n\to\infty} H}$$

*in **norm resolvent sense**, with self-energy counterterm*

$$E_n := \sum_{j,j'=1}^{N} B_j^* B_{j'} \int \frac{\overline{f_{j,n}(k)} f_{j',n}(k)}{\omega(k)} \mathrm{d}k$$

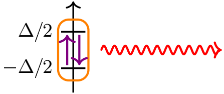
Applications

- **Van Hove model**: $N = 1$, $D = 1$,

$$H^{\text{bare}} = \mathrm{d}\Gamma(\omega) + a^*(f) + a(f)$$

- **Standard spin–boson model**: $N = 1$, $D = 2$, with $\Delta > 0$

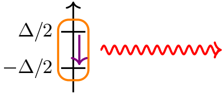
$$K = \frac{\Delta}{2}\sigma_z = \begin{pmatrix} \frac{\Delta}{2} & 0 \\ 0 & -\frac{\Delta}{2} \end{pmatrix}, \quad B = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



$\sigma_x, \sigma_y, \sigma_z$: Pauli matrices. Here, $BB^* = B^*B$

- **Rotating wave approximation (RWA)** spin–boson model:

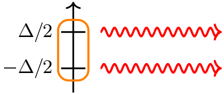
$$K = \frac{\Delta}{2}\sigma_z, \quad B = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$



Here, $BB^* \neq B^*B$, but $B^2 = 0$

- **Dephasing** spin–boson model:

$$K = \frac{\Delta}{2}\sigma_z, \quad B = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

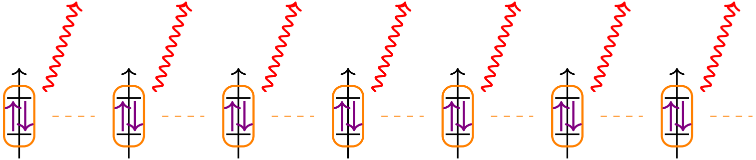


(= direct sum of 2 van Hove models, up to constants)

Applications

- **Multi-spin–boson models**: $N \in \mathbb{N}$, $D = 2^N$, fix some $B \in \mathbb{C}^{2\times 2}$,

$$B_j := 1 \otimes \ldots \otimes \underbrace{B}_{j\text{-th}} \otimes \ldots \otimes 1$$



- In particular, here $[B_j, B_{j'}] = 0 \; \forall j \neq j'$
- Then, $K \in \mathbb{C}^{2^N}$ models interactions between the spins.

Literature

- Case 0 is thoroughly studied [Bach, Ballesteros, Bruneau, Dereziński, Gérard, Hasler, Herbst, Hinrichs, Jakšić, Könenberg, Menrath, Siebert, ...]
- Cases 1 and 2 for similar models via cutoff-renormalization are well-studied [Nelson 1964], [Eckmann 1970], [Fröhlich 1973], [Sloan 1973]
- Cases 1 and 2 for similar models without cutoffs: [Lampart, Henheik, Posilicano, Schmidt, Teufel, Tumulka] via "interior–boundary conditions" (IBC)
- Case 3: **van Hove model** is thoroughly studied [Dereziński 2003], [Fewster, Rejzner 2020], [Falconi, Hinrichs 2025].
  Standard spin–boson model gets trivial, i.e., $H_n \to \mathrm{d}\Gamma(\omega)$ [Dam, Møller 2020].
  Beyond that, very few works on similar models exist, e.g. [Gross 1973].

- **GSB Cases 1 and 2**: [Lonigro 2021-23] constructs H for specific B_j
- **GSB Case 2**: [Hinrichs, Lampart, Martín 2025] normal B_j and $B_j^2 = 0$
- **GSB Case 3**: [Falconi, Hinrichs, Martín 2025] $N = 1$, normal B

Proof Ideas (Case 2)

- First note that $\|K\| < \infty$, so adding/subtracting K does not affect self-adjointness. Formally, we write:

$$
\begin{aligned}
&H^{\text{bare}} - K \\
&= \mathrm{d}\Gamma(\omega) + \sum_j B_j a^*(f_j) + \sum_j B_j^* a(f_j) \\
&= \int \Big(\omega(k) a_k^* a_k + \sum_j B_j f_j(k) a_k^* + \sum_j B_j^* \overline{f_j(k)} a_k \Big) \mathrm{d}k \\
&= \int \omega(k) \Big(a_k + \sum_j B_j \frac{f_j(k)}{\omega(k)} \Big)^* \underbrace{\Big(a_k + \sum_j B_j \frac{f_j(k)}{\omega(k)} \Big)}_{=:\hat{a}_k} \mathrm{d}k \underbrace{- \sum_{j,j'} B_j^* B_{j'} \int \frac{\overline{f_j(k)} f_{j'}(k)}{\omega(k)} \mathrm{d}k}_{=:-E_\infty}
\end{aligned}
$$

$$
\Rightarrow \quad \boxed{H^{\text{bare}} - E_\infty = K + \int \omega(k) \hat{a}_k^* \hat{a}_k \mathrm{d}k}
$$

- Note: $\hat{a}_k^*, \hat{a}_k$ generally do not satisfy the CCR.
- However, if $B_j B_j^* = B_j^* B_j$, then $\hat{a}_k^*, \hat{a}_k$ satisfy the CCR.

- We will define the r.h.s. as the renormalized Hamiltonian H:

$$\boxed{H^{\text{bare}} - E_\infty = K + \int \omega(k)\hat{a}_k^* \hat{a}_k \mathrm{d}k =: H}$$

- Step 1: Construct dressing trafo $T = \exp\Big(- \sum_{j=1}^N B_j a^*(\frac{f_j}{\omega})\Big)$, such that for any $v \in \mathbb{C}^d$ we have $\hat{a}_k T(v \otimes \Omega) = 0$.
- Step 2: Fix suitable ONB $(e_\ell)_{\ell \in \mathbb{N}} \subset \mathcal{H}_2$, show $\|H\psi\| < \infty$ for

$$\psi \in \mathcal{D} := \mathrm{Span}\{a^*(e_{\ell_1}) \dots a^*(e_{\ell_n}) T(v \otimes \Omega)\} \subset \mathfrak{H}$$

- Step 3: Show that $\mathcal{D}$ is dense.
- Obviously, $\int \omega(k)\hat{a}_k^* \hat{a}_k \mathrm{d}k \geqslant 0$, so $H \geqslant -\|K\|$ is self-adjoint by Friedrichs' extension.

- Proof of norm resolvent convergence: For $z \in \mathbb{C}$, $\mathrm{Re}(z)$ small, resolvents are $R(z) := (H - z)^{-1}$ and $R_n(z) := (H_n - z)^{-1}$
- Compute via resolvent identity with $\hat{a}_n(g) := a(g) + \sum_j B_j \langle g, \frac{f_{j,n}}{\omega} \rangle$:

$$
\begin{aligned}
&\left\| R_n(z) - R(z) \right\| \\
&\leqslant \sum_{j=1}^{N} \left\| R_n(z) \big(B_j^* \hat{a}(f_j - f_{j,n}) + \hat{a}_n^*(f_j - f_{j,n}) B_j \big) R(z) \right\| \\
&\leqslant C \sum_{j=1}^{N} \| B_j \| \Big(\left\| \hat{a}(f_j - f_{j,n}) R(z) \right\| + \left\| R_n(z) \hat{a}_n^*(f_j - f_{j,n}) \right\| \Big)
\end{aligned}
$$

- Conclude with $\left\| \hat{a}(f_j - f_{j,n}) R(z) \right\| \leqslant C \left\| \frac{f_j - f_{j,n}}{\omega} \right\| \xrightarrow{n \to \infty} 0$, and likewise $\left\| R_n(z) \hat{a}_n^*(f_j - f_{j,n}) \right\| \xrightarrow{n \to \infty} 0$ $\square$

Warning: In the proof, $R_n(z)(H - H_n)\psi$ with $\psi \in \mathrm{dom}(H)$ appears. But typically $\mathrm{dom}(H) \cap \mathrm{dom}(H_n) = \{0\}$. Solution: Use Hilbert space riggings to extend $H_n : \mathrm{dom}(H_n) \to \mathfrak{H}$ to $H_n : \mathfrak{H} \to \mathfrak{H}_{n,-}$. Then, $R_n(z) : \mathfrak{H}_{n,-} \to \mathfrak{H}$.

Remarks

$$\boxed{H^{\mathsf{bare}} - E_\infty = K + \int \omega(k) \hat{a}_k^* \hat{a}_k \mathrm{d}k =: H}$$

- Generally, $\int \frac{|f_j|^2}{\omega} = \infty$ so E_∞ contains divergent integral.
  $\Rightarrow E_\infty$ and H^{bare} are not operators on $\mathfrak{H}$
- Heuristically, H^{bare} was "too large by an infinite term", which we removed.
- Still, we could rigorously define H^{bare} and E_∞ via Fock space extensions as in [L. 2022-25].
- In case $B_j B_j^* = B_j^* B_j$, a unitary Weyl trafo $W : \mathfrak{H} \to \mathfrak{H}$ exists with

$$W^* \hat{a}_k W = a_k \quad \Rightarrow \quad W^* H W = W^* K W + \mathrm{d}\Gamma(\omega)$$

So $W^* H W$ is a bounded perturbation of a trivial model.

Outlook

- We expect IBC technique as in [Lampart, Schmidt] to work in Case 2 without $[B_j, B_{j'}] = 0$ $\rightarrow$ future research
- Resolvent expansion technique by [Alvarez, Møller 2022-2024] should also work in Case 2 without $[B_j, B_{j'}] = 0$ $\rightarrow$ future research
- Case 3 without $B_j B_j^* = B_j^* B_j$ will require finding suitable dressing transformation. This may be very hard $\rightarrow$ future research
- Case 3 - triviality argument of [Dam, Møller 2020] relies on $B_j B_j^* = B_j^* B_j$. In general, we do not expect triviality.

Thank you for your attention!

Outlook

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