The gnocchi phase in nuclear pasta

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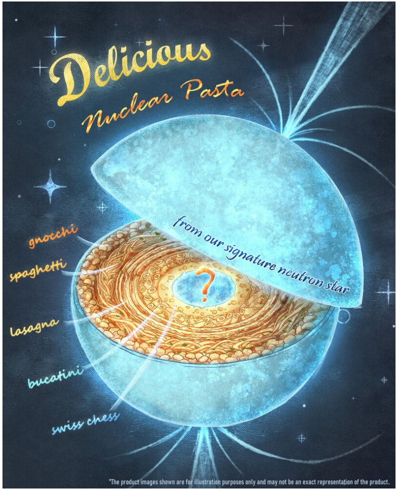
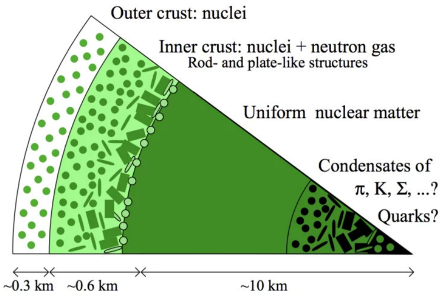


https://www.artstation.com/artwork/Ye54Dd

Nuclear pasta

- postulated to exist within the crust of neutron stars
- believed to be the strongest material in the universe
- nuclear attraction and Coulomb repulsion forces comparable magnitude
  $\rightsquigarrow$ neutrons+protons form a variety of complex geometric structures (delocalized electrons)



H. Heiselberg, *arXiv:astro-ph/0201465*, 2002

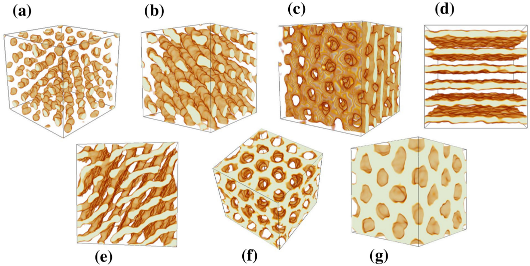
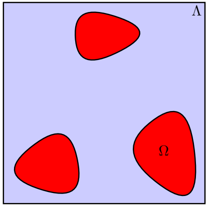


FIG. 3 Nuclear pasta configurations produced in our MD simulations with 51,200 nucleons: (a) gnocchi, (b) spaghetti, (c) waffles, (d) lasagna, (e) defects, (f) antispaghetti, and (g) antignocchi (Horowitz *et al.*, 2015; Schneider *et al.*, 2014, 2013).

M. E. Caplan and C. J. Horowitz, *Rev. Mod. Phys.* 2017

Liquid drop model

For $0 < \rho < 1$ and $\Omega \subset \Lambda \subset \mathbb{R}^3$ with $|\Omega| = \rho|\Lambda|$



$$\mathcal{E}_\Lambda[\rho,\Omega] := \mathrm{Per}(\Omega) + \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(\mathbb{1}_\Omega - \rho\mathbb{1}_\Lambda)(x)(\mathbb{1}_\Omega - \rho\mathbb{1}_\Lambda)(y)}{|x-y|}\,dx\,dy$$

- uniform liquid of protons+neutrons in Ω, with $\mathrm{Per}(\Omega)$ = surface area ($\sim$nuclear attraction)
- uniform gas of electrons with (relative) density ρ
- Gamow (1929), Bohr-Wheeler (1939), Ohta-Kawasaki (1986) for diblock copolymers

$$E_\Lambda(\rho) := \min_{\substack{\Omega\subset\Lambda \\ |\Omega|=\rho|\Lambda|}} \mathcal{E}_\Lambda[\rho,\Omega]$$

Conjecture (Ravenhall-Pethick-Wilson '83, Hashimoto-Seki-Yamada '84)

In the limit $\Lambda \nearrow \mathbb{R}^3$ with $\rho = |\Omega|/|\Lambda|$ fixed, previous pasta phases for Ω when ρ is varied in $[0,1]$.

[symmetry $\Omega \leftrightsquigarrow \Lambda\setminus\Omega$ corresponding to $\rho \leftrightsquigarrow 1-\rho$]

Main results

Theorem (Thermodynamic limit)

Let Λ_n be a sequence of smooth sets, so that $B(0,\ell_n/C)\subset\Lambda_n\subset B(0,\ell_n)$ for some $\ell_n\to\infty$. The following limit exists and does not depend on the sequence Λ_n:

$$e(\rho):=\lim_{n\to\infty}\frac{E_{\Lambda_n}(\rho)}{|\Lambda_n|}$$

- Adaptation of Lieb-Lebowitz-Narnhofer (1972–75)
- Periodic boundary conditions: Alberti-Choksi-Otto (2009). Cubes Λ_n: Emmert-Frank-König (2020)

Theorem (Low density regime)

$$e(\rho)=\mu_*\,\rho+m_*^{\frac{2}{3}}\,e_{\rm Jel}\,\rho^{\frac{4}{3}}+o(\rho^{\frac{4}{3}})_{\rho\to0}$$

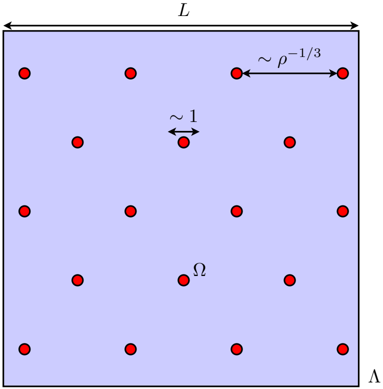
where μ_*, m_* = energy, mass of an isolated nuclear droplet, and $e_{\rm Jel}$ = Jellium energy.

- Choksi-Peletier (2010), Knüpfer-Muratov-Novaga (2016), Emmert-Frank-König (2020) when $\rho_n\to0$
- shape of Ω_n? many unsolved conjectures for $\mu_*, m_*, e_{\rm Jel}$!

Gnocchi phase

$$e(\rho) = \underbrace{\mu_* \, \rho}_{\text{shape}} + \underbrace{m_*^{\frac{2}{3}} \, e_{\rm Jel} \, \rho^{\frac{4}{3}}}_{\text{arangement of gnocchis}} + o(\rho^{\frac{4}{3}})_{\rho\to 0}$$

Gnocchi phase: $\Omega \approx$ union of infinitely many sets of size 1 placed at distance $\sim \rho^{-1/3}$



Two poorly understood famous minimization problems:

- gnocchis should be identical balls of radius $R_* = (15/8\pi)^{1/3}$, conjectured optimizers of **isolated drop problem**,

$$m_* \stackrel{?}{=} |B_{R_*}|, \qquad \mu_* \stackrel{?}{=} \mathrm{Per}(B_{R_*}) + \frac{1}{2}\iint_{B_{R_*}\times B_{R_*}} \frac{dx\,dy}{|x-y|}$$

- should be placed on Body-Centered-Cubic lattice of side length $\sim \rho^{-1/3}$, conjectured minimizer of **Jellium problem**

$$e_{\rm Jel} \stackrel{?}{=} \zeta_{\rm BCC}(1) \simeq -1.44 \qquad \text{(Epstein zeta function)}$$

Isolated droplets

$$\mu_* := \min_{\substack{\Omega\subset\mathbb{R}^3\\ |\Omega|>0}} \frac{I[\Omega]}{|\Omega|}, \qquad I[\Omega] := \mathrm{Per}(\Omega) + \frac{1}{2}\iint_{\Omega^2}\frac{dx\,dy}{|x-y|}$$

- minimizers exist and are bounded, with a volume $|\Omega_*| \in [m_{**}, m_*]$ with $5/2 \leq m_{**} \leq m_* \leq 8$
- conjectured to be balls of radius $R_* = (15/8\pi)^{1/3}$, in which case $m_* = 5/2$

Knüpfer-Muratov 2013, Frank-Lieb 2015, Knüpfer-Muratov-Novaga 2016, Frank-Killip-Nam 2016

Theorem (Minimizing sequences)

For any sequence $\{\Omega_n\}$ such that $|\Omega_n|^{-1}I[\Omega_n] \to \mu_*$ with $|\Omega_n| \to m > 0$, we can find minimizers $\Omega_*^{(1)}, ..., \Omega_*^{(K)}$ and sequences $a_n^{(j)} \in \mathbb{R}^3$ with $|a_n^{(j)} - a_n^{(k)}| \to_{n\to\infty} \infty$ such that, up to a subsequence,

- $(\Omega_n - a_n^{(j)}) \cap B(0,R) \to \Omega_*^{(j)}$ *(in L^1 for the corresponding characteristic fns);*
- $\left|\Omega_n \setminus \bigcup_{j=1}^K B(a_n^{(j)}, R)\right| + \mathrm{Per}\left(\Omega_n \setminus \bigcup_{j=1}^K B(a_n^{(j)}, R)\right) \to 0$ *and hence* $m = \sum_{j=1}^K |\Omega_*^{(j)}|$;
- $I[\Omega_n] \geq \mu_*|\Omega_n| + \displaystyle\sum_{1\leq j<k\leq K} \frac{|\Omega_*^{(j)}|\,|\Omega_*^{(k)}| + o(1)}{|a_n^{(j)} - a_n^{(k)}|}$

Jellium model: Wigner crystal

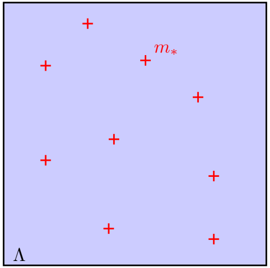
$$e_{\rm Jel}(m_*,\rho) := \lim_{\Lambda\nearrow\mathbb{R}^3} \min_{\substack{x_1,\dots,x_N\\ Nm_*=\rho|\Lambda|}} |\Lambda|^{-1}\left(\sum_{1\le j<k\le N}\frac{m_*^2}{|x_j-x_k|} - m_*\rho\sum_{j=1}^N\int_\Lambda\frac{dy}{|x_j-y|} + \rho^2\iint_{\Lambda^2}\frac{dx\,dy}{|x-y|}\right)$$

- N point particles of charge m_* in a uniform background of charge ρ
- $e_{\rm Jel}(m_*,\rho) = m_*^{\frac23}\rho^{\frac43}\underbrace{e_{\rm Jel}(1,1)}_{=:e_{\rm Jel}}$
- existence of thermodynamic limit by Lieb-Lebowitz-Narnhofer 1973–75
- also called one-component plasma or plum pudding model
- **Crystallization conjecture (Wigner, 1934):** particles placed on BCC lattice, in which case one finds

$$e_{\rm Jel} \stackrel{?}{=} \zeta_{\rm BCC}(1) = \sum_{z\in{\rm BCC}\setminus\{0\}} \frac{1}{|z|^s}\Big|_{\Re(s)>3\ \rightsquigarrow\ s=1} \qquad \text{(Epstein zeta function)}$$

- crystallization only known in dimensions $d\in\{1,8,24\}$

Kunz 1974, Ventevogel 1978, Cohn-Kumar-Miller-Radchenko-Viazovska 2022, Petrache–Serfaty 2020, L. 2022



Infinite Jellium ground states

Definition (Infinite ground states for the Jellium problem, $m_* = \rho = 1$)

$X = \{x_j\}$ with $|x_j - x_k| \geq \eta > 0$ and there exists a $V : \mathbb{R}^3 \to \mathbb{R}$ such that $-\Delta V = 4\pi(\sum_j \delta_{x_j} - 1)$ with $V(x) - \sum_j \frac{\mathbb{1}(|x-x_j| \leq \delta/2)}{|x-x_j|} \in L^\infty(\mathbb{R}^3)$ (or less), such that $\forall R > 0$, $X \cap B_R$ minimizes the inside energy

$$Y \mapsto \sum_{j<k} \frac{1}{|y_j - y_k|} - \sum_j \int_{B_R} \frac{dy}{|y_j - y|} + \sum_j V_{B_R^c}(y_j)$$

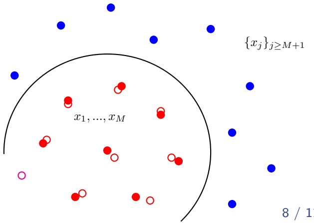
- **(canonical)** among all $Y \subset B_R$ with $\#Y = \#(X \cap B_R)$
- **(grand-canonical)** among all $Y \subset B_R$

where $V_{B_R^c}(y) := V(y) - \sum_{x_j \in B_R} \frac{1}{|y-x_j|} + \int_{B_R} \frac{dz}{|y-z|}$ is the potential generated from the outside.

$\approx$ Dobrushin-Lanford-Ruelle, Sinai '82, Radin '84, Radin-Bellissard-Schlosman '10, L. '22

Theorem (Existence – L. 2022)

Infinite grand-canonical Jellium ground states exist.



- **Conjecture 1:** BCC lattice is an infinite ground state
- **Conjecture 2:** liquid drop $\Omega \approx \bigcup_j \{\rho^{-\frac{1}{3}} x_j + \Omega_j^*\}$

Main steps for proof of low density expansion I

- **Step 1: reducing to sets of size $\gtrsim \rho^{-1/3}$**

Theorem (Graf-Schenker inequality '94)

Let $\{\Delta_j\}$ be the tiling of $\mathbb{R}^3$ obtained from $\ell\mathbb{Z}^3 + (-\ell/2, \ell/2)^3$ by splitting each cube into 24 congruent tetrahedra. Then for any $f \in L^1 \cap L^{6/5}(\mathbb{R}^3)$ we have

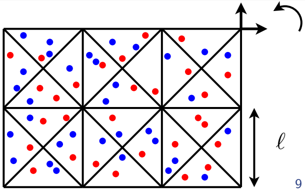
$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{f(x)f(y)}{|x-y|}\,dx\,dy \geq \frac{1}{\ell^3}\int_{[0,\ell]^3\times SO(3)} \left(\sum_j \iint_{g\cdot\Delta_j\times g\cdot\Delta_j} \frac{f(x)f(y)}{|x-y|}\,dx\,dy\right) dg$$

In particular there exists a translation+rotation of the tiling g such that

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{f(x)f(y)}{|x-y|}\,dx\,dy \geq \sum_j \iint_{g\cdot\Delta_j\times g\cdot\Delta_j} \frac{f(x)f(y)}{|x-y|}\,dx\,dy$$

Hainzl-L.-Solovej 2009

- $\frac{1-|\Delta|^{-1}\mathbb{1}_\Delta * \mathbb{1}_{-\Delta}(x)}{|x|}$ has positive Fourier transform
- similar version for point particles
- $\Longrightarrow$ reduces the liquid drop to tetrahedra of size $A\rho^{-1/3}$ with an error $\rho^{4/3}/A$ coming from the localization of the perimeter **(grand-canonical)**



Main steps for proof of low density expansion II

- **Step 2: study of the liquid drop in a tetrahedron of size $A\rho^{-1/3}$ (grand-canonical)**
  - background has a finite charge $\rho(A\rho^{-\frac{1}{3}})^3|\Delta| = A^3|\Delta| \Longrightarrow$ expect finitely many gnocchis
  - proof that Ω has volume $|\Omega| \leq 8 + 16\pi A^3 \text{diam}(\Delta)^3$ following Frank-Killip-Nam '16
  - theorem on minimizing sequences $\Longrightarrow$ Ω essentially made of finitely many minimizers for the isolated droplet model
  - repulsion between the droplets $\Longrightarrow$ Jellium model for finitely many particles in $A\Delta$
  - largest possible mass m_* by concavity of the Jellium energy w.r.t. m

- **Step 3: thermodynamic limit for Jellium**
  - grand-canonical $\equiv$ canonical
  - gives the lower bound

- **Step 4: Upper bound**
  - place isolated droplets on an approximate (periodic) minimizer for the Jellium problem
  - use that periodic BC lead to same Jellium energy e_{Jel}

Conclusion

Summary:

- studied a continuous model for nuclear matter where competition between attractive geometric forces and Coulomb repulsion leads to phase transitions with a lot of geometry
- in the low density regime $\rho \to 0$, Ω should be close to the union of balls placed at distance $\rho^{-1/3}$ on a BCC lattice, in the uniform sea of electrons (gnocchi in tomato sauce)

Open problems:

- better understand μ_* and $e_{\rm Jel}$
- $\Omega =$ union of minimizers of μ_* placed on an infinite ground state for $e_{\rm Jel}$, whatever they are
- phase transitions when ρ is increased
- numerics