Large deviations for quantum trajectories: a look at the Keep–Switch instrument*

September 15, 2025, *Quantissima sur Oise*, Cergy

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*ongoing work with **T. Benoist, J.-L. Fatras, C. Pellegrini and P. Petit**

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- Example: Keep-Switch instrument

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- Ancilla: $V_a = (\mathbf{1} \otimes \langle a|)W(\mathbf{1} \otimes |p\rangle)$ with W unitary on $\mathcal{H} \otimes \mathcal{H}_p$

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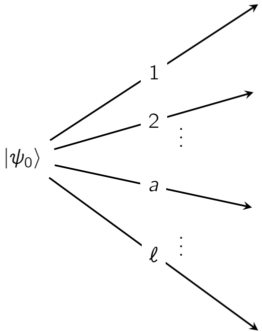

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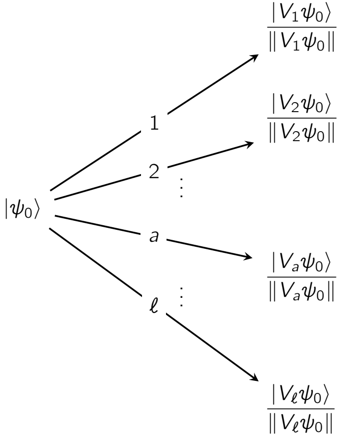

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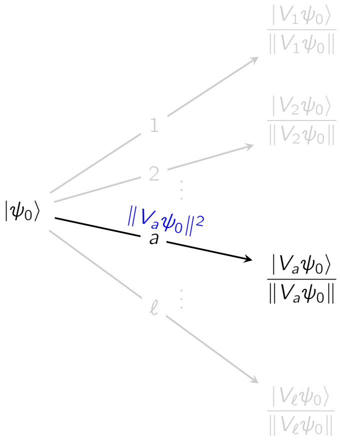

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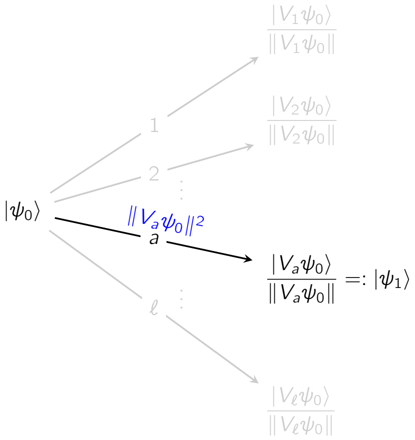

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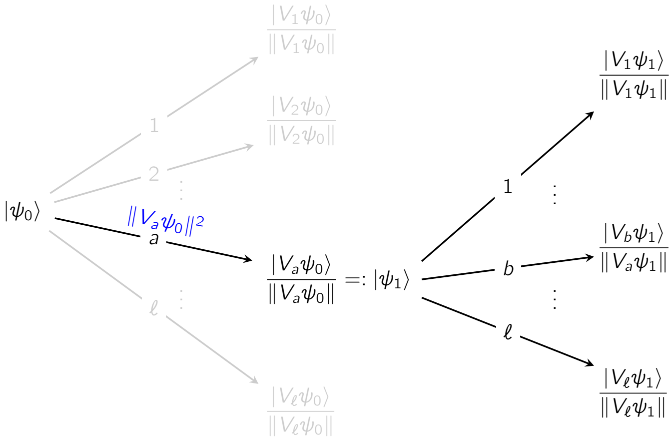

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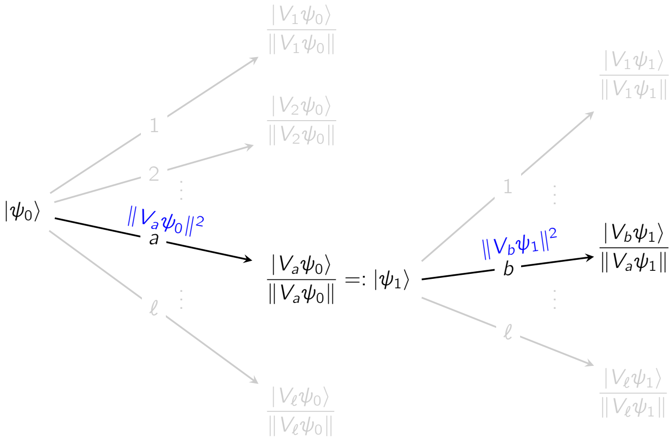

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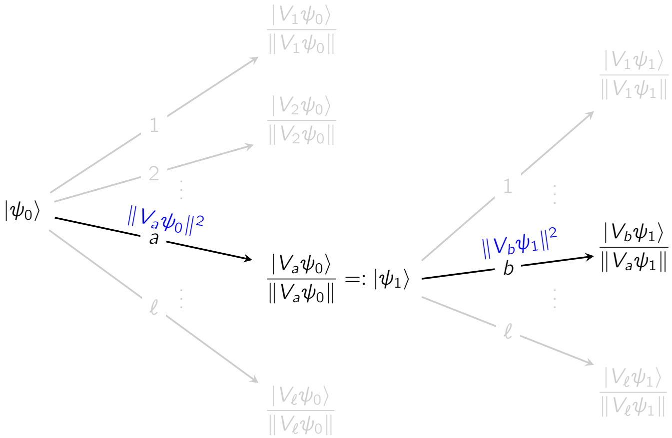

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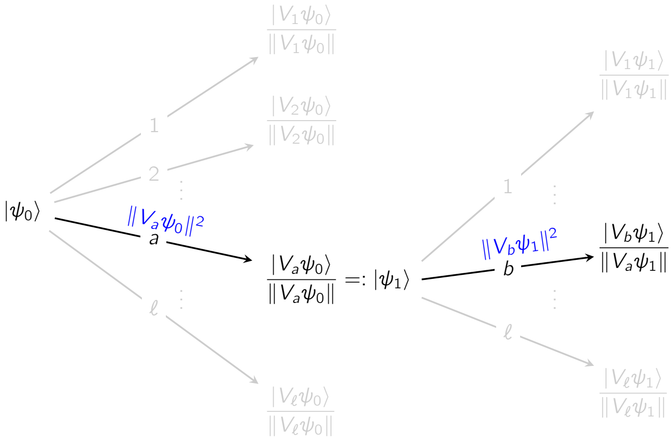

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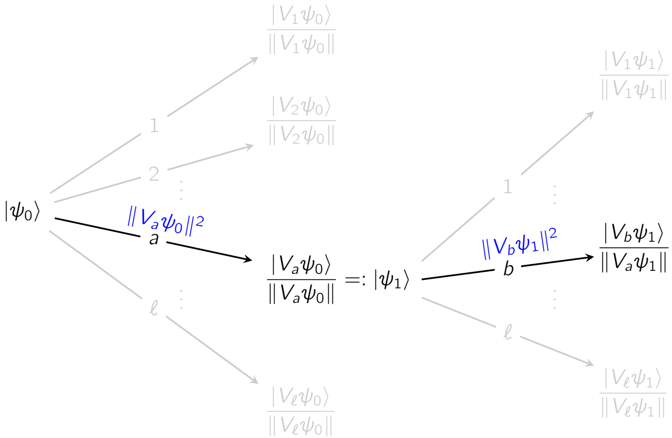

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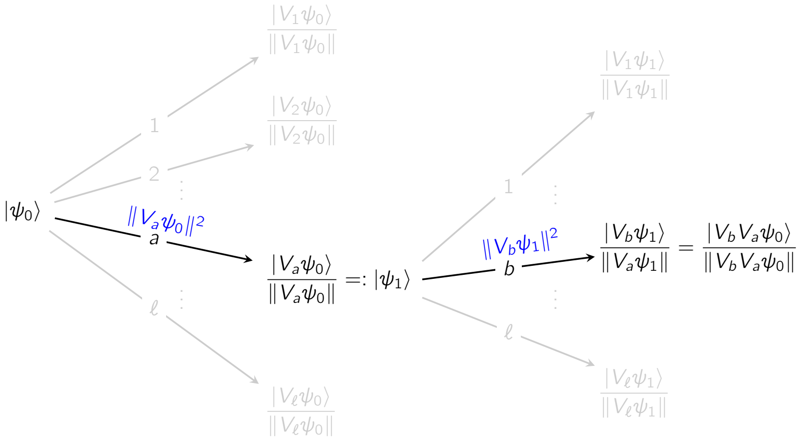

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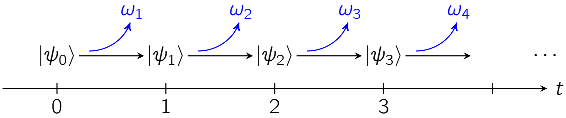

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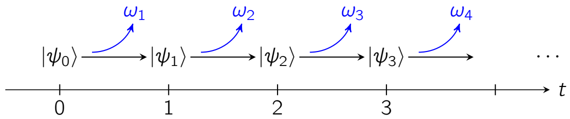

- **Sequence of outcomes**: $\omega_1, \omega_2, \ldots$. Large deviations in [BJPP18], [CJPS19], [BCJP21]
- **Quantum trajectory**: $|\psi_0\rangle, |\psi_1\rangle, \ldots$. Focus of this talk

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- ✗ Or: ‘chaotic’ functions

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  - $\Rightarrow$ density matrices attracted to the set of pure states

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$$\nu P^n \xrightarrow{W_1} \nu_{\text{inv}} \quad \text{exponentially fast}$$

Here:

- $\nu P^n(A) = \int P^n(x, A)\nu(dx)$
- W_1 is the 1-Wasserstein metric

$$W_1(\mu, \nu) \;=\; \inf_{\pi:\text{coupl.}} \int_{\mathbf{P}(\mathbb{C}^d)\times\mathbf{P}(\mathbb{C}^d)} d(x, y)\, d\pi(x, y),$$

with $d(|\psi\rangle, |\varphi\rangle) = (1 - |\langle\psi|\varphi\rangle|^2)^{\frac{1}{2}}$ (all unit vectors)

LLN, CLT, LIL and LDP

Theorem. Benoist, Fatras, Pellegrini '23, Benoist, Hautecœur, Pellegrini '25
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- **Local LDP** on $(\partial_\theta^+ \Lambda(\theta_-), \partial_\theta^- \Lambda(\theta_+)) \ni \langle g \rangle$

Large Deviation Principle

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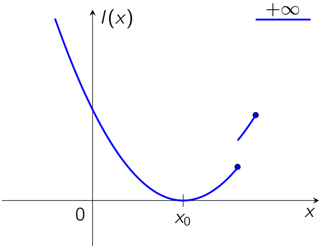

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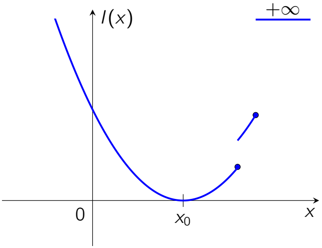

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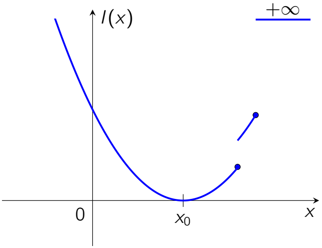

- **Local LDP** on $J \subset X$: both bounds required only for $A \subset J$
- **Weak LDP**: lower bound for all A, upper bound for all precompact A

What we would like

We would like an LDP for the **empirical measures**

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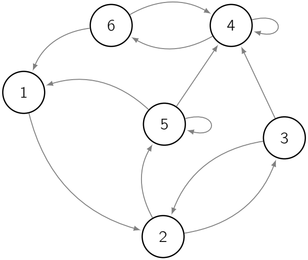
$$\mathbb{I}_{DV}(\mu) = \sup_{u \geq 1} \int_{\mathbf{P}(\mathbb{C}^d)} \ln\left(\frac{u}{Pu}\right) d\mu$$

A digression about transients and large deviations



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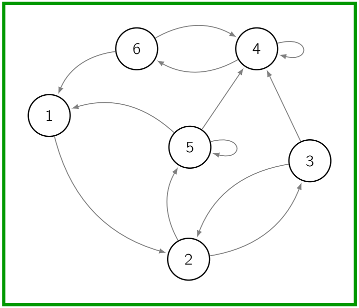

$\mathbb{I}_*$

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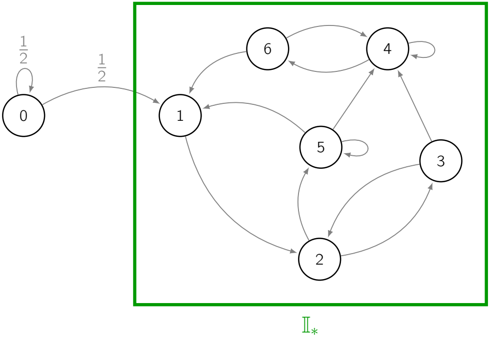

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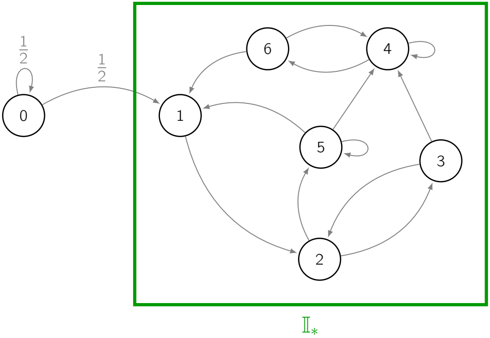

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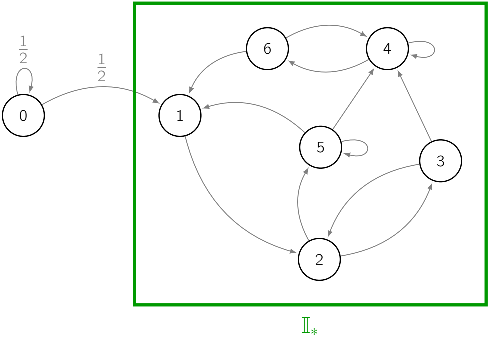

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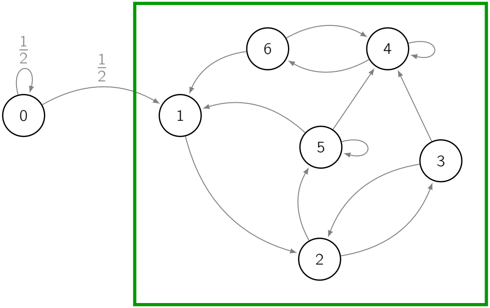

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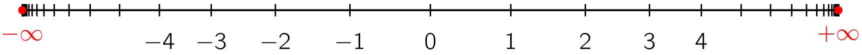
$-\infty$
-4
-3
-2
-1
0
1
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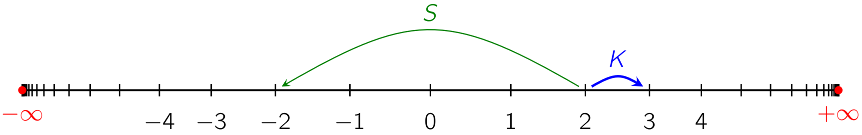
3
4
$+\infty$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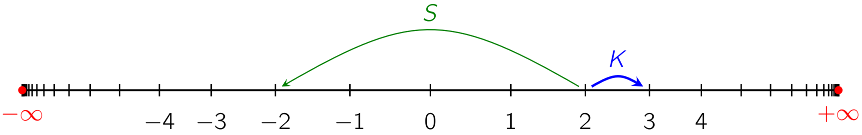
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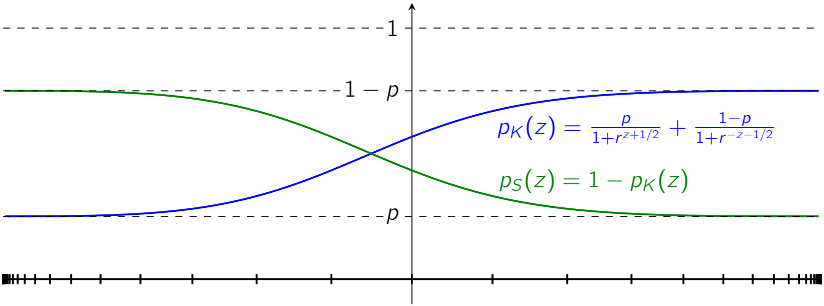
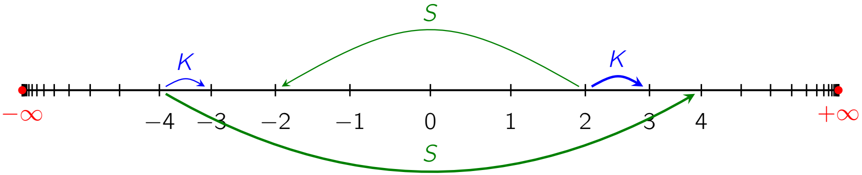
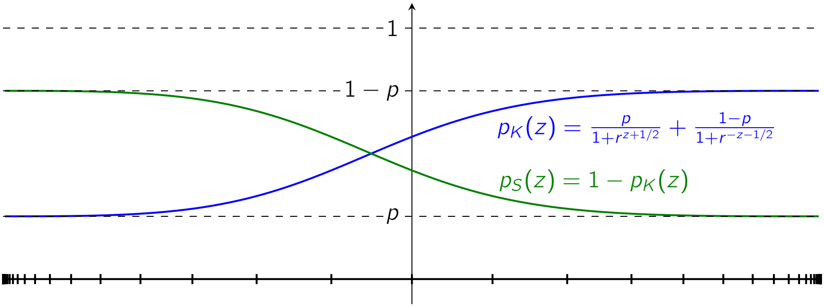

1
$1-p$
p
$p_K(z) = \frac{p}{1+r^{z+1/2}} + \frac{1-p}{1+r^{-z-1/2}}$
$p_S(z) = 1 - p_K(z)$


S
K
K
$-\infty$
-4 -3 -2 -1 0 1 2 3 4
$+\infty$
S


1
$1-p$
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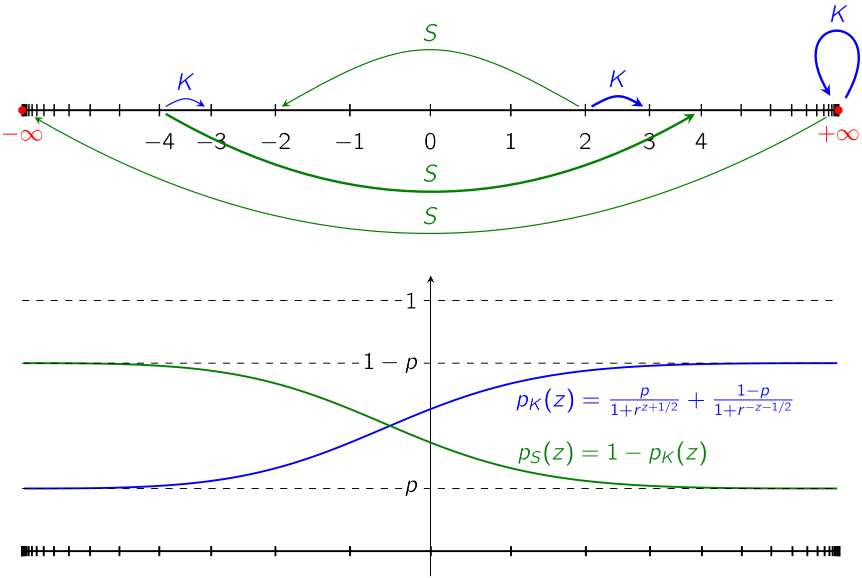
S
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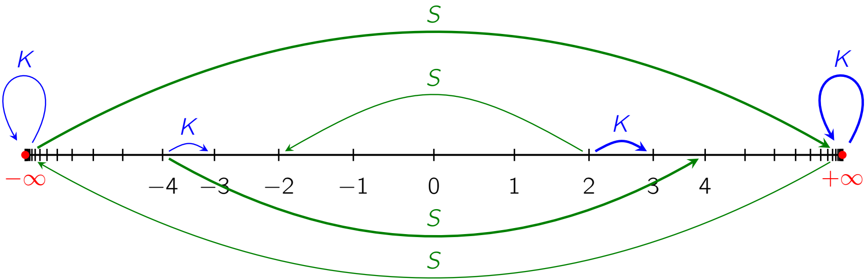

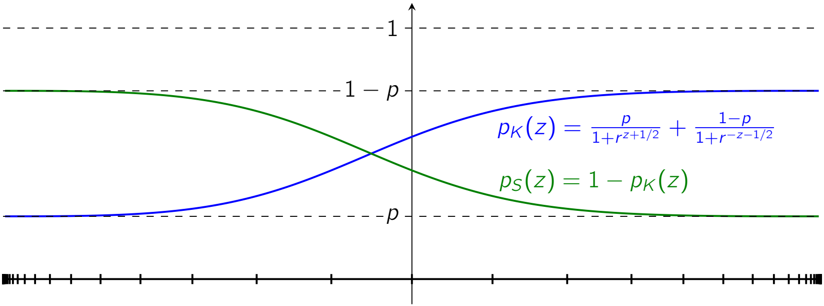

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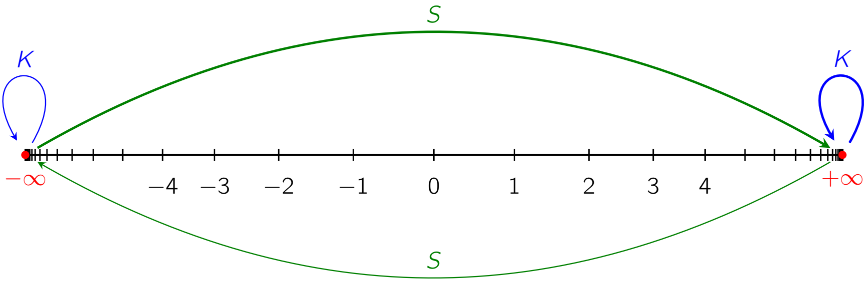
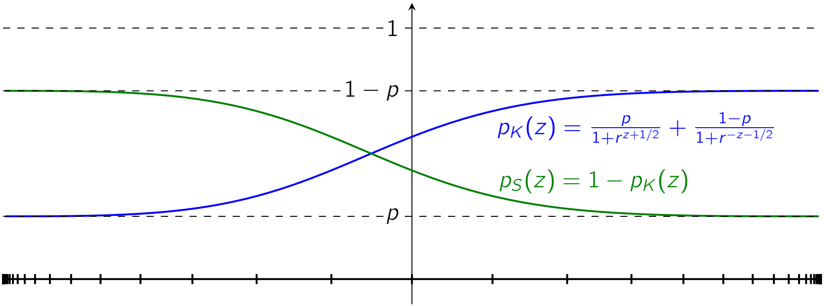

S
K
K
$-\infty$
$+\infty$
-4 -3 -2 -1 0 1 2 3 4
S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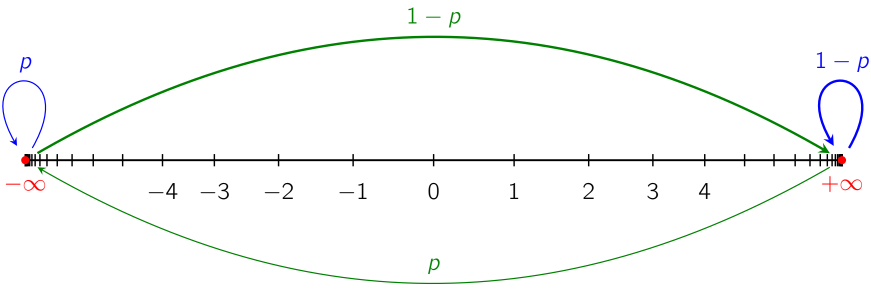
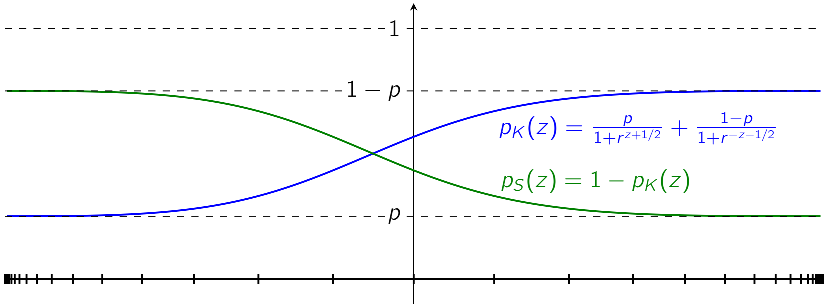

1 − p
p
1 − p
−∞
−4 −3 −2 −1 0 1 2 3 4
+∞
p


1
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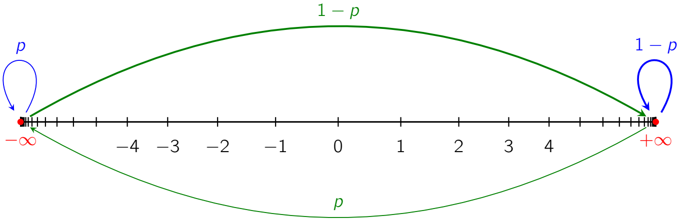
Warm-up: large deviations on $\{-\infty, +\infty\}$



If the initial condition z_0 is $\pm\infty$,

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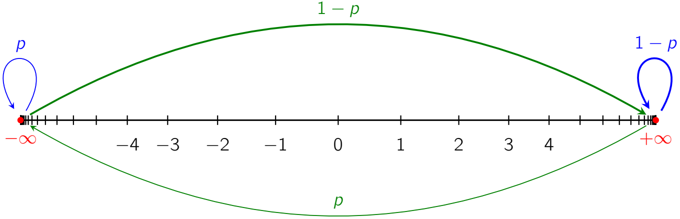

If the initial condition z_0 is $\pm\infty$,

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$$S(\mu|\nu_{\text{inv}}) := \mu(-\infty) \ln \frac{\mu(-\infty)}{p} + \mu(+\infty) \ln \frac{\mu(+\infty)}{1-p}$$

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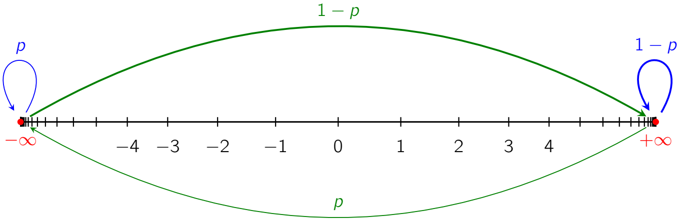

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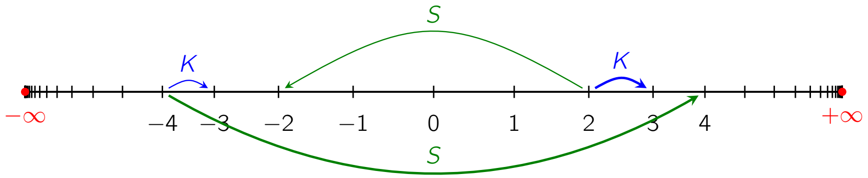
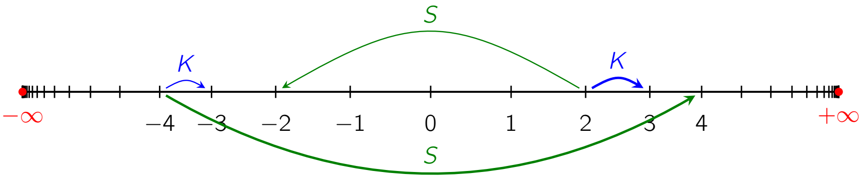
Large deviations away from $\pm\infty$



For $z_0 \in \mathbb{R}$, the set $S_{z_0} := \{z_0, -z_0\} + \mathbb{Z}$ is invariant. The chain on S_{z_0} is irreducible.

Large deviations away from $\pm\infty$



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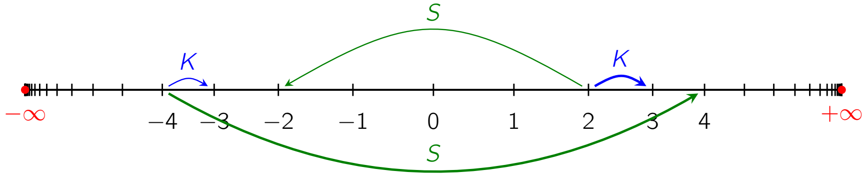

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$$\mathbb{I}_{\mathbb{Z}}(\mu) = \mathbb{I}_{DV}(\mu) = \sup_{u \geq 1} \sum_{i \in \mathbb{Z}} \ln\left(\frac{u(i)}{Pu(i)}\right)\mu(i)$$

See Donsker–Varadhan '75-'76, Daures '25

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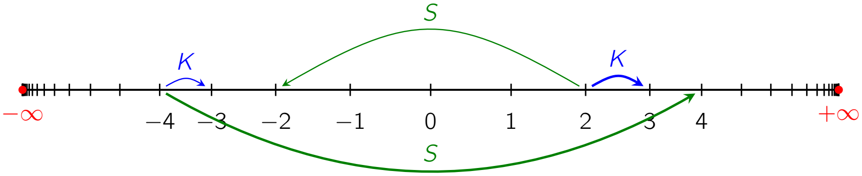

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  - On $\mathbb{Z}$: the theory only gives a weak LDP

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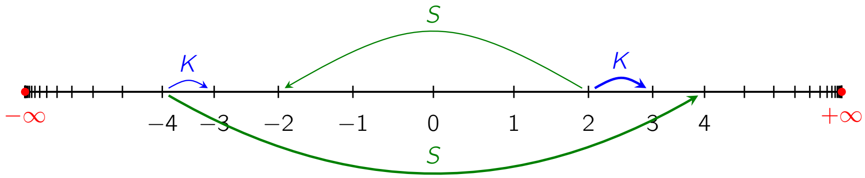

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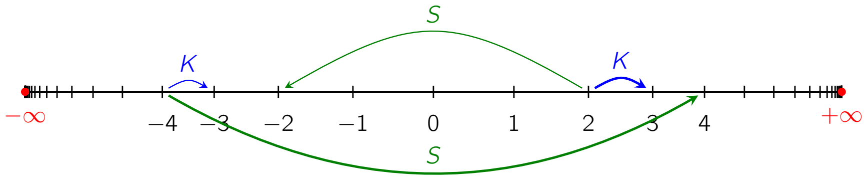
Large deviations away from $\pm\infty$



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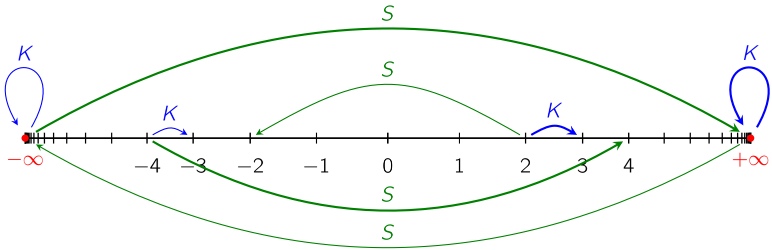
  - On $\overline{\mathbb{R}}$: even less irreducible

Mixing the two



Theorem. Benoist, C, Pellegrini, Petit, 2025.
If $z_0 \in \mathbb{Z}$, then LDP on $\mathcal{M}_1(\overline{\mathbb{R}})$, i.e. $\mathbb{P}(\ell_n \approx \mu) \sim e^{-n\,\mathbb{I}(\mu)}$,

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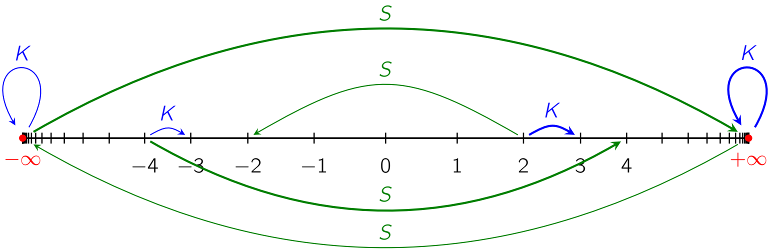

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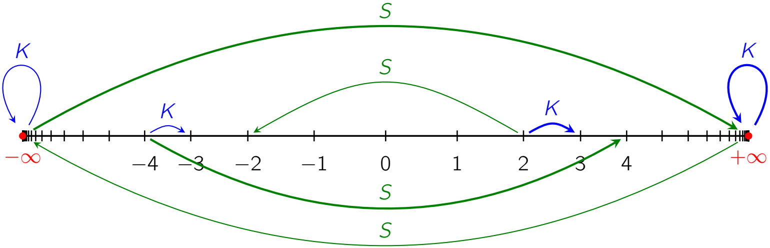

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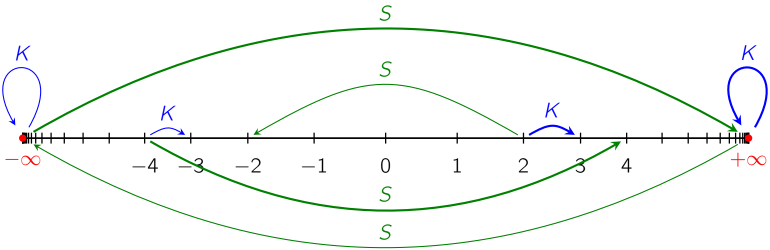

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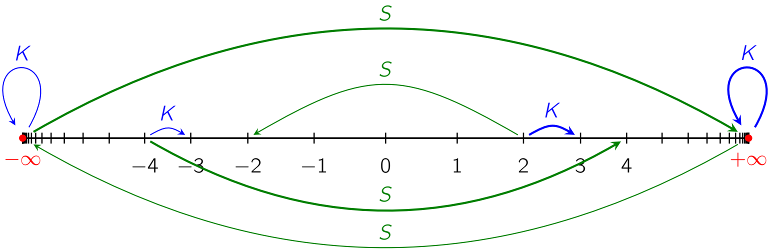

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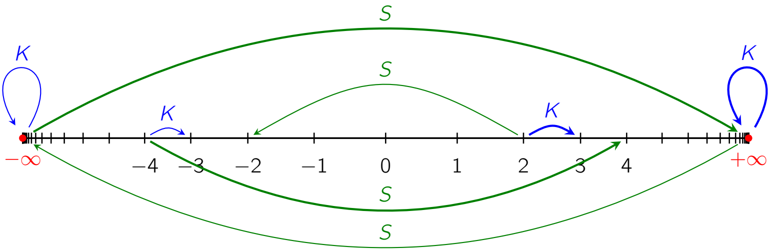

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Actually, LDP for any $z_0 \in \overline{\mathbb{R}}$, but **NOT** for random initial conditions!

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