Ancilla tomography estimation of quantum entropic fluctuations

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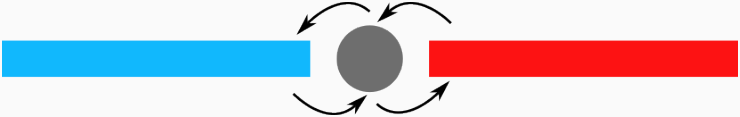
Typical physical setup



Study the fluctuations of heat or entropy production.

$$H = H_c + H_h + V.$$

$$\mathrm{ep} = \frac{1}{T_c} J_{Q_c} + \frac{1}{T_h} J_{Q_h} \geq 0.$$

Classical stochastic thermodynamics definitions

- Phase space: $\mathcal{X}$,
- Dynamics: $(\phi^t)_t$, $\phi^t : \mathcal{X} \to \mathcal{X}$,
- Entropy observable: $\mathcal{X} \ni x \mapsto -\log \rho(x)$ with ρ a reference probability density

$$\rho(x) \propto \exp(-\beta_c h_c(x) - \beta_h h_h(x)).$$

- Entropy production rate: $\sigma = -\partial_t \log \rho \circ \phi^{-t}|_{t=0}$,
- Entropy production: $\Delta S_t = \int_0^t \sigma \circ \phi^{-s} ds = -\log \rho \circ \phi^{-t} + \log \rho$.

Evans-Searles vs. Gallavotti-Cohen

Evans-Searles:

$$\mathcal{F}_{ES,t}(\alpha) = \int_{\mathcal{X}} e^{-\alpha \Delta S_t} \rho(x) dx = \int_{\mathcal{X}} \rho^{1-\alpha}(x) \rho^{\alpha}_{-t}(x) dx.$$

with $\rho_{-t} = \rho \circ \phi^{-t}$.

Gallavotti-Cohen:
NESS: $\rho_+ = \lim_{t\to\infty} \frac{1}{t} \int_0^t \rho \circ \phi^s ds$.

$$\mathcal{F}_{GC,t}(\alpha) = \int_{\mathcal{X}} e^{-\alpha \Delta S_t} \rho_+(x) dx = \int_{\mathcal{X}} \rho^{-\alpha}(x) \rho^{\alpha}_{-t}(x) \rho_+(x) dx.$$

If ϕ^t and ρ are time reversal invariant:

$$\mathcal{F}_{ES,t}(\alpha) = \mathcal{F}_{ES,t}(1-\alpha).$$

Equivalent to

$$\text{Probab}_{ES}(\Delta S_t = -s) = e^{-st}\,\text{Probab}_{ES}(\Delta S_t = s).$$

In general

$$\mathcal{F}_{GC,t}(\alpha) \neq \mathcal{F}_{GC,t}(1-\alpha).$$

Large deviation level

Under strict regularity conditions on ϕ:

$$\lim_{t\to\infty} \frac{1}{t} \log \mathcal{F}_{GC,t}(\alpha) = \lim_{t\to\infty} \frac{1}{t} \log \mathcal{F}_{GC,t}(1-\alpha).$$

Actually,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathcal{F}_{GC,t}(\alpha) = \lim_{t\to\infty} \frac{1}{t} \log \mathcal{F}_{ES,t}(\alpha).$$

This is the Principle of Regular Entropic Fluctuations (PREF).

Quantum stochastic thermodynamics

- Hilbert space: $\mathcal{H}$,
- Dynamics: $(U^t)_t$ unitary group ($U^t = e^{-itH}$),
- Entropy observable: $-\log\rho$ with ρ a reference state

$$\rho \propto \exp(-\beta_c H_c - \beta_h H_h).$$

- Entropy production rate: $\sigma = -i[H, \log\rho]$?
- Entropy production: $\Delta S_t = \int_0^t U^{-s}\sigma U^s ds = -U^{-t}\log\rho U^t + \log\rho$?

Issues:

- No clear physical interpretation of the measurement of ΔS_t,
- Continuous measurement of σ can lead to Zeno effect,
- No fluctuation relation except in few trivial cases.

Two time measurement definition

Interpret the formula $\mathcal{F}(\alpha) = \int_{\mathcal{X}} \rho_{-t}^{\alpha}\rho^{-\alpha} d\nu$ as measurements at the end and beginning.

$$-\log \rho = \sum_e e P_e.$$

1. Start in a state ν and measure $-\log \rho$ obtaining result e,

$$\nu \rightsquigarrow P_e \nu P_e / \operatorname{tr}(P_e \nu), \quad \operatorname{Probab}(e|\nu) = \operatorname{tr}(P_e \nu),$$

2. Let evolve with U^t,

$$P_e \nu P_e / \operatorname{tr}(P_e \nu) \rightsquigarrow U^t P_e \nu P_e / \operatorname{tr}(P_e \nu) U^{-t},$$

3. Measure $-\log \rho$, obtaining e',

$$\operatorname{Probab}(e'|\nu, e, t) = \operatorname{tr}\left(P_{e'} U^t P_e \nu P_e U^{-t}\right) / \operatorname{tr}(P_e \nu).$$

As a result:

$$\operatorname{Probab}_{\nu,t}(\Delta S_t = s) = \sum_{e'-e=s} \operatorname{tr}\left(P_{e'} U^t P_e \tilde{\nu} U^{-t}\right)$$

with

$$\tilde{\nu} = \sum_e P_e \nu P_e.$$

Two time measurement definition

Let,

$$\mathcal{F}_{\nu,t}(\alpha) = \mathrm{tr}\big(\rho_{-t}^{\alpha}\rho^{\alpha}\tilde{\nu}\big)$$

with $\rho_{-t} = U^{-t}\rho U^{t}$ and

$$\tilde{\nu} = \lim_{R\to\infty}\frac{1}{R}\int_0^R \varsigma_\rho^\theta(\nu)d\theta$$

with $\varsigma_\rho^\theta = \rho^{i\theta}\nu\rho^{-i\theta}$ the modular dynamic of ρ.

Bruneau-Panati formula:

$$\mathcal{F}_{\nu,t}(\alpha) = \lim_{R\to\infty}\frac{1}{R}\int_0^R \nu\circ\varsigma_\rho^\theta([D\rho_{-t} : D\rho]_\alpha)d\theta.$$

Connes' cocycle: $[D\nu : D\rho]_\alpha : O \mapsto \nu^\alpha\rho^{-\alpha}O\rho^{-\alpha}\rho^\alpha = \nu^\alpha\rho^{-\alpha}O.$

- Clear physical interpretation,
- Automatic fluctuation relation when $\nu = \rho$ (Evans-Searles),
- It makes sense in the thermodynamic limit even for $\nu \neq \rho$.

Two time measurement definition: Thermodynamic limit

- Observable C*-algebra: $\mathcal{O}$,
- Dynamic: $(\tau^t)_t$ group of $*$-automorphism of $\mathcal{O}$, $\partial_t \tau^t = \tau^t \circ (\delta_c + \delta_h + i[V, \cdot])$, $V \in \mathcal{O}$,
- Free dynamic: $(\tau_{fr}^t)_t$, $\partial_t \tau_{fr}^t = \tau_{fr}^t \circ (\delta_c + \delta_h)$,
- Reference state: $\rho \circ \tau_{fr}^t = \rho$, $\rho_t = \rho \circ \tau^t$,
- Under standard assumptions, $[D\rho_{-t} : D\rho]_\alpha \in \mathcal{O}$.

Assuming

$$\mathcal{F}_{\nu,t}(\alpha) = \lim_{R \to \infty} \frac{1}{R} \int_0^R \nu \circ \varsigma_\rho^\theta([D\rho_{-t} : D\rho]_\alpha) d\theta$$

exists, it is the Laplace transform of the probability measure

$$\mathrm{Probab}_{\nu,t}(|\Delta S_t - s| \leq \varepsilon).$$

Two time measurement definition: Thermodynamic limit

Assume ν is a thermodynamic limit: $\nu = \lim_{L\to\infty} \nu_L$.

Theorem (B., Bruneau, Jakšić, Panati, Pillet 2024)
Under standard assumptions, for any $\nu \ll \rho$, $\mathcal{F}_{\nu,t}(\alpha)$ is well defined and

$$\lim_{L\to\infty} \mathcal{F}_{\nu_L,t}(\alpha) = \mathcal{F}_{\nu,t}(\alpha).$$

$\nu \ll \rho$ means it is a weak* limit of $\rho(A^* \cdot A)$, $A \in \mathcal{O}$.

If $\nu \mapsto \mathcal{F}_{\nu,t}$ is weak* continuous, by density of ρ-normal states, $\mathcal{F}_{\nu,t}$ extended to any state ν by continuity.

Super stability of the the two time measurement definition

Theorem (B., Bruneau, Jakšić, Panati, Pillet 2024)
Assume ς_ρ is ergodic, then for any state ν,

$$\mathcal{F}_{\nu,t}(\alpha) = \mathcal{F}_{\rho,t}(\alpha).$$

Assume $\mathcal{O} = \mathcal{O}_S \otimes \mathcal{O}_R$ with $\mathcal{O}_S$ finite dimensional, $\tau^t_{fr,R} : \mathcal{O}_R \to \mathcal{O}_R$ ergodic and $\rho \propto \operatorname{tr} \otimes \rho_R$, then,

$$\mathcal{F}_{\nu,t}(\alpha) = \mathcal{F}_{\nu_S \otimes \rho_R}(\alpha)$$

with $\nu_S = \nu|_{\mathcal{O}_S}$. If $\nu_S > 0$, there exists $C > 0$ such that for α real,

$$C^{-1}\mathcal{F}_{\nu,t}(\alpha) \leq \mathcal{F}_{\rho,t}(\alpha) \leq C\mathcal{F}_{\nu,t}(\alpha).$$

Remark
The decoherence due to the first $-\log \rho$ measurement forces the initial state back to the reference state.

Assumption of perfect measurement implies its timescale is at least similar to the return to equilibrium time scale.

Principle of Regular Entropic Fluctuations (PREF)

Let α real be such that $\lim_{t\to\infty} \frac{1}{t} \log \mathcal{F}_{\rho,t}(\alpha)$ exists.

Corollary
If $\rho_{T,S} > 0$, then

$$\lim_{t\to\infty} \frac{1}{t} \log \mathcal{F}_{\rho_T,t}(\alpha) = \lim_{t\to\infty} \frac{1}{t} \log \mathcal{F}_{\rho,t}(\alpha).$$

Assume moreover that $\rho_+ = \lim_{T\to\infty} \rho_T$ exists.

Corollary
If $\rho_{+,S} > 0$,

$$\lim_{t\to\infty} \lim_{T\to\infty} \frac{1}{t} \log \mathcal{F}_{\rho_T,t}(\alpha) = \lim_{T\to\infty} \lim_{t\to\infty} \frac{1}{t} \log \mathcal{F}_{\rho_T,t}(\alpha).$$

Remark
Evans-Searles and Gallavotti-Cohen versions of fluctuation relation are equivalent. The PREF holds trivially due to the strong decoherence/return to equilibrium effect of the first measurement of $-\log \rho$.

Two-time measurement issues

- Trivializes the thermodynamic,
- Invasive extended infinitely precise measurement,
- Experimentally inaccessible.

Idea: perform a physical Fourier transform.
Initial idea in 2011 by Jakšić and Pillet in private communications. Emerged independently in physics in 2014 (see De Chiara *et al.* 2018 for a review).

A toy example of physical Fourier transform

Observable O probability distribution $\mathbb{P}_{\nu,O}$ and Fourier transform

$$\mathcal{F}_{\nu,O} : \alpha \mapsto \int e^{-\alpha o} \mathrm{d}\mathbb{P}_{\nu,O}(o).$$

Goal: Sample $\mathcal{F}_{\nu,O}$.

Procedure:

1. Adding a qbit: $\mathcal{H} \rightsquigarrow \mathcal{H} \otimes \mathbb{C}^2$,
2. Initialize the qbit: $\nu \rightsquigarrow \nu \otimes \rho_a$, where $\rho_a = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$,
3. Let the system and qbit interact for time $i\alpha$ with interaction Hamiltonian $O \otimes \sigma_z$:

   $$\rho_a(\alpha) = \mathrm{tr}_{\mathcal{H}}[e^{-\frac{\alpha}{2} O \otimes \sigma_z}(\nu \otimes \rho_a) e^{\frac{\alpha}{2} O \otimes \sigma_z}].$$

   Then,

   $$\rho_a(\alpha) = \frac{1}{2}\begin{pmatrix} 1 & \mathcal{F}_{\nu,O}(\alpha) \\ \overline{\mathcal{F}_{\nu,O}(\alpha)} & 1 \end{pmatrix},$$

4. Tomography of $\rho_a(\alpha) \rightsquigarrow \mathcal{F}_{\nu,O}(\alpha)$ sampling,
5. Repeating for several α and an inverse Fourier transform gives $\mathbb{P}_{\rho,O}$.

Ancilla tomography for entropic fluctuations

Physical Fourier transform corresponding to measurement⇝evolution⇝measurement.

- $\rho_a(t,\alpha) = \mathrm{tr}_{\mathcal{H}}[U_t(\alpha)(\nu \otimes \rho_a)U_t(\alpha)^*]$ with

$$U_t(\alpha) = e^{\frac{\alpha}{2}\log\rho\otimes\sigma_z}\left(U^t \otimes \mathbb{1}_{\mathbb{C}^2}\right)e^{-\frac{\alpha}{2}\log\rho\otimes\sigma_z},$$

- $\rho_a(t,\alpha) = \frac{1}{2}\begin{pmatrix} 1 & \mathcal{G}_{\nu,t}(\alpha) \\ \overline{\mathcal{G}_{\nu,t}(\alpha)} & 1 \end{pmatrix}.$

Remarks
If $\nu = \rho$, $\mathcal{G}_{\rho,t} = \mathcal{F}_{\rho,t}$.

Not true if $\nu \neq \rho$. $\mathcal{G}_{\nu,t}$ may not even be the Fourier transform of a probability measure.

The interaction $\log\rho\otimes\sigma_z$ is non local. Does it make sense in the thermodynamic limit?

Thermodynamic limit

- $U_t(\alpha) = \exp\Big(-it(H_c \otimes \mathbb{1}_{\mathbb{C}^2} + H_h \otimes \mathbb{1}_{\mathbb{C}^2} + e^{-\frac{\alpha}{2}\log\rho\otimes\sigma_z}(V \otimes \mathbb{1}_{\mathbb{C}^2})e^{\frac{\alpha}{2}\log\rho\otimes\sigma_z})\Big)$,
- $U_t(\alpha) \rightsquigarrow \tau_\alpha^t$, a group of $*$-automorphism of $\mathcal{O}$,
- $\partial_t \tau_\alpha^t = \tau_\alpha^t \left(\delta_c \otimes \mathbb{1}_{M_2} + \delta_h \otimes \mathbb{1}_{M_2} + i\left[\frac{1}{2}\varsigma_\rho^{-i\alpha}(V) \otimes P_e + \frac{1}{2}\varsigma_\rho^{i\alpha}(V) \otimes P_g,\ \cdot\right]\right)$,
- $\varsigma_\rho^{-i\alpha}(V) \in \mathcal{O}$ (quasi-)local,

$$\rho_a(t,\alpha) = \nu \otimes \rho_a \circ \tau_\alpha^t|_{M_2(\mathbb{C})}.$$

Proposition

$$\rho_a(t,\alpha) = \frac{1}{2}\begin{pmatrix} \frac{1}{\mathcal{G}_{\nu,t}(\alpha)} & \mathcal{G}_{\nu,t}(\alpha) \\ & 1 \end{pmatrix}$$

with $\mathcal{G}_{\nu,t}(\alpha) = \nu([D\rho_{-t} : D\rho]^*_{\overline{\alpha}/2}[D\rho_{-t} : D\rho]_{\alpha/2})$.

Ancilla tomography

- If $\nu = \rho$, $\mathcal{G}_{\rho,t}(\alpha) = \mathcal{F}_{\rho,t}(\alpha)$
- If $\nu \neq \rho$, the function $\mathcal{G}_{\nu,t}$ may be a Fourier transform,
- But is not necessarily the characteristic function of a random variable,
- $(\tau_\alpha^t)_t$ differs from $(\tau^t)_t$ only through the local ancilla-system interaction.

Questions:

- Is this method of estimation stable with respect to initial time?
- What can we say when $\nu = \rho_+$?
- Do we have PREF?

Definition
The PREF holds for $\mathcal{G}$ if

$$\lim_{t\to\infty}\lim_{T\to\infty}\frac{1}{t}\log\mathcal{G}_{\rho_T,t}(\alpha) = \lim_{T\to\infty}\lim_{t\to\infty}\frac{1}{t}\log\mathcal{G}_{\rho_T,t}(\alpha)$$

for α in a real neighborhood of 0.

Ancilla tomography stability

Theorem (B., Bruneau, Jakšić, Panati, Pillet 2024)
There exist $C > 0$ such that, for any $T > 0$ and α in a real neighborhood of 0,

$$C^{-T}\mathcal{G}_{\rho_T,t}(\alpha) \le \mathcal{F}_{\rho,t}(\alpha) \le C^{T}\mathcal{G}_{\rho_T,t}(\alpha).$$

So that, for all $T > 0$,

$$\lim_{t\to\infty} \frac{1}{t}\log \mathcal{G}_{\rho_T,t}(\alpha) = \lim_{t\to\infty} \frac{1}{t}\log \mathcal{F}_{\rho,t}.$$

Moreover, for some paradigmatic models

$$\lim_{t\to\infty}\lim_{T\to\infty} \frac{1}{t}\log \mathcal{G}_{\rho_T,t}(\alpha) = \lim_{T\to\infty}\lim_{t\to\infty} \frac{1}{t}\log \mathcal{G}_{\rho_T,t}(\alpha).$$

Hence, the PREF holds and

$$\lim_{t\to\infty}\lim_{T\to\infty} \frac{1}{t}\log \mathcal{G}_{\rho_T,t}(\alpha) = \lim_{t\to\infty} \frac{1}{t}\log \mathcal{F}_{\rho,t}(\alpha).$$

The second part is proved under the standard hypotheses of the Liouvillean spectral (resonances) approach to existence and uniqueness of NESS (developed at the end of the 90's and in the 2000's).

Typically applies to the spin-fermion model for a high enough temperature and small enough $\|V\|$. Also, $\alpha \mapsto \varsigma_\rho^{i\alpha}(V)$ has to be analytic in a neighborhood of 0 (UV regularity).

GNS representation

Let $(\mathfrak{h}, \pi, \Omega)$ be the GNS representation of $(\mathcal{O}, \rho)$.

Let Δ and J be the associated modular operator and modular conjugation:

$$J\Delta_\rho^{\frac{1}{2}}\pi(A)\Omega = \pi(A^*)\Omega.$$

Liouvilean formulation of the evolution: $L : \mathfrak{H} \to \mathfrak{H}$, $L^* = L$, $JL = -LJ$ and

$$\pi(\tau^t(A)) = e^{itL}\pi(A)e^{-itL}$$

for any $A \in \mathcal{O}$. Similarly, $L_{fr} = L_{fr}^*$, $JL_{fr} = -L_{fr}J$

$$\pi(\tau_{fr}^t(A)) = e^{itL_{fr}}\pi(A)e^{-itL_{fr}}.$$

$$L = L_{fr} + \pi(V) - J\pi(V)J.$$

Actually, $\forall W \in \mathcal{O}$ such that $W = W^*$,

$$e^{it(L+J\pi(W)J)}\pi(A)e^{-it(L+J\pi(W)J)} = \pi(\tau^t(A)).$$

Liouvilean expressions for $\mathcal{F}$ and $\mathcal{G}$

Two one parameter families of Liouvilleans:

- For any $A \in \mathcal{O}$, $\pi(\tau^t(A)) = e^{itL_\alpha}\pi(A)e^{-itL_\alpha}$ with

$$L_\alpha = L - J\pi(V)J + J\pi(\varsigma_\rho^{-i\overline{\alpha}}(V))J$$

  Remark that $L_0 = L$,

- $\widehat{L}_\alpha = \Delta_\rho^{-\alpha/2} L_{1/2-\alpha} \Delta_\rho^{\alpha/2}$, or

$$\widehat{L}_\alpha = L - \pi(V) + J\pi(V)J + \pi(\varsigma_\rho^{i\alpha/2}(V)) - J\pi(\varsigma_\rho^{-i(1-\overline{\alpha})/2}(V))J$$

Proposition

$$\mathcal{F}_{\rho,t}(\alpha) = \langle \Omega, e^{itL_{1/2-\alpha}}\Omega \rangle,$$

$$\mathcal{G}_{\rho_T,t}(\alpha) = \langle \Omega, e^{iTL_{1/2}} e^{it\widehat{L}_\alpha}\Omega \rangle.$$

It remains to analyse the spectrum of these Liouvilleans.

Spectral theory: Analytic deformation

On $\mathfrak{H}$, the spectrum of L has no gap. But there exist $(S^\theta)_{\theta\in\mathbb{C}}$ in $\mathcal{L}(\mathfrak{h})$ such that

$$S^\theta\Omega = \Omega \quad \text{and} \quad L_{fr}(\theta) = S^\theta L_{fr} S^{-\theta} = L_{fr} + \theta N$$

and

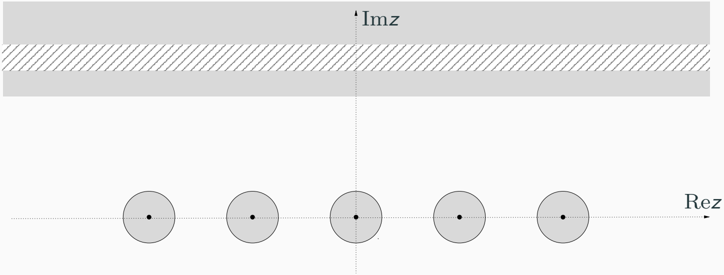
$$\operatorname{spec} L_{fr} \subset \mathbb{R}, \quad \operatorname{spec} L_{fr}(\theta) = \operatorname{spec}([H_{sys.}, \cdot]) \cup (\theta\mathbb{N}^* + \mathbb{R}).$$

On $D = \cap_{|\mathrm{Im}\theta|<r} \operatorname{Dom} S^\theta$ a gap opens between the discrete and essential spectrum for $L_{fr}(\theta)$.

Then, using

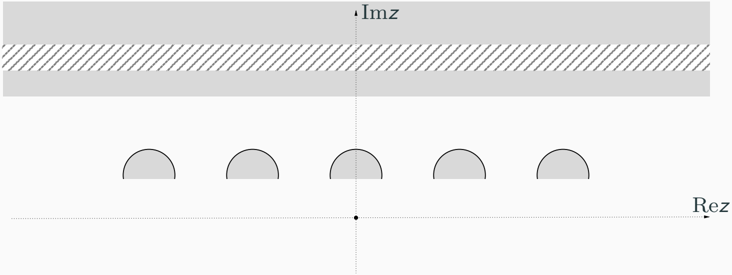
$$L_\alpha(\theta) = S^\theta L_\alpha S^{-\theta} = L_{fr} + \theta N + \lambda W(\alpha,\theta)$$

and perturbation theory in (θ,λ), the gap remains open.



Simplicity from perturbation theory

From the weak coupling limit (Davies Lindbladian), for λ small enough, the dominating eigenvalue of $S^{\theta} L_{\alpha} S^{-\theta}$ is simple and purely imaginary. A similar argument yields the same for $\widehat{L}_{\alpha}$.



Hence, for α real, both L_{α} and $\widehat{L}_{\alpha}$ have some simple purely imaginary resonances, $\mathcal{E}(\alpha)$ and $\widehat{\mathcal{E}}(\alpha)$ respectively, dominating their spectrum.

Limit CGF for two-time measurement

There exist $\gamma > 0$ such that,

$$\langle \Omega, e^{itL_{1/2-\alpha}}\Omega\rangle = \langle \Omega, e^{itL_{1/2-\alpha}(\theta)}\Omega\rangle = ce^{it\mathcal{E}(1/2-\alpha)} + O(e^{t(\mathrm{Im}(\mathcal{E}(1/2-\alpha)-\gamma))}).$$

Hence,

$$\lim_{t\to\infty} \frac{1}{t}\log \mathcal{F}_t(\alpha) = i\mathcal{E}(1/2-\alpha).$$

Limit CGF and PREF for ancilla tomography

$$\mathcal{G}_{\rho_T,t}(\alpha) = \langle \Omega, e^{itL_{1/2}} e^{it\widehat{L}_\alpha} \Omega \rangle = \langle \Omega, e^{itL_{1/2}(\theta)} e^{it\widehat{L}_\alpha(\theta)} \Omega \rangle.$$

Hence,

$$\lim_{T\to\infty} \mathcal{G}_{\rho_T,t}(\alpha) = \langle \Omega, X(\alpha,\theta) e^{it\widehat{L}_\alpha(\theta)} \Omega \rangle.$$

It follows that, there exist $\gamma > 0$ such that,

$$\lim_{T\to\infty} \mathcal{G}_{\rho_T,t}(\alpha) = \widehat{c} e^{it\widehat{\mathcal{E}}(\alpha)} + O(e^{t(\mathrm{Im}(\widehat{\mathcal{E}}(\alpha) - \gamma))}).$$

Then $\mathcal{E}(1/2 - \alpha) = \widehat{\mathcal{E}}(\alpha)$ yields the theorem.

Summary

For two time measurement definition of entropy fluctuations:

- Direct measurement implies PREF and stability by thermodynamic trivialization,
- Ancilla tomography is stable (under UV assumptions) and under standard spectral assumptions, PREF holds.
- However ancilla does not provide access to each fluctuation individualy.

Some open questions

- Proof of PREF using the scattering approach (asymptotic abelianess, XY-model, Electronic Black Box...)?
- Taking into account imperfect measurements of $-\log \rho$ and check the consistency of a perfect measurement limit?
- Other definitions of stochastic entropy production? Continuous measurement in a non-Markovian setting?

Thank you!