Mathematical aspects of quantum Hall physics in microscopic models of interacting fermions

Lectures 4 and 5

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Course overview

1. Physics background, mathematical modelling, and questions we would like to answer
2. Mathematical tools: local approximation of quasi-local operators, the quasi-local inverse of gapped Liouvillians
3. Adiabatic theorem and non-equilibrium almost stationary states (NEASS) as a foundation for understanding (linear) response of gapped systems
4. **Nearly linear Hall current response that is constant in gapped phases (microscopic and macroscopic)**
5. Remarks and perspectives: gapped phases, integer quantization, Hall current density in non-periodic systems

A super-adiabatic theorem for NEASS in infinite volume

Theorem: Becker, T., Wesle '25 (also T. CMP '20 and Henheik, T. FMσ '22)

Then for each $t \in I$, $\eta \in [0,1]$ and $\varepsilon \in [0,1]$ there exists $(S_t^{\varepsilon,\eta})_{t\in I} \in B_{\infty,I}^{(\infty)}$ such that the super-adiabatic state defined by

$$\omega_t^{\varepsilon,\eta} := \omega_t \circ \beta_t^{\varepsilon,\eta} := \omega_t \circ \mathrm{e}^{\mathrm{i}\mathcal{L}_{S_t^{\varepsilon,\eta}}} ,$$

almost intertwines the time evolution $\mathfrak{U}_{t_0,t}^{\varepsilon,\eta}$ in the following sense:

For every $n \in \mathbb{N}$ there exists a constant C_n such that for all $t,\ t_0 \in I$, $\varepsilon \in [0,\ 1]$, $\eta \in (0,\ 1]$, and $A \in \mathcal{A}_\infty$ the following bound holds:

$$|\omega_t^{\varepsilon,\eta}(A) - \omega_{t_0}^{\varepsilon,\eta}(\mathfrak{U}_{t_0,t}^{\varepsilon,\eta}\,A)| \leq C_n \frac{\varepsilon^{n+1} + \eta^{n+1}}{\eta^{4d+2}} \left(1 + |t - t_0|^{4d+2}\right) \|A\|_{2d+1,x} \,.$$

We now apply this general super-adiabatic theorem for NEASS to the problem of adiabatically switching on a perturbation.

Response of gapped systems to adiabatic switching of perturbations

Recall that we are interested in the dynamics generated by a Hamiltonian of the form

$$H^{\varepsilon,\eta}(t) := \frac{1}{\eta}\left(H_0 + \varepsilon\, f(t)\,(\Phi + V)\right) ,$$

where $H_0, \Phi \in B_\infty$ and $V \in \mathcal{V}$, H_0 has a gapped ground state ω_0, and $f : \mathbb{R} \to [0,1]$ is a smooth function such that $f(t) = 1$ for $t \geq 0$ and $f(t) = 0$ for $t \leq -1$.

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Since in this case the ground state ω_0 of H_0 does not depend on time, the regularity assumptions of the general theorem are trivially satisfied and we obtain the following

Corollary

Let H_0, ω_0, Φ, and V as above. Then the NEASS state ω_ε of the previous theorem satisfies that for all $n, m \in \mathbb{N}$ there is $c_{n,m} > 0$ such that for all $t \geq 0$ and $A \in \mathcal{A}_\infty$

$$\sup_{\eta\in[\varepsilon^m,\varepsilon^{1/m}]} \left|\omega_0 \circ \mathfrak{U}^{\varepsilon,\eta}_{-1,t}(A) - \omega_\varepsilon(A)\right| \leq c_{n,m}\,\varepsilon^n\,(1+t^{4d+2}) \inf_{x\in\mathbb{Z}^d} \|A\|_{2d+1,x} \,.$$

Response of gapped systems to adiabatic switching of perturbations

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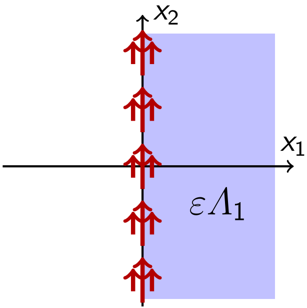
This result justifies using the NEASS ω_ε for computing the current response (or any other response) of the system to all orders.

Defining the Hall conductance resp. conductivity

One can consider the current response

to a **potential step**

$$V = \Lambda_1 := \sum_{x \in \mathbb{N}_0 \times \mathbb{Z}} n_x$$



or to a **potential gradient**

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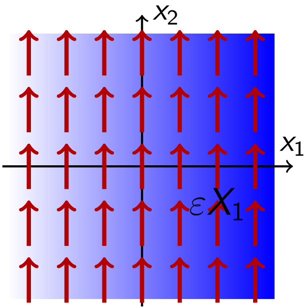

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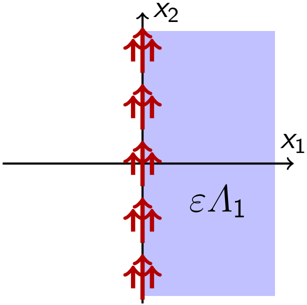

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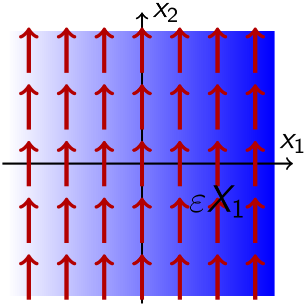
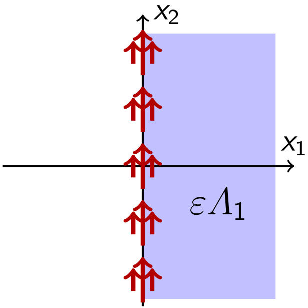


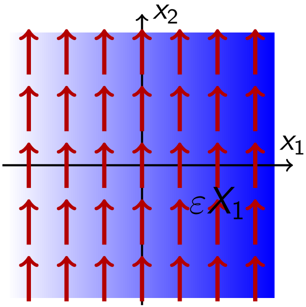
The **response observable** is

the current through any horizontal line

the current density into the x_2-direction

Defining the Hall conductance resp. conductivity





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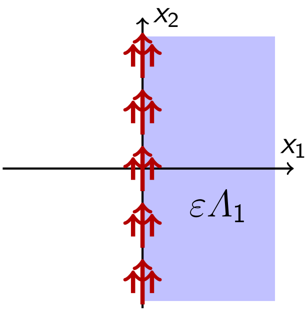

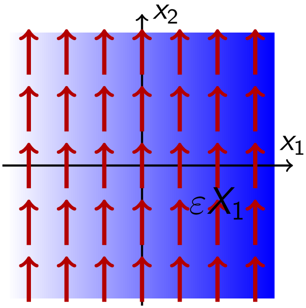

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The perturbation **does not close the gap** and the response is microscopic.

The perturbation **closes the gap** and the response is macroscopic.

Current response of an insulator to a **voltage drop**

Theorem: T., Wesle (2025) arXiv:2506.13581

Assume that $H_0 \in B_\infty$ and that H_0 has a gapped ground state ω_0.

Then for the NEASS ω_ε associated with $H_\varepsilon = H_0 + \varepsilon \Lambda_1$ and ω_0 it holds that

$$\omega_\varepsilon(I_2) - \omega_0(I_2) \;=\; -\varepsilon \underbrace{\omega_0\left(\mathrm{i}\,[\Lambda_1^{\mathrm{OD}}, \Lambda_2^{\mathrm{OD}}]\right)}_{=:c_{\mathrm{H}}} \;+\; \mathcal{O}(\varepsilon^\infty)\,,$$

where c_{H} is the "microscopic" Hall conductance.

Here

$$I_2 := \mathrm{i}[H_0, \Lambda_2] = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{\mathrm{i}\mathcal{L}_{H_0} t} \Lambda_2|_{t=0}$$

is the "current flowing into the upper half-plane operator".

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Note that no periodicity or homogeneity is assumed and that c_{H} is independent of the choice of the origin and of the orientation of the half-planes.

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Moreover, $\mathrm{i}\,[\Lambda_1^{\mathrm{OD}}, \Lambda_2^{\mathrm{OD}}] \in \mathcal{A}_\infty \subset \mathcal{A}$ is a quasi-local observable localized near the origin and can be considered a local Chern marker.

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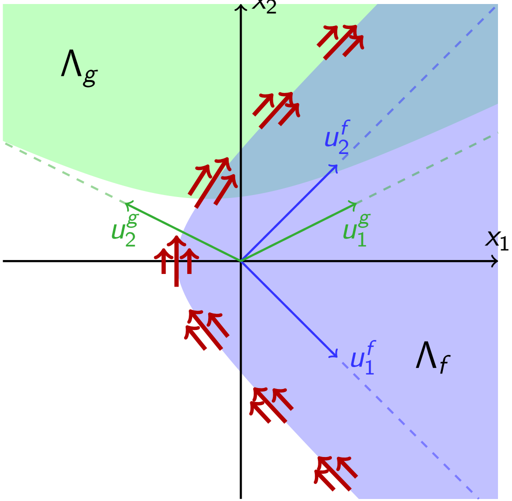
where c_{H} is the "microscopic" Hall conductance.

If ω_ε is a gapped ground state of H_ε, i.e., if the perturbation does not close the gap, then the response is exactly linear:

$$\omega_\varepsilon(I_2) - \omega_0(I_2) \;=\; -\varepsilon\,\omega_0\left(\mathrm{i}\,[\Lambda_1^{\mathrm{OD}}, \Lambda_2^{\mathrm{OD}}]\right).$$

Current response of an insulator to a **voltage drop**

The result actually holds for any pair of generalised step functions on sets containing transversal cones, always with the same c_{H}.



Translation invariant systems

Definition: Translations

A translation is a map $T : \mathbb{Z}^d \to \mathrm{Aut}(\mathcal{A})$ that associates to each possible shift vector γ a $*$-automorphism T_γ and satisfies the following properties:

(i) For all $\gamma \in \mathbb{Z}^d$ and $M \subseteq \mathbb{Z}^d$ it holds that $T_\gamma(\mathcal{A}_M) = \mathcal{A}_{M+\gamma}$.

(ii) For all $\gamma \in \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, it holds that $T_\gamma n_x = n_{x+\gamma}$,

where $n_x := \sum_{i=1}^n a_{x,i}^* a_{x,i}$.

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where $n_x := \sum_{i=1}^n a_{x,i}^* a_{x,i}$.

Note that T does **not** have to be a group homomorphism, i.e. $T_\gamma \circ T_{\tilde{\gamma}} = T_{\gamma+\tilde{\gamma}}$ need not hold for all $\gamma, \tilde{\gamma} \in \mathbb{Z}^d$.

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where $n_x := \sum_{i=1}^n a^*_{x,i} a_{x,i}$.

Definition: Translation invariance

A state ω is called T-invariant, iff

$$\omega \circ T_\gamma = \omega \qquad \text{for all } \gamma \in \mathbb{Z}^d$$

An interaction H is called T-invariant, iff

$$\mathcal{L}_H \circ T_\gamma = T_\gamma \circ \mathcal{L}_H \qquad \text{for all } \gamma \in \mathbb{Z}^d$$

Current response of a periodic insulator to a **constant electric field**

Theorem: Wesle, Marcelli, Miyao, Monaco, T. '24 (CMP 2025)

Assume that $H_0, \Phi \in B_\infty$ are periodic and that H_0 has a periodic gapped ground state ω_0

Then for the NEASS ω_ε associated with $H_\varepsilon = H_0 + \varepsilon(\Phi + X_1)$ it holds that

$$\overline{\omega_\varepsilon}(J_{\varepsilon,1}) \;=\; \mathcal{O}(\varepsilon^\infty)$$

and

$$\overline{\omega_\varepsilon}(J_{\varepsilon,2}) \;=\; -\varepsilon \underbrace{\overline{\omega_0}\left(\mathrm{i}\,[X_1^{\mathrm{OD}}, X_2^{\mathrm{OD}}]\right)}_{=:\sigma_{\mathrm{H}}} \;+\; \mathcal{O}(\varepsilon^\infty)\,,$$

where σ_{H} is the Hall conductivity and $\sigma_{\mathrm{H}} = c_{\mathrm{H}}$.

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where σ_{H} is the Hall conductivity and $\sigma_{\mathrm{H}} = c_{\mathrm{H}}$.

Here $J_{\varepsilon,k} := \mathrm{i}[H_\varepsilon, X_k] \in B_\infty$ is the kth component of the current operator and

$$\overline{\omega_\varepsilon}(O) := \lim_{\Lambda \to \mathbb{Z}^2} \frac{1}{|\Lambda|}\, \omega_\varepsilon(O|_\Lambda)$$

denotes the density of an extensive observable.

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where σ_{H} is the Hall conductivity and $\sigma_{\mathrm{H}} = c_{\mathrm{H}}$.

Note that the weakly interacting Hofstadter model satisfies the assumptions of this theorem and the previous one for all magnetic fields.

Sketch of the proof

The proof basically consists in proving all steps in the following computation:

$$\overline{\omega_\varepsilon}([H_0, X_2]) \;=\; -\omega_\varepsilon\left(\mathcal{L}_{X_2}(H_0)_0\right)$$

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&= -\varepsilon\,\overline{\omega_\varepsilon}\left([X_1^{\mathrm{OD}\varepsilon}, X_2^{\mathrm{OD}\varepsilon}]\right) + \mathcal{O}(\varepsilon^\infty)\,.
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This is the double-commutator formula, but for ω_ε instead of ω_0.

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This is the double-commutator formula, but for ω_ε instead of ω_0.

$\Rightarrow$ We need to prove that the double-commutator expression is invariant under locally generated automorphisms of the algebra.

The invariance lemma

Let $\alpha \in \mathrm{Aut}(\mathcal{A})$ be such that $\alpha(\mathcal{A}_\infty) = \mathcal{A}_\infty$ and $H \in B_\infty$. For $A \in \mathcal{A}_\infty$ we define

$$A^{\mathrm{OD}\alpha} := \alpha^{-1}(\alpha(A)^{\mathrm{OD}}),$$

such that for $\omega_\alpha := \omega \circ \alpha$ we have

$$\begin{aligned} \omega_\alpha([A^{\mathrm{OD}\alpha}, B]) &= \omega(\alpha[A^{\mathrm{OD}\alpha}, B]) = \omega(\alpha[\alpha^{-1}(\alpha(A)^{\mathrm{OD}}), B]) \\ &= \omega([\alpha(A)^{\mathrm{OD}}, \alpha(B)]) = \omega([\alpha(A), \alpha(B)]) \\ &= \omega_\alpha([A, B]). \end{aligned}$$

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Invariance lemma

Let $H \in B_\infty$ be periodic with periodic gapped ground state ω and $\alpha_{s,t}$ a cocycle of automorphisms generated by a periodic $S_t \in B^{(0)}_{\infty,I}$.

Then it holds for all $s, t \in I$ that

$$\overline{\omega_{\alpha_{s,t}}}\Big([X_1^{\mathrm{OD}\alpha_{s,t}}, X_2^{\mathrm{OD}\alpha_{s,t}}]\Big) = \overline{\omega_0}\Big([X_1^{\mathrm{OD}}, X_2^{\mathrm{OD}}]\Big).$$

Some corollaries and consequences

- For the "measured" Hall conductance it follows that

$$c_{\mathrm{H}}^{\mathrm{exp}} := \lim_{L\to\infty} \omega_\varepsilon\left(\frac{J_L}{\Delta U_L}\right) = \sigma_{\mathrm{H}} + \mathcal{O}(\varepsilon^\infty)$$

and that it is deterministic

$$\mathrm{var}(c_{\mathrm{H}}^{\mathrm{exp}}) := \lim_{L\to\infty}\left(\omega_\varepsilon\left(\left(\frac{J_L}{\Delta U_L}\right)^2\right) - \omega_\varepsilon\left(\frac{J_L}{\Delta U_L}\right)^2\right) = 0\,.$$

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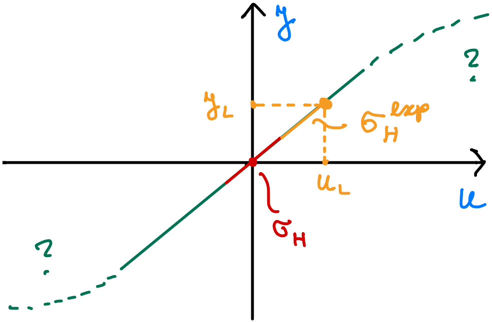
Ohm's law:



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- The invariance lemma also implies that the Hall conductivity (resp. conductance) σ_{H} is constant within gapped phases defined by automorphic equivalence.

Some corollaries and consequences

- For the "measured" Hall conductance it follows that

$$c_{\mathrm{H}}^{\mathrm{exp}} := \lim_{L\to\infty} \omega_\varepsilon\left(\frac{J_L}{\Delta U_L}\right) = \sigma_{\mathrm{H}} + \mathcal{O}(\varepsilon^\infty)$$

and that it is deterministic

$$\mathrm{var}(c_{\mathrm{H}}^{\mathrm{exp}}) := \lim_{L\to\infty}\left(\omega_\varepsilon\left(\left(\frac{J_L}{\Delta U_L}\right)^2\right) - \omega_\varepsilon\left(\frac{J_L}{\Delta U_L}\right)^2\right) = 0\,.$$

- The invariance lemma also implies that the Hall conductivity (resp. conductance) σ_{H} is constant within gapped phases defined by automorphic equivalence.
- The Hall conductivity is the same for any pair of orthogonal directions $a, b \in \mathbb{R}^2$, i.e. one can replace X_1, X_2 by $X_a := a_1X_1 + a_2X_2$, $X_b := b_1X_1 + b_2X_2$ and the current response $\mathrm{i}[H_\varepsilon, X_b]$ to the driving X_a is again nearly linear with the same linear coefficient σ_{H}.

Related results

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  Announced: **Mazzini ,Monaco** '25 also for $B \notin 2\pi\mathbb{Q}$
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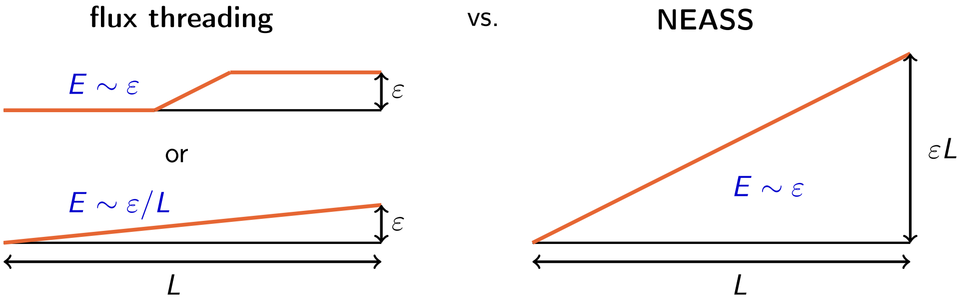

Course overview

1. Physics background, mathematical modelling, and questions we would like to answer
2. Mathematical tools: local approximation of quasi-local operators, the quasi-local inverse of gapped Liouvillians
3. Adiabatic theorem and non-equilibrium almost stationary states (NEASS) as a foundation for understanding (linear) response of gapped systems
4. Nearly linear Hall current response that is constant in gapped phases (microscopic and macroscopic)
5. **Remarks and perspectives: gapped phases, integer quantization, Hall current density in non-periodic systems**

Gapped phases of the weakly-interacting Hofstadter model

For $B, \mu \in \mathbb{R}$ the non-interacting Hofstadter model

$$H_{(B,\mu,0)} := \mathrm{d}\Gamma(\mathfrak{h}_B - \mu\, 1_{\ell^2(\mathbb{Z}^2,\mathbb{C}^n)}) := \sum_{x,y\in\mathbb{Z}^2} \mathfrak{h}_B(x,y)\, a_x^* a_y - \mu \sum_{x\in\mathbb{Z}^2} n_x$$

has a gapped ground state, whenever $\mu \notin \sigma(\mathfrak{h}_B)$. Here

$$\mathfrak{h}_B(x,y) = \mathrm{e}^{\mathrm{i}\frac{x_2+y_2}{2}B(x_1-y_1)}\, \mathfrak{h}_0(x-y)$$

is the kernel of the one-body Hofstadter Hamiltonian.

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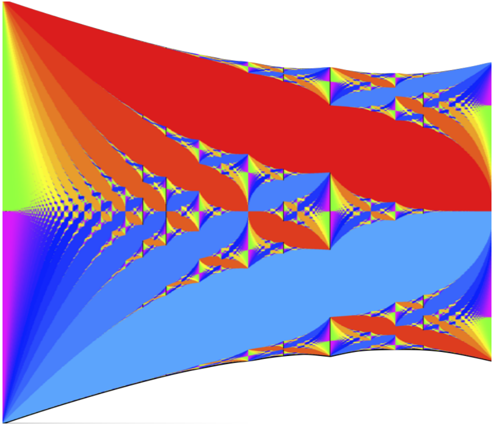
This gap is locally stable (Giuliani, Mastropietro, Porta '17; Hastings '19; De Roeck, Salmhofer '19) , i.e. if $\mu \notin \sigma(\mathfrak{h}_B)$ then for $\lambda \in \mathbb{R}$ small enough and $\Phi \in B_\infty$, also the weakly interacting Hofstadter model

$$H_{(B,\mu,\lambda)} := H_{(B,\mu,0)} + \lambda\Phi$$

has a gapped ground state.

Gapped phases of the weakly-interacting Hofstadter model

The coloured butterfly: gapped phases of the non-interacting Hofstadter model (**Osadchy, Avron arXiv:math-ph/0101019**)



Is also the Hall conductivity stable under weak perturbations?

Gapped phases of the weakly-interacting Hofstadter model

Giuliani, Mastropietro, Porta CMP '17, Giuliani '20 prove that the Hall conductivity does not change when adding a sufficiently small periodic perturbation to a gapped periodic non-interacting system. For the Hofstadter model this result applies, however, only for $B \in 2\pi\mathbb{Q}$.

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Problem: Changes in the direction of B are not given by an automorphism that is generated by a B_∞ interaction, because

$$\begin{aligned} \partial_B H_{(B,\mu,0)}(\{x,y\}) &= \partial_B \left(\mathrm{e}^{\mathrm{i}\frac{x_2+y_2}{2} B(x_1-y_1)}\, \mathfrak{h}_0(x-y)\, a_x^* a_y \right) \\ &= \tfrac{\mathrm{i}}{2} \underbrace{(x_2+y_2)}_{\text{unbounded}} (x_1-y_1)\, \mathrm{e}^{\mathrm{i}\frac{x_2+y_2}{2} B(x_1-y_1)}\, \mathfrak{h}_0(x-y)\, a_x^* a_y \,. \end{aligned}$$

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But one can directly show that $\sigma_{\mathrm{H}}^B = \overline{\omega_B}\left(\mathrm{i}\,[X_1^{\mathrm{OD}B}, X_2^{\mathrm{OD}B}]\right)$ is continuous in B. Integer quantization for rational values then gives integer quantization for all values.

Gapped phases of the weakly-interacting Hofstadter model

Theorem (Wesle, Marcelli, Miyao, Monaco, T., in preparation)

For every $\mu_0, B_0 \in \mathbb{R}$ such that $\mu_0 \notin \sigma(h_{B_0})$ there is $\delta > 0$ such that for all $(\mu, B, \lambda) \in \mathbb{R}^3$ with $\|(\mu, B, \lambda) - (\mu_0, B_0, 0)\| < \delta$

$$\sigma_{\mathrm{H}}(\mu, B, \lambda) = \sigma_{\mathrm{H}}(\mu_0, B_0, 0) \in \frac{1}{2\pi}\mathbb{Z}.$$

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However, for non-interacting systems (or more generally systems with an invertible ground state) σ_{H} is always an integer multiple of $1/2\pi$ (cf. **Kapustin, Sopenko JMP 2020**) and thus also in all gapped phases that contain a non-interacting Hamiltonian.

Macroscopic response without periodicity

Recall that neither the construction of the NEASS nor the result on the microscopic current response required any assumption in addition to the gap assumption. In particular, no periodicity was assumed.

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It turns out yes (almost), basically because one can show that

$$\lim_{\Lambda\to\mathbb{Z}^2} \frac{1}{|\Lambda|}\, \omega_\varepsilon(J_{\varepsilon,2}|_\Lambda) = \lim_{\Lambda\to\mathbb{Z}^2} \frac{1}{|\Lambda|}\, \omega_0\left(\mathrm{i}\,[X_1^{\mathrm{OD}}, X_2^{\mathrm{OD}}]|_\Lambda\right) \;+\; \mathcal{O}(\varepsilon^\infty)$$

and that

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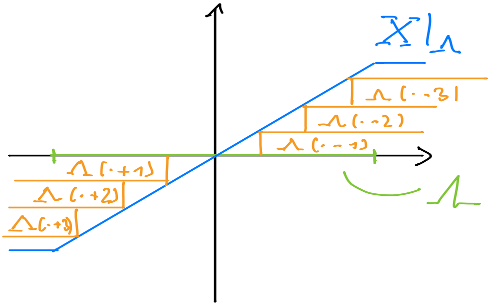

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Theorem (Miyao, Wesle, T., in preparation)

Assume that $H_0 \in B_\infty$ has a gapped ground state ω_0 and let ω_ε be the corresponding NEASS for the perturbation X_1. Then for $i \in \{1, 2\}$

$$\limsup_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \omega_\varepsilon(J_i|_\Lambda) - \liminf_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \omega_\varepsilon(J_i|_\Lambda) = \mathcal{O}(\varepsilon^\infty)$$

and $\overline{\omega_\varepsilon}(J_i) := \frac{1}{2}(\limsup_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \omega_\varepsilon(J_i|_\Lambda) + \liminf_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \omega_\varepsilon(J_i|_\Lambda))$ satisfies

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For non-interacting systems it was already shown in (the appendices of) **Elgart, Graf, Schenker CMP '05** and **Marcelli, Moscolari, Panati AHP '23** that the non-interacting analogue of $\overline{\omega_0}\left([X_1^{\mathrm{OD}}, X_2^{\mathrm{OD}}]\right)$ exists without any periodicity or ergodicity assumptions as long as H is gapped.

Summary and conclusion

Under the sole assumption that H_0 has short-range interactions and a gapped ground state ω_0, we can show that when at zero temperature a constant external field is applied, then

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Thanks for your attention and for the great questions and comments!