Mathematical aspects of quantum Hall physics in microscopic models of interacting fermions

Lecture 2

Stefan Teufel
Fachbereich Mathematik, Universität Tübingen

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Course overview

1. Physics background, mathematical modelling, and questions we would like to answer
2. **Mathematical tools: local approximation of quasi-local operators, the quasi-local inverse of gapped Liouvillians**
3. Adiabatic theorem and non-equilibrium almost stationary states (NEASS) as a foundation for understanding (linear) response of gapped systems
4. Nearly linear Hall current response that is constant in gapped phases (microscopic and macroscopic)
5. Remarks and perspectives: gapped phases, integer quantization, Hall current density in non-periodic systems

Quantum mechanics of infinitely many particles: fermions on a lattice

We consider fermions on $M \subseteq \mathbb{Z}^d$. The N-particle Hilbert space for such a system is

$$\mathcal{H}_N^-(M) := \ell^2(M, \mathbb{C}^n)^{\wedge N},$$

and Fock space is defined by

$$\mathfrak{F}^-(M) := \bigoplus_{N=0}^{\infty} \mathcal{H}_N^-(M) \ni \psi = (\psi_0, \psi_1, \psi_2, \dots) \quad \text{with} \quad \|\psi\|^2 := \sum_{N=0}^{\infty} \|\psi_N\|^2_{\mathcal{H}_N^-(M)}.$$

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We write $\mathfrak{F}^- := \mathfrak{F}^-(\mathbb{Z}^d)$ and note that for any $M \subseteq \mathbb{Z}^d$

$$\mathfrak{F}^- \subset \mathfrak{F}^-(M) \otimes \mathfrak{F}^-(M^c).$$

One defines the algebra $\mathcal{A}_M$ as the C*-sub-algebra of $\mathcal{B}(\mathfrak{F}^-)$ generated by the fermionic creation and annihilation operators $a^*_{x,i}$ and $a_{x,i}$ with $x \in M$ and $i \in \{1, \ldots, n\}$.

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Clearly, for $M_1 \subset M_2$ we have $\mathcal{A}_{M_1} \subset \mathcal{A}_{M_2}$.

Quantum mechanics of infinitely many particles: fermions on a lattice

The algebra of local observables is defined as

$$\mathcal{A}_{\mathrm{loc}} := \left\{ A \in \mathcal{B}(\mathfrak{F}^-) \,\middle|\, A \in \mathcal{A}_M \text{ for some } M \text{ with } |M| < \infty \right\} ,$$

and the quasi-local algebra or CAR algebra is the unital C^*-algebra

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$$g_\varphi(a^*_{x,i}) = \mathrm{e}^{i\varphi}\, a^*_{x,i}, \quad \text{for all } \ x \in \mathbb{Z}^d, \ i \in \{1, \ldots, n\} \ .$$

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One defines the set of even quasi-local operators

$$\mathcal{A}^+ := \{A \in \mathcal{A} \mid g_\pi(A) = A\} , \qquad \mathcal{A}^+_M := \mathcal{A}^+ \cap \mathcal{A}_M$$

and the set of gauge-invariant quasi-local operators

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For disjoint regions $M_1, M_2 \subseteq \mathbb{Z}^d$, $M_1 \cap M_2 = \emptyset$, operators $A \in \mathcal{A}^+_{M_1}$ and $B \in \mathcal{A}_{M_2}$ commute,

$$[A, B] = 0 \,.$$

Local approximations of quasi-local operators

The CAR-algebra $\mathcal{A}$ has a unique state ω^{tr} that satisfies $\omega^{\mathrm{tr}}(AB) = \omega^{\mathrm{tr}}(BA)$ for all $A, B \in \mathcal{A}$, called the tracial state.

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For $A \in \mathcal{A}_M$ with $|M| < \infty$ just define

$$\omega^{\mathrm{tr}}(A) := \frac{1}{n} \operatorname{tr}_{\mathcal{F}^-(M)}(A)$$

and extend by continuity to $\mathcal{A}$.

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One can now define for any region $M \subset \mathbb{Z}^d$ and $A \in \mathcal{A}$ the "partial trace" of A over the complement M^c of M

$$\mathbb{E}_M(A) \text{ ``}:= \mathrm{tr}_{\mathcal{F}^-(M^c)}(A)\text{''} \;\in\; \mathcal{A}_M\,.$$

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Proposition (Araki, Moriya RMP 2003): For each $M \subseteq \mathbb{Z}^d$ there exists a unique linear map $\mathbb{E}_M : \mathcal{A} \to \mathcal{A}_M$, called the conditional expectation with respect to ω^{tr}, such that

$$\forall A \in \mathcal{A} \;\; \forall B \in \mathcal{A}_M \; : \quad \omega^{\mathrm{tr}}(AB) = \omega^{\mathrm{tr}}(\mathbb{E}_M(A)B)\,.$$

It is unital, positive and has the properties

$$\forall M \subseteq \mathbb{Z}^d \;\; \forall A, C \in \mathcal{A}_M \;\; \forall B \in \mathcal{A} : \qquad \mathbb{E}_M(A\,B\,C) = A\,\mathbb{E}_M(B)\,C$$

$$\forall M_1, M_2 \subseteq \mathbb{Z}^d : \qquad \mathbb{E}_{M_1} \circ \mathbb{E}_{M_2} = \mathbb{E}_{M_1 \cap M_2}$$

$$\forall M \subseteq \mathbb{Z}^d : \qquad \mathbb{E}_M \mathcal{A}^+ \subseteq \mathcal{A}^+ \quad \text{and} \quad \mathbb{E}_M \mathcal{A}^{\mathcal{N}} \subseteq \mathcal{A}^{\mathcal{N}}$$

$$\forall M \subseteq \mathbb{Z}^d \;\; \forall A \subseteq \mathcal{A} : \qquad \|\mathbb{E}_M(A)\| \leq \|A\|\,.$$

Local approximations of quasi-local operators

Definition: For $\nu \in \mathbb{N}_0$, $x \in \mathbb{Z}^d$ and $A \in \mathcal{A}$ let

$$\|A\|_{\nu,x} := \|A\| + \sup_{k \in \mathbb{N}_0} \|A - \mathbb{E}_{B_k(x)}(A)\| \, (1+k)^{\nu} \,,$$

where $B_k(x) := \{y \in \mathbb{Z}^d \mid \|x - y\| \leq k\}$ is the box with side-length $2k$ around x with respect to the maximum norm on $\mathbb{Z}^d$. We denote the Fréchet space of all $A \in \mathcal{A}$ with $\|A\|_{\nu,x} < \infty$ for some (and therefore all) $x \in \mathbb{Z}^d$ and all $\nu \in \mathbb{N}_0$ by $\mathcal{A}_\infty$.

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"Operators localized in different regions have small commutators":

Lemma: Let $A, B \in \mathcal{A}_\infty$, such that either A or B is even, $\nu, m \in \mathbb{N}_0$ and $x, y \in \mathbb{Z}^d$. It holds that

$$\|[A,B]\|_{\nu,x} \leq 4^{\nu+m+3} \frac{\|A\|_{\nu+m,y} \, \|B\|_{\nu+m,x}}{(1 + \|x-y\|)^m} \,.$$

Interactions

An interaction is a map $\Phi : P_0(\mathbb{Z}^d) \to \mathcal{A}^{\mathcal{N}}$, such that $\Phi(\emptyset) = 0$ and for all $M \in P_0(\mathbb{Z}^d)$ it holds that $\Phi(M) \in \mathcal{A}_M$, $\Phi(M)^* = \Phi(M)$, and the following sum converges unconditionally:

$$\sum_{\substack{K \in P_0(\mathbb{Z}^d) \\ M \cap K \neq \emptyset}} \Phi(K) \, .$$

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Definition: For an interaction Φ and $\nu \in \mathbb{N}_0$ let

$$\|\Phi\|_\nu := \sup_{x \in \mathbb{Z}^d} \sum_{\substack{M \in P_0(\mathbb{Z}^d) \\ x \in M}} (1 + \mathrm{diam}(M))^\nu \|\Phi(M)\|\ .$$

The Fréchet space of interactions with finite $\|\cdot\|_\nu$ for all $\nu \in \mathbb{N}_0$ is denoted by B_∞.

Interactions, 0-chains, and Lipschitz potentials

We define $R_x \subseteq P_0(\mathbb{Z}^d)$ to be set of all finite subsets of $\mathbb{Z}^d$ that have their center in x. Given an interaction Φ define the interaction at x as

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Then for any interaction Φ and lattice point x the quasi local observable $\Phi_x \in \mathcal{A}$ is well-defined and $\|\Phi_x\|_{\nu,x} \leq 3 \, \|\Phi\|_\nu$ for all $\nu \in \mathbb{N}_0$.

Such a map $\mathbb{Z}^d \to \mathcal{A}$, $x \mapsto \Phi_x$ is also called a 0-chain.

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A Lipschitz potential V is an on-site interaction, i.e. only supported on one-element sets, such that for all $x \in \mathbb{Z}^d$ it holds that $V(\{x\}) = v(x)\, n_x$, where $v : \mathbb{Z}^d \to \mathbb{R}$ is a function that satisfies

$$\exists\, C_v \in \mathbb{R} \quad \forall x, y \in \mathbb{Z}^d \;:\quad |v(x) - v(y)| \leq C_v\, \|x - y\|$$

and $n_x := \sum_{i=1}^n a_{x,i}^* a_{x,i}$. We denote the set of all Lipschitz potentials by $\mathcal{V}$.

Commutators with interactions, 0-chains, and Lipschitz potentials

Lemma: Let $\Phi \in B_\infty$ and $V \in \mathcal{V}$ be a Lipschitz potential. Then $\mathcal{A}_\infty \subseteq D(\mathcal{L}_{\Phi+V})$:

For all $A \in \mathcal{A}_\infty$ the sums

$$\sum_{M \in P_0(\mathbb{Z}^d)} [\Phi(M) + V(M), A] \quad \text{and} \quad \sum_{x \in \mathbb{Z}^d} [\Phi_x + V_x, A]$$

converge absolutely and

$$\mathcal{L}_{\Phi+V} A := \sum_{M \in P_0(\mathbb{Z}^d)} [\Phi(M) + V(M), A] = \sum_{x \in \mathbb{Z}^d} [\Phi_x + V_x, A].$$

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The commutator with a B_∞-interaction or Lipschitz potential is a continuous map on $\mathcal{A}_\infty^{\mathcal{N}}$:
For each $\nu \in \mathbb{N}_0$ there is a constant c_ν, independent of Φ and V, such that for all $x \in \mathbb{Z}^d$ and $A \in \mathcal{A}^{\mathcal{N}}$

$$\|\mathcal{L}_{\Phi+V} A\|_{\nu,x} \leq c_\nu \left(\|\Phi\|_{\nu+d+1} \|A\|_{\nu+d+1,x} + C_V \|A\|_{\nu+d+2,x} \right).$$

Dynamics: cocycles of automorphisms generated by interactions

Let $I \subseteq \mathbb{R}$ be an interval. A family $(\alpha_{u,v})_{(u,v)\in I^2}$ of automorphisms of $\mathcal{A}$ is called a cocycle if it satisfies

$$\forall t, u, v \in I, \quad \alpha_{t,u}\, \alpha_{u,v} = \alpha_{t,v}\,.$$

We say the cocycle is generated by the family of interactions $(\Phi_v)_{v\in I}$, if for all $A \in \mathcal{A}_0$ it holds that

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We call $(\alpha_{u,v})_{(u,v)\in I^2}$ locally generated if it is generated by some family of interactions in

$$B^{(0)}_{\infty,I} := C(I, B_\infty)\,.$$

Dynamics: cocycles of automorphisms generated by interactions

Lemma: Let $I \subseteq \mathbb{R}$ be an interval and $(\Phi_v)_{v \in I}$ a family of interactions in $B^{(0)}_{\infty,I}$ and $(V_v)_{v \in I}$ a family of interactions in $\mathcal{V}^{(0)}_I$.
Then there exists a unique cocycle of automorphisms $(\alpha_{u,v})_{(u,v) \in I^2}$ generated by $(\Phi_v + V_v)_{v \in I}$.

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Then there exists a unique cocycle of automorphisms $(\alpha_{u,v})_{(u,v)\in I^2}$ generated by $(\Phi_v + V_v)_{v\in I}$.

Moreover, for all $\nu \in \mathbb{N}_0$ there exists an increasing function $b_\nu : \mathbb{R} \to \mathbb{R}$, that grows at most polynomially, such that

$$\|\alpha_{u,v} A\|_{\nu,x} \leq b_\nu(|v-u|)\|A\|_{\nu,x} \quad \text{for all } x \in \mathbb{Z}^d,\ A \in \mathcal{A}_\infty,\ u,v \in I\,.$$

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For all $C > 0$ the function b_ν can be chosen uniformly for all $(\Phi_v)_{v\in I}$ with $\sup_{s\in I}\|\Phi_v\|_{4\nu+9d+4} < C$.

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This statement is a consequence of the finite-volume version of the following Lieb-Robinson bound.

Dynamics: cocycles of automorphisms generated by interactions

Theorem (Lieb-Robinson bound): Let $(\Phi_v)_{v\in I}$ be a family of interactions in $B^{(0)}_{\infty,I}$ and $(V_v)_{v\in I}$ a family of interactions in $\mathcal{V}^{(0)}_I$ and let $(\alpha_{u,v})_{(u,v)\in I^2}$ be the family of automorphisms generated by $(\Phi_v + V_v)_{v\in I}$.

Then for each $\nu \in \mathbb{N}_0$ there exists a constant $c_\nu > 0$ and an increasing and at most polynomially growing function $f_\nu\colon \mathbb{R}_+ \to \mathbb{R}_+$, such that for all finite disjoint sets $X, Y \subseteq \mathbb{Z}^d$ and all $A \in \mathcal{A}^+_X$, $B \in \mathcal{A}_Y$, it holds that

$$\|[\alpha_{u,v} A, B]\| \leq \|A\|\,\|B\|\,|Y|\,\frac{f_\nu(|v-u|)}{(1+\max(0,\mathrm{dist}(X,Y)^{1/2} - c_\nu\,|v-u|))^\nu}\,.$$

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Furthermore the dependence on $(\Phi_v)_{v\in I}$ of f_ν and c_ν is such that for each $C > 0$ they can be chosen uniformly for all $(\Phi_v)_{v\in I}$ with $\sup_{v\in I} \|\Phi_v\|_{2\nu+d} < C$.

Motivation: Perturbation theory for isolated eigenvalues

Let H_0, Φ be self-adjoint operators on a finite-dimensional Hilbert space $\mathcal{H}$ and $E_0 \in \mathbb{R}$ a simple eigenvalue with spectral projection P_0, i.e., $H_0 P_0 = E_0 P_0$.

Then there exists $\varepsilon_0 > 0$ and analytic functions $\varepsilon \mapsto E_\varepsilon$ and $\varepsilon \mapsto P_\varepsilon$ defined on $(-\varepsilon_0, \varepsilon_0)$ such that E_ε is a simple eigenvalue for $H_\varepsilon := H_0 + \varepsilon\Phi$ and

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One can determine P_ε perturbatively using the ansatz

$$\widetilde{P}_\varepsilon := \mathrm{e}^{-\mathrm{i}S^\varepsilon} P_0 \mathrm{e}^{\mathrm{i}S^\varepsilon} = \mathrm{e}^{-\mathrm{i}\mathcal{L}_{S^\varepsilon}} P_0 \qquad \text{with} \qquad S^\varepsilon = \sum_{i=1}^{\infty} \varepsilon^i K_i \qquad \text{and all } K_i \text{ self-adjoint}\,.$$

This ansatz guarantees that $\|\widetilde{P}_\varepsilon - P_0\| = \mathcal{O}(\varepsilon)$ and thus $\widetilde{P}_\varepsilon = P_\varepsilon$ iff $[H_\varepsilon, \widetilde{P}_\varepsilon] = 0\,.$

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$$\widetilde{P}_\varepsilon := \mathrm{e}^{-\mathrm{i}S^\varepsilon} P_0 \mathrm{e}^{\mathrm{i}S^\varepsilon} = \mathrm{e}^{-\mathrm{i}\mathcal{L}_{S^\varepsilon}} P_0 \qquad \text{with} \qquad S^\varepsilon = \sum_{i=1}^{\infty} \varepsilon^i K_i \qquad \text{and all } K_i \text{ self-adjoint}\,.$$

This ansatz guarantees that $\|\widetilde{P}_\varepsilon - P_0\| = \mathcal{O}(\varepsilon)$ and thus $\widetilde{P}_\varepsilon = P_\varepsilon$ iff $[H_\varepsilon, \widetilde{P}_\varepsilon] = 0\,.$

This condition can now be used to determine S^ε order by order:

$$0 = [H_\varepsilon, \mathrm{e}^{-\mathrm{i}\mathcal{L}_{S^\varepsilon}} P_0] = \varepsilon([\Phi, P_0] - \mathrm{i}[H_0, \mathcal{L}_{K_1} P_0]) + \mathcal{O}(\varepsilon^2) = \varepsilon(\mathcal{L}_\Phi P_0 - \mathrm{i}\mathcal{L}_{H_0}\mathcal{L}_{K_1} P_0) + \mathcal{O}(\varepsilon^2)$$

Motivation: Perturbation theory for isolated eigenvalues

Let H_0, Φ be self-adjoint operators on a finite-dimensional Hilbert space $\mathcal{H}$ and $E_0 \in \mathbb{R}$ a simple eigenvalue with spectral projection P_0, i.e., $H_0 P_0 = E_0 P_0$.

Then there exists $\varepsilon_0 > 0$ and analytic functions $\varepsilon \mapsto E_\varepsilon$ and $\varepsilon \mapsto P_\varepsilon$ defined on $(-\varepsilon_0, \varepsilon_0)$ such that E_ε is a simple eigenvalue for $H_\varepsilon := H_0 + \varepsilon\Phi$ and

$$H_\varepsilon P_\varepsilon = E_\varepsilon P_\varepsilon \,.$$

One can determine P_ε perturbatively using the ansatz

$$\widetilde{P}_\varepsilon := \mathrm{e}^{-\mathrm{i}S^\varepsilon} P_0 \mathrm{e}^{\mathrm{i}S^\varepsilon} = \mathrm{e}^{-\mathrm{i}\mathcal{L}_{S^\varepsilon}} P_0 \qquad \text{with} \qquad S^\varepsilon = \sum_{i=1}^\infty \varepsilon^i K_i \qquad \text{and all } K_i \text{ self-adjoint}\,.$$

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$$\Rightarrow \quad \mathrm{i}\mathcal{L}_{H_0}\mathcal{L}_{K_1} P_0 = \mathcal{L}_\Phi P_0 \quad \Rightarrow \quad \mathcal{L}_{K_1} P_0 = -\mathrm{i}\mathcal{L}_{H_0}^{-1}\mathcal{L}_\Phi P_0$$

Motivation: Inverting the Liouvillian

The inner product

$$\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \to \mathbb{C}\,, \quad (A, B) \mapsto \langle A, B \rangle := \mathrm{tr}(A^* B)\,,$$

turns $\mathcal{B}(\mathcal{H})$ into a Hilbert space and the Liouvillian

$$\mathcal{L}_{H_0} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\,, \quad A \mapsto \mathcal{L}_{H_0} A := [H_0, A]$$

is self-adjoint,

$$\begin{aligned} \langle \mathcal{L}_{H_0} A, B \rangle &= \mathrm{tr}([H_0, A]^* B) \;=\; \mathrm{tr}((A^* H_0 - H_0 A^*) B) \\ &= \mathrm{tr}(A^* H_0 B - A^* B H_0) \;=\; \mathrm{tr}(A^* [H_0, B]) \;=\; \langle A, \mathcal{L}_{H_0} B \rangle\,. \end{aligned}$$

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Thus, $\mathrm{ran}(\mathcal{L}_{H_0}) = \ker(\mathcal{L}_{H_0})^\perp$ and

$$\mathcal{L}_{H_0} : \ker(\mathcal{L}_{H_0})^\perp \to \ker(\mathcal{L}_{H_0})^\perp$$

is invertible.

Motivation: Diagonal and off-diagonal operators

Given the spectral projection P_0 of H_0, we can split $\mathcal{B}(\mathcal{H})$ into diagonal and off-diagonal parts w.r.t. P_0, $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H})^{\mathrm{D}} \oplus \mathcal{B}(\mathcal{H})^{\mathrm{OD}}$, where any operator A is split according to

$$A = \underbrace{P_0 A P_0 + P_0^{\perp} A P_0^{\perp}}_{=: A^{\mathrm{D}}} + \underbrace{P_0 A P_0^{\perp} + P_0^{\perp} A P_0}_{=: A^{\mathrm{OD}}} .$$

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Then $\ker \mathcal{L}_{H_0} \subset \mathcal{B}(\mathcal{H})^{\mathrm{D}}$, since $[H_0, A] = 0$ implies also $[P_0, A] = 0$, and thus $\mathcal{B}(\mathcal{H})^{\mathrm{OD}} \subset \ker(\mathcal{L}_{H_0})^{\perp}$.

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$$\mathcal{L}_{H_0} : \mathcal{B}(\mathcal{H})^{\mathrm{OD}} \to \mathcal{B}(\mathcal{H})^{\mathrm{OD}}$$

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More precisely,

$$\mathcal{L}_{H_0}^{-1}\big|_{\mathcal{B}(\mathcal{H})^{\mathrm{OD}}} = \mathcal{L}_{R_0}\big|_{\mathcal{B}(\mathcal{H})^{\mathrm{OD}}}$$

with

$$\|R_0\| = \|(H_0 - E_0)^{-1} P_0^\perp\| = \mathrm{dist}(E_0, \sigma(H_0) \setminus \{E_0\})^{-1} = \mathrm{gap}^{-1}.$$

Motivation: Back to perturbation theory

By defining

$$\mathcal{L}_{K_1} := -\mathrm{i}\mathcal{L}_{R_0}\mathcal{L}_{\Phi}$$

we thus obtain

$$\mathcal{L}_{K_1} P_0 = -\mathrm{i}\mathcal{L}_{R_0} \underbrace{\mathcal{L}_{\Phi} P_0}_{=[\Phi, P_0] \in \mathcal{B}(\mathcal{H})^{\mathrm{OD}}} = -\mathrm{i}\mathcal{L}_{H_0}^{-1}\mathcal{L}_{\Phi} P_0 \,,$$

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Problem: How to invert the densely defined derivation $\mathcal{L}_H$ on the quasi-local algebra $\mathcal{A}$ given by a local interaction H with a gapped ground state ω_0 ?

Gapped ground states and the GNS representation

Theorem (GNS representation): For any state ω on $\mathcal{A}$ there exists a $*$-representation π_ω on $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\omega$ with a distinguished unit cyclic vector Ω_ω such that

$$\forall A \in \mathcal{A} : \quad \omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle_{\mathcal{H}_\omega} .$$

One calls $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ a corresponding GNS triple. It is unique up to unitary equivalence.

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Recall that for a densely defined derivation $\mathcal{L}_H$ on $\mathcal{A}$ a state $\omega \in \mathcal{A}^*$ is a (locally unique) gapped ground state, iff there exists $g > 0$ such that

$$\omega(A^*\mathcal{L}_H A) \geq g \left(\omega(A^*A) - |\omega(A)|^2\right) \quad \text{for all } A \in D(\mathcal{L}_H) .$$

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Moreover, let α_t^H be the automorphism group generated by $\mathcal{L}_H$, then

$$\forall A \in \mathcal{A} \,, t \in \mathbb{R} \ : \quad \pi_\omega(\alpha_t^H A) = \mathrm{e}^{\mathrm{i} H_\omega t} \, \pi_\omega(A) \, \mathrm{e}^{-\mathrm{i} H_\omega t} \,.$$

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No, the GNS representations for ω_0 and ω_ε are, in general, only unitarily equivalent if the perutrbation Φ is quasi-local, i.e. if $\Phi \in \mathcal{A}$.

The quasi-local inverse of the Liouvillian

The key ingredient for the following constructions is a version of $\mathcal{L}_{R_0}$ defined by **Hastings JSM 2007** and **Bachmann, Michalakis, Nachtergaele, Sims CMP 2011** in finite volume and by **Moon, Ogata JFA 2020** in infinite volume.

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For $g > 0$ let $w_g \in L^1(\mathbb{R})$ be **non-negative, normalized, and even** such that

(i) the Fourier transform $\widehat{w}_g$ of w_g is supported in $[-g, g]$,

(ii) $\sup_{s\in\mathbb{R}} |s|^n \, w_g(s) < \infty$ for all $n \in \mathbb{N}_0$.

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For $H \in B_\infty$ and $A \in \mathcal{A}_\infty$ we define

$$\mathcal{I}_H(A) := \int_{\mathbb{R}} \mathrm{d}t \, w_g(t) \int_0^t \mathrm{d}s \, \mathrm{e}^{\mathrm{i}s\mathcal{L}_H} \, A \,,$$

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$$A^{\mathrm{OD}} := -\mathrm{i}\, \mathcal{L}_H \, \mathcal{I}_H(A) \,, \qquad \text{and} \qquad A^{\mathrm{D}} := A - A^{\mathrm{OD}} \,.$$

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Proposition: Let $H \in B_\infty$ and $g > 0$. Then the maps

$$\mathcal{I}_H : \mathcal{A}_\infty \to \mathcal{A}_\infty\,, \quad (\cdot)^{\mathrm{OD}} : \mathcal{A}_\infty \to \mathcal{A}_\infty \quad \text{and} \quad (\cdot)^{\mathrm{D}} : \mathcal{A}_\infty \to \mathcal{A}_\infty$$

are well-defined, linear and continuous.

The quasi-local inverse of the Liouvillian

Theorem: Let $H \in B_\infty$ and let ω be a gapped ground state of $\mathcal{L}_H$ with gap $g > 0$. Then for all $A \in \mathcal{A}_\infty$

$$\omega(A^{\mathrm{OD}}) = 0 \quad \text{and equivalently} \quad \omega(A) = \omega(A^{\mathrm{D}})$$

and for all $B \in \mathcal{A}$

$$\omega(A^{\mathrm{D}} B) = \omega(A)\omega(B) \quad \text{and thus} \quad \omega([A, B]) = \omega([A^{\mathrm{OD}}, B])\,. \qquad (*)$$

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Recall that we want to solve the analogue of

$$\begin{aligned}
\mathrm{i}\mathcal{L}_{H_0}\mathcal{L}_{K_1} P_0 = \mathcal{L}_\Phi P_0 \quad &\Leftrightarrow \quad \forall A \,:\, \mathrm{i}\,\mathrm{tr}(A\,\mathcal{L}_{H_0}\mathcal{L}_{K_1} P_0) = \mathrm{tr}(A\,\mathcal{L}_\Phi P_0) \\
&\Leftrightarrow \quad \forall A \,:\, \mathrm{i}\,\mathrm{tr}(P_0 \mathcal{L}_{K_1}\mathcal{L}_{H_0} A) = -\,\mathrm{tr}(P_0\,\mathcal{L}_\Phi A) \\
&\Leftrightarrow \quad \forall A \,:\, \omega_{P_0}(\mathcal{L}_{K_1}\mathcal{L}_{H_0} A) = \mathrm{i}\,\omega_{P_0}([\Phi, A]) \\
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&\Leftrightarrow \quad \forall A : \ \omega_{P_0}([\mathcal{L}_{H_0} K_1, A]) = \mathrm{i}\,\omega_{P_0}([\Phi, A])
\end{aligned}$$

Picking $K_1 = \mathcal{I}_{H_0}(\Phi)$ we find

$$\omega_{P_0}([\mathcal{L}_{H_0}\mathcal{I}_{H_0}(\Phi), A]) \ = \ \mathrm{i}\,\omega_{P_0}([\Phi^{\mathrm{OD}}, A]) \ \overset{(*)}{=} \ \mathrm{i}\,\omega_{P_0}([\Phi, A])\,.$$

The quasi-local inverse of the Liouvillian

Proof: Since $\omega \circ \mathcal{L}_H = 0$ (as we saw in the GNS representation) it follows immediately that

$$\omega(A^{\mathrm{OD}}) = -\mathrm{i}\,\omega(\mathcal{L}_H\,\mathcal{I}_H(A)) = 0\,.$$

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$$\omega(A^{\mathrm{OD}}) = -\mathrm{i}\,\omega(\mathcal{L}_H \mathcal{I}_H(A)) = 0\,.$$

In the GNS representation we obtain

$$\mathrm{i}\,\omega((\mathrm{e}^{\mathrm{i}s\mathcal{L}_H}\mathcal{L}_H\,A)B) = \tfrac{\mathrm{d}}{\mathrm{d}s}\omega((\mathrm{e}^{\mathrm{i}s\mathcal{L}_H}\,A)B) = \tfrac{\mathrm{d}}{\mathrm{d}s}\langle \Omega_\omega, \mathrm{e}^{\mathrm{i}sH_\omega}\,\pi_\omega(A)\,\mathrm{e}^{-\mathrm{i}sH_\omega}, \pi_\omega(B)\,\Omega_\omega\rangle$$

and thus, using that $\widehat{w}(H_\omega) = |\Omega_\omega\rangle\langle\Omega_\omega|$, that

$$\begin{aligned}
\omega(A^{\mathrm{OD}}B) &= -\mathrm{i}\int_{\mathbb{R}} \mathrm{d}t\, w(t) \int_0^t \mathrm{d}s\, \omega((\mathrm{e}^{\mathrm{i}s\mathcal{L}_H}\mathcal{L}_H\,A)B) \\
&= -\int_{\mathbb{R}} \mathrm{d}t\, w(t) \left(\langle \Omega, \mathrm{e}^{\mathrm{i}tH}\,A\,\mathrm{e}^{-\mathrm{i}tH} B\Omega\rangle - \langle \Omega, AB\Omega\rangle\right) \\
&= \langle \Omega, AB\Omega\rangle - \langle \Omega, A\,\widehat{w}(H)\,B\Omega\rangle \\
&= \langle \Omega, AB\Omega\rangle - \langle \Omega, A\Omega\rangle\langle \Omega, B\Omega\rangle \;=\; \omega(AB) - \omega(A)\omega(B)\,,
\end{aligned}$$

where, for better readability, we dropped the representation π in the notation. □

Course overview

1. Physics background, mathematical modelling, and questions we would like to answer
2. **Mathematical tools: local approximation of quasi-local operators, the quasi-local inverse of gapped Liouvillians**
3. Adiabatic theorem and non-equilibrium almost stationary states (NEASS) as a foundation for understanding (linear) response of gapped systems
4. Nearly linear Hall current response that is constant in gapped phases (microscopic and macroscopic)
5. Remarks and perspectives: gapped phases, integer quantization, Hall current density in non-periodic systems