Mathematical aspects of quantum Hall physics in microscopic models of interacting fermions

Lecture 1

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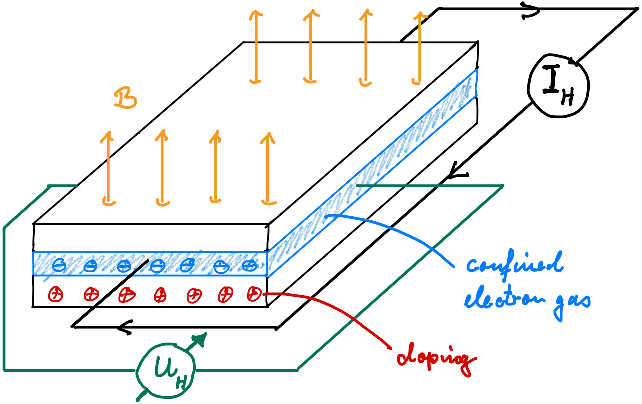
Quantissima sur Oise – Cergy 2025

LONG-RANGE INTERACTING
QUANTUM
SPIN SYSTEMS
OUT OF EQUILIBRIUM
EXPERIMENT, THEORY AND MATHEMATICS

CRC
TRR
352

Quantum Hall effect: current response of gapped systems



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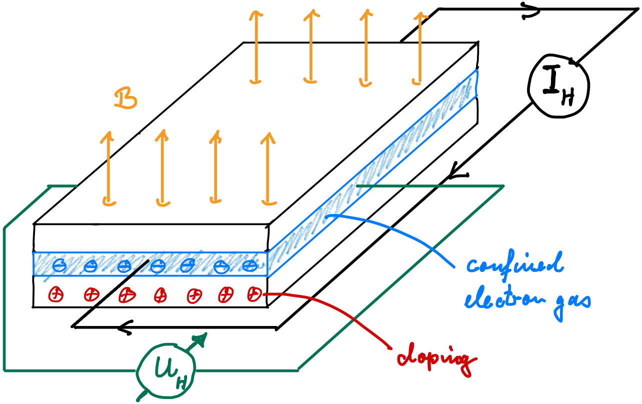

Measured Hall resistance

$$\rho_{xy}^{\text{exp}} = \frac{U_{\text{H}}}{I_{\text{H}}}$$

and Hall conductance

$$c_{\text{H}}^{\text{exp}} = \frac{I_{\text{H}}}{U_{\text{H}}} \, .$$

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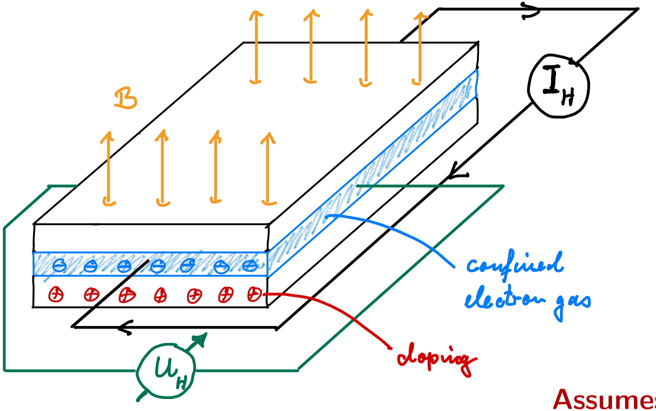

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Assumes Ohm's law for Hall currents!

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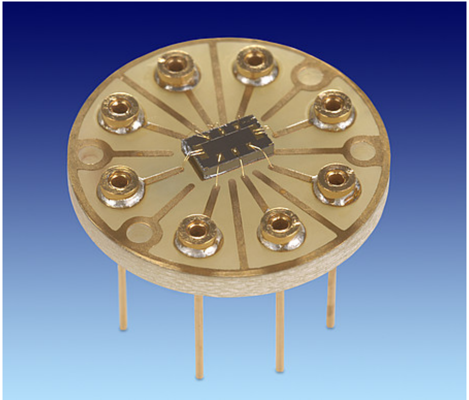

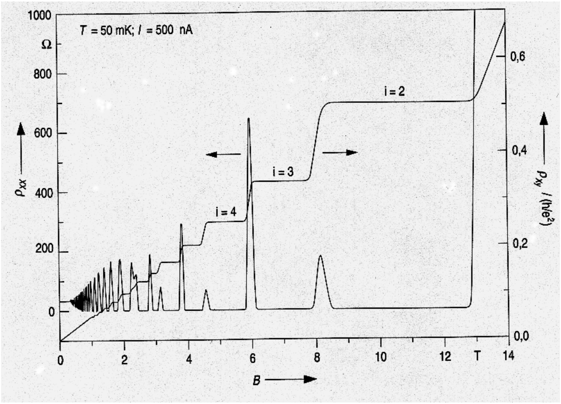

Linear dimensions up to 10 mm

Hall voltages up to 5 V

Hall currents up to 200 μA

Typical gap: 10 – 100 meV

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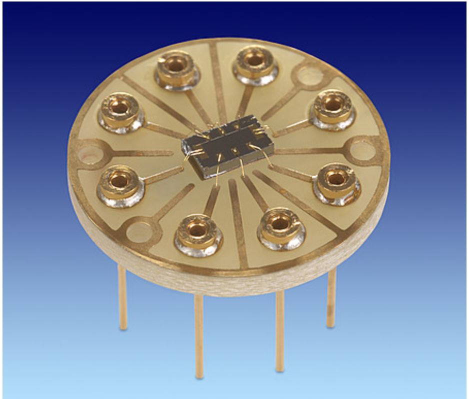
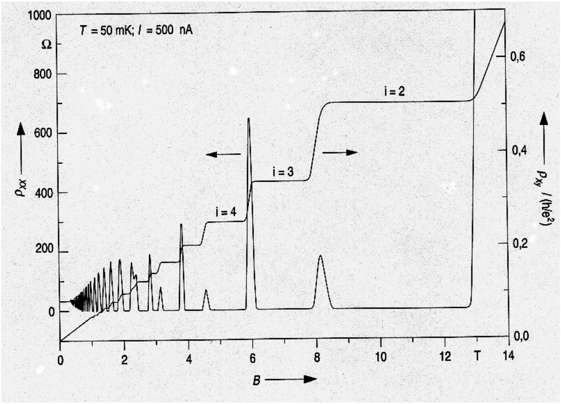


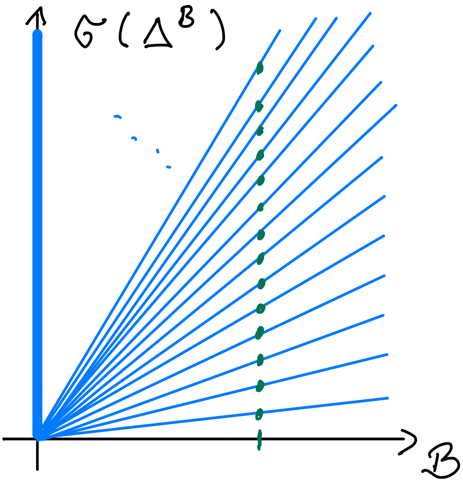


Best experiments: $\mathrm{dist}(c_{\mathrm{H}}, \frac{e^2}{h}\mathbb{Z})/(\frac{e^2}{h}) \leq 10^{-10}$

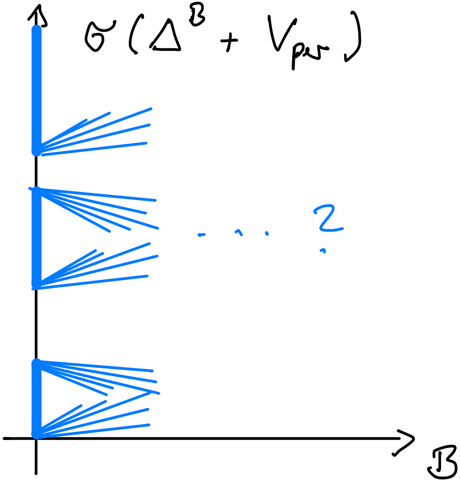
$\Rightarrow$ QHE experiments are now used as a standard for resistance measurements.

Magnetic field and spectrum: Landau levels in **non-interacting** systems

Landau levels



Bloch-Landau levels



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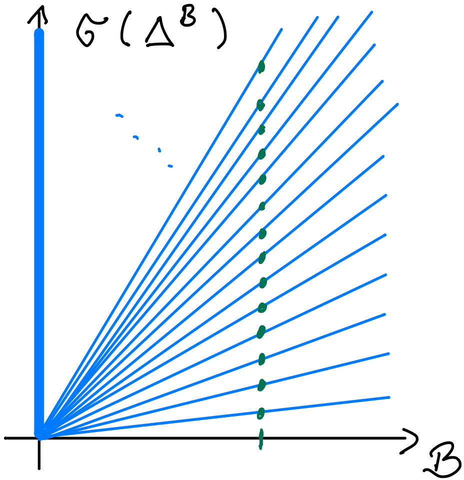

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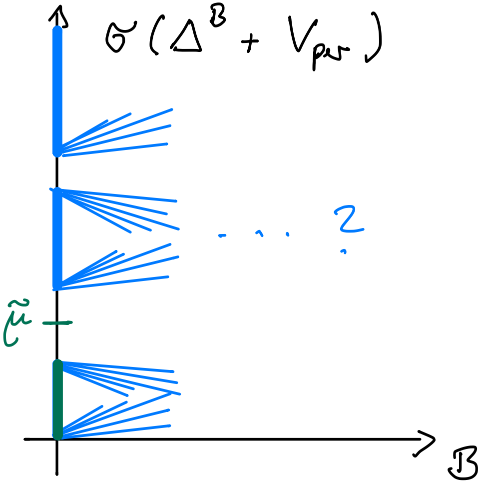

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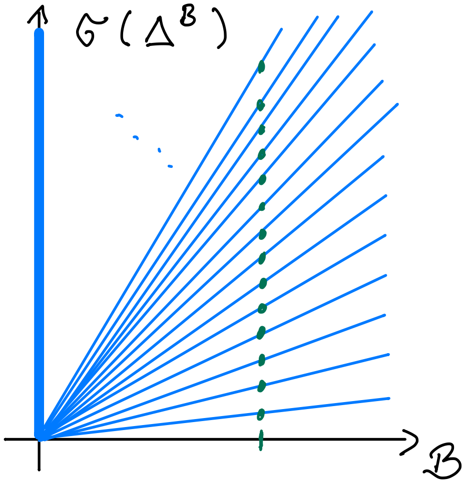

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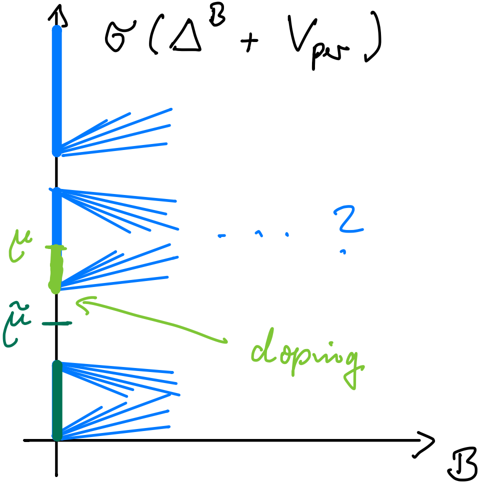

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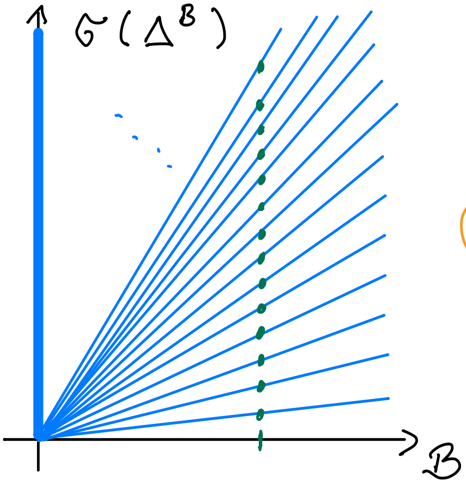

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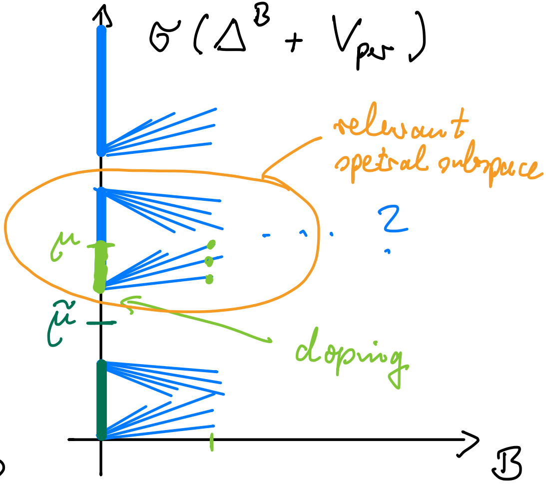
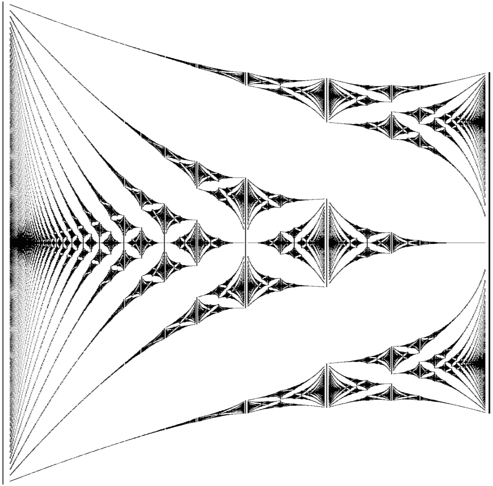


Spectrum of the Hofstadter Hamiltonian and gapped phases



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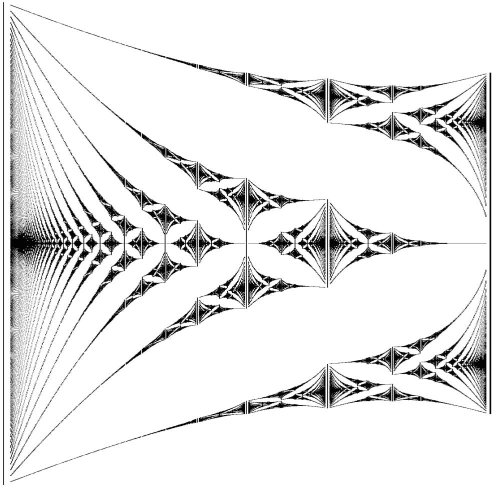
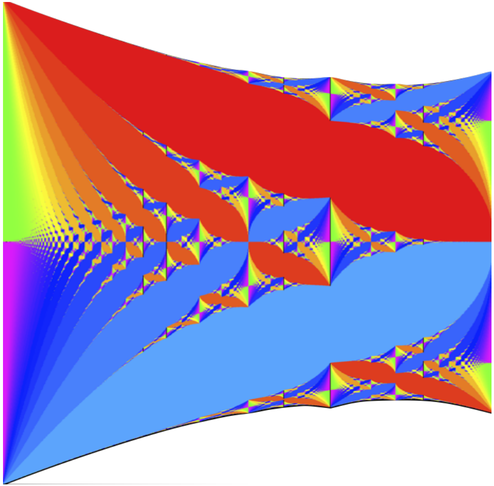
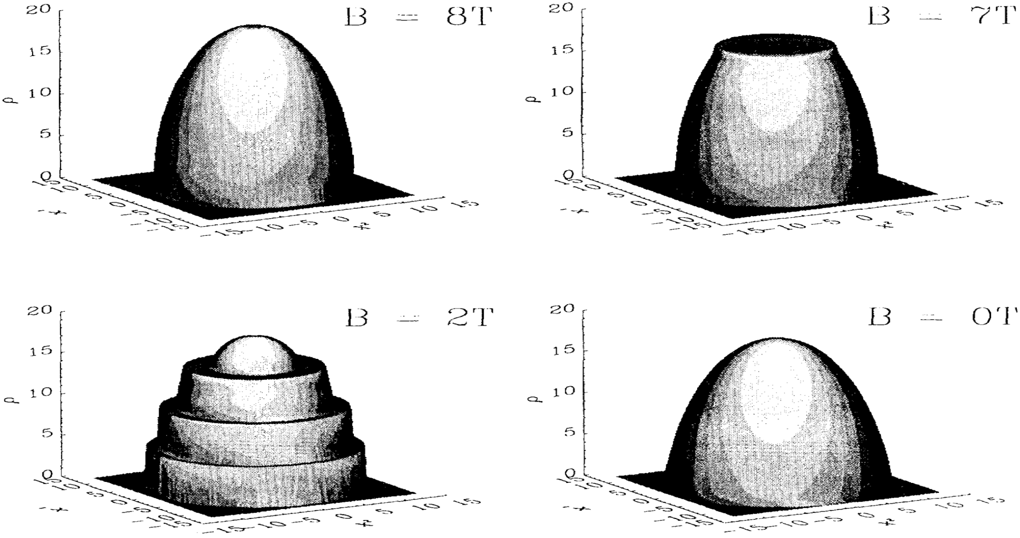


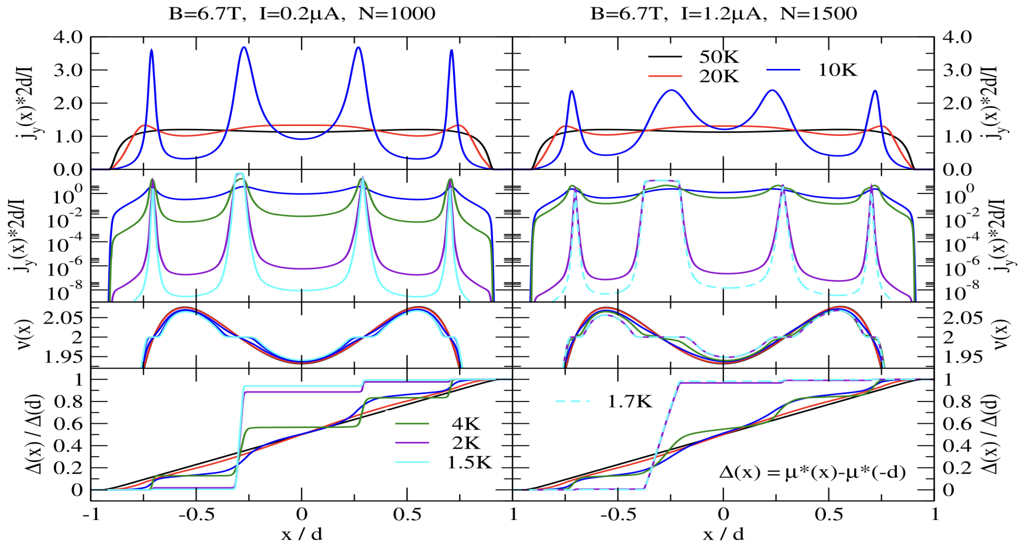


Screening and electron density in strong magnetic fields



Lieb, Solovej, Yngvason, Phys. Rev. B 1994

Electron density and Hall currents in (narrow) Hall bars



R. Gerhardts, New Journal of Physics 2019.

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Why infinitely extended?

- One can ignore effects of boundaries without assuming a torus geometry.

  This has at least two advantages:
  - The constant external magnetic field resp. the magnetic flux per unit area is a continuous variable
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- Many approximate properties become exact in the thermodynamic limit. This allows to define phases and phase transitions.
- We do not need to make assumptions about finite volume approximations.

Course overview

1. **Physics background, mathematical modelling, and questions we would like to answer**
2. Mathematical tools: local approximation of quasi-local operators, the quasi-local inverse of gapped Liouvillians
3. Adiabatic theorem and non-equilibrium almost stationary states (NEASS) as a foundation for understanding (linear) response of gapped systems
4. Approximate Ohm's law: nearly linear Hall current response that is constant in gapped phases (microscopic and macroscopic)
5. Remarks and perspectives: gapped phases, integer quantization, Hall current density in non-periodic systems

Quantum mechanics of infinitely many particles: fermions on a lattice

We consider fermions on $\mathbb{Z}^d$. The N-particle Hilbert space for such a system is

$$\mathcal{H}_N^- := \ell^2(\mathbb{Z}^d, \mathbb{C}^n)^{\wedge N},$$

and Fock space is defined by

$$\mathfrak{F}^- := \bigoplus_{N=0}^{\infty} \mathcal{H}_N^- \ni \psi = (\psi_0, \psi_1, \psi_2, \dots) \quad \text{with} \quad \|\psi\|^2 := \sum_{N=0}^{\infty} \|\psi_N\|^2_{\mathcal{H}_N^-}.$$

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For any subset $M \subseteq \mathbb{Z}^d$ one defines the algebra $\mathcal{A}_M$ as the C*-sub-algebra of $\mathcal{B}(\mathfrak{F}^-)$ generated by the fermionic creation and annihilation operators $a^*_{x,i}$ and $a_{x,i}$ with $x \in M$ and $i \in \{1, \ldots, n\}$.

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The latter satisfy the canonical anti-commutation relations (CAR)

$$\{a^*_{x,i}, a_{y,j}\} := a^*_{x,i}\, a_{y,j} + a_{y,j} a^*_{x,i} = \delta_{x,y}\delta_{i,j}, \quad \{a^*_{x,i}, a^*_{y,j}\} = \{a_{x,i}, a_{y,j}\} = 0.$$

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The algebra of local observables is defined as

$$\mathcal{A}_{\mathrm{loc}} := \left\{ A \in \mathcal{B}(\mathfrak{F}^-) \,|\, A \in \mathcal{A}_M \text{ for some } M \text{ with } |M| < \infty \right\} ,$$

and the quasi-local algebra or CAR algebra is the unital C^*-algebra

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One defines the set of even quasi-local operators

$$\mathcal{A}^+ := \{ A \in \mathcal{A} \mid g_\pi(A) = A \} , \qquad \mathcal{A}_M^+ := \mathcal{A}^+ \cap \mathcal{A}_M$$

and the set of gauge-invariant quasi-local operators

$$\mathcal{A}^{\mathcal{N}} := \{ A \in \mathcal{A} \mid \forall \varphi \in \mathbb{R} : \; g_\varphi(A) = A \} , \qquad \mathcal{A}_M^{\mathcal{N}} := \mathcal{A}^{\mathcal{N}} \cap \mathcal{A}_M \, .$$

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For disjoint regions $M_1, M_2 \subseteq \mathbb{Z}^d$, $M_1 \cap M_2 = \emptyset$, operators $A \in \mathcal{A}^+_{M_1}$ and $B \in \mathcal{A}_{M_2}$ commute,

$$[A, B] = 0 \, .$$

Quantum mechanics of infinitely many particles: fermions on a lattice

The Heisenberg time evolution is generated by densely defined derivations $\mathcal{L}_H$ on $\mathcal{A}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t) = \mathrm{i}\mathcal{L}_H A(t)\,,$$

typically of the sum-of-local-terms form

$$\mathcal{L}_H A = \sum_{x \in \mathbb{Z}^d} [H_x, A] \quad \text{where } H_x = H_x^* \in \mathcal{A}^{\mathcal{N}} \text{ for all } x \in \mathbb{Z}^d\,.$$

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In contrast to vector states or density matrices on $\mathcal{B}(\mathfrak{F}^-)$, such states can describe systems with infinitely many particles.

Insulators are characterized by gapped ground states

For a densely defined derivation $\mathcal{L}_H$ on $\mathcal{A}$ a state $\omega \in \mathcal{A}^*$ is a (locally unique) gapped ground state, iff there exists $g > 0$ such that

$$\omega(A^* \mathcal{L}_H A) \geq g \left(\omega(A^* A) - |\omega(A)|^2 \right) \quad \text{for all } A \in D(\mathcal{L}_H) \,.$$

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Finite systems: For $H \in \mathcal{L}(\mathbb{C}^n)$ a self-adjoint matrix and $\rho \in \mathcal{L}(\mathbb{C}^n)$ a density matrix this reads

$$\mathrm{tr}(\rho A^*[H, A]) \geq g(\mathrm{tr}(\rho A^* A) - |\mathrm{tr}(\rho A)|^2) \quad \text{for all } A \in \mathcal{L}(\mathbb{C}^n)\,,$$

which you can easily check is equivalent to ρ being the rank-one ground state projection of H and $E_1 - E_0 \geq g$.

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Example: For $B, \mu \in \mathbb{R}$ the non-interacting Hofstadter model

$$H_{(B,\mu,0)} := \mathrm{d}\Gamma(\mathfrak{h}_B - \mu\, 1_{\ell^2(\mathbb{Z}^2,\mathbb{C}^n)}) := \sum_{x,y\in\mathbb{Z}^2} \mathfrak{h}_B(x,y)\, a_x^* a_y - \mu \sum_{x\in\mathbb{Z}^2} n_x$$

has a gapped ground state, whenever $\mu \notin \sigma(\mathfrak{h}_B)$. Here

$$\mathfrak{h}_b(x,y) = \mathrm{e}^{\mathrm{i}\frac{x_2+y_2}{2} b(x_1-y_1)}\, \mathfrak{h}_0(x-y)$$

is the kernel of the one-body Hofstadter Hamiltonian.

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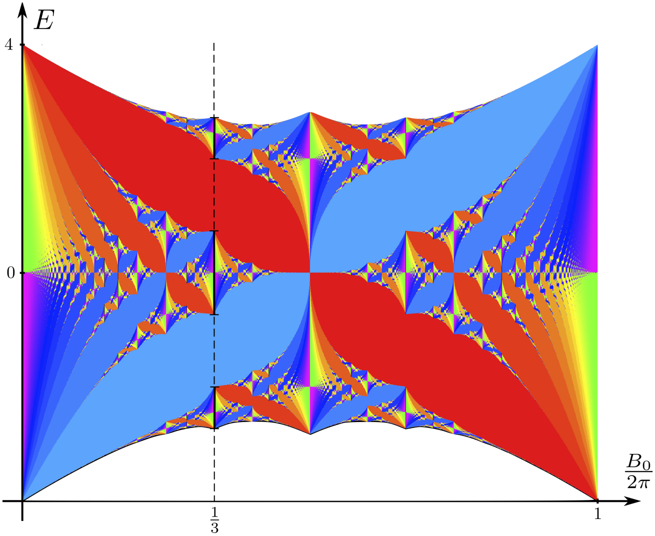
This gap is locally stable, i.e. if $\mu \notin \sigma(\mathfrak{h}_B)$ then for $\lambda \in \mathbb{R}$ small enough and $V \in B_\infty$, also the weakly interacting Hofstadter model

$$H_{(B,\mu,\lambda)} := H_{(B,\mu,0)} + \lambda V$$

has a gapped ground state.

Gapped phases in the Hofstadter model

The coloured butterfly: gapped phases of the non-interacting Hofstadter model



Gapped ground states: some questions for mathematical physics

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- Stability: Is the gap stable under (small) perturbations of $\mathcal{L}_H$?
- Response to perturbations: How do systems in gapped ground states respond to external driving? Can we explain the observations in quantum Hall experiments:
  - Ohm's law?
  - Quantization of conductance?
  - Vanishing fluctuations?

Gapped ground states: some questions for mathematical physics

- Existence: Can one show for a given $\mathcal{L}_H$ that it has a gapped ground state?
- Uniqueness: Can one show that a gapped ground state is unique?
- Stability: Is the gap stable under (small) perturbations of $\mathcal{L}_H$?
- Response to perturbations: How do systems in gapped ground states respond to external driving? Can we explain the observations in quantum Hall experiments:
  - Ohm's law?
  - Quantization of conductance?
  - Vanishing fluctuations?
- Classification: What are good definitions of gapped phases (equivalence classes of gapped ground states) and can one characterize them by characteristic data?

How does a system in a gapped ground state respond to a constant field?

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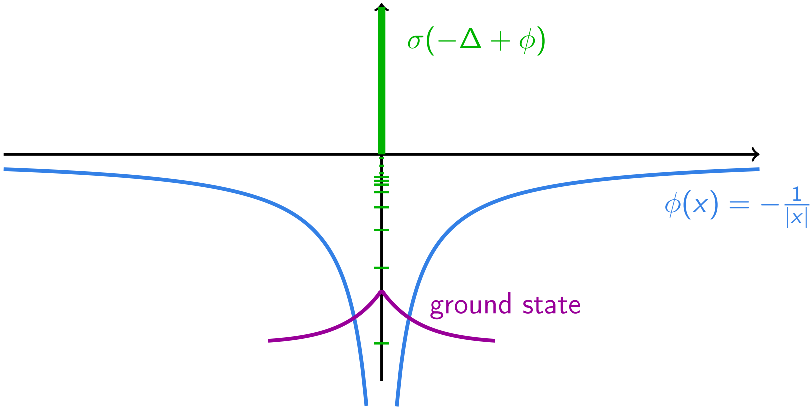
Example: The Stark Hamiltonian



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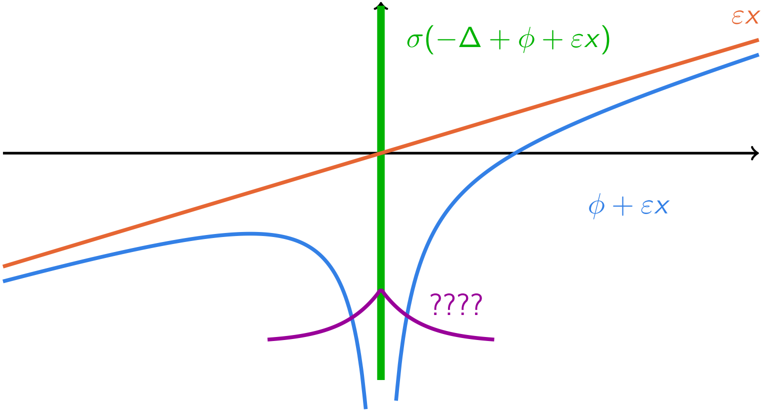

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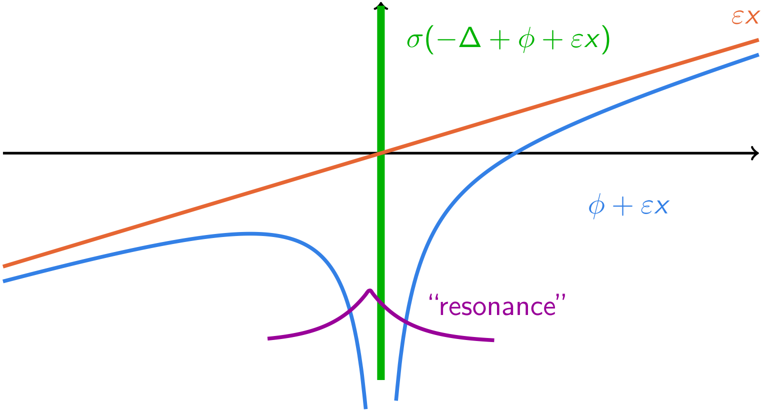

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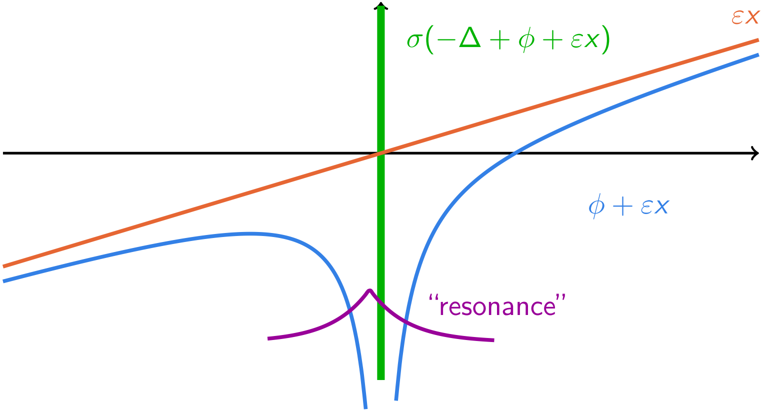

Adiabatic theorems for resonances were established e.g. by
Abou Salem, Fröhlich CMP '07 and by **Elgart, Hagedorn** CPAM '11.

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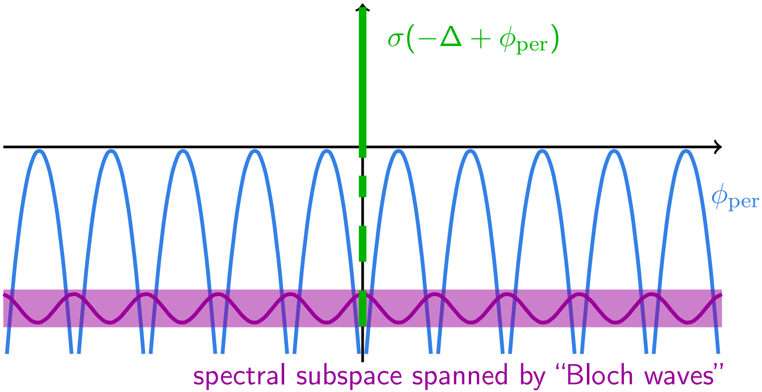
Example: An "Extended Stark Hamiltonian"



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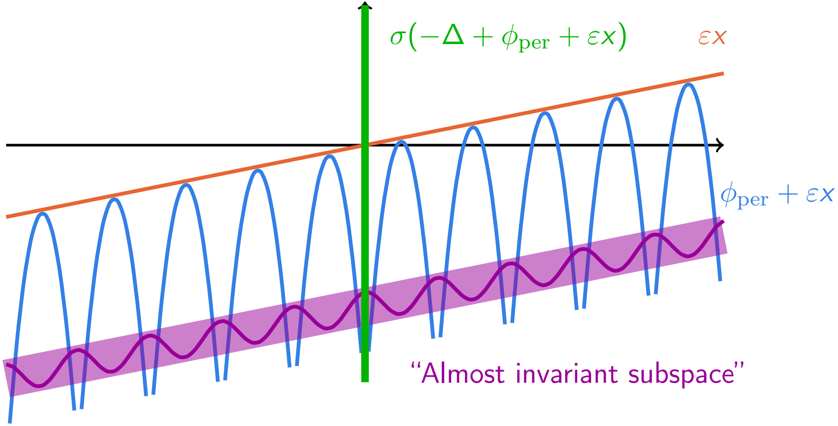

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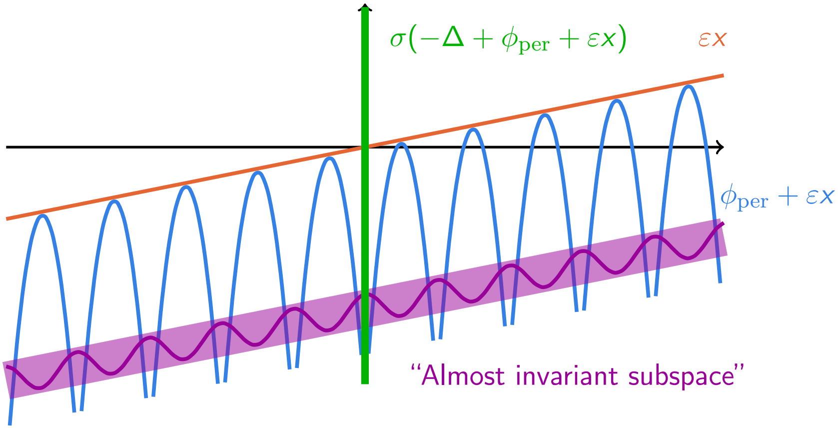

"Infinite dimensional resonances" were first discussed and named almost invariant subspaces by **Nenciu** CMP '81, JMP '02.

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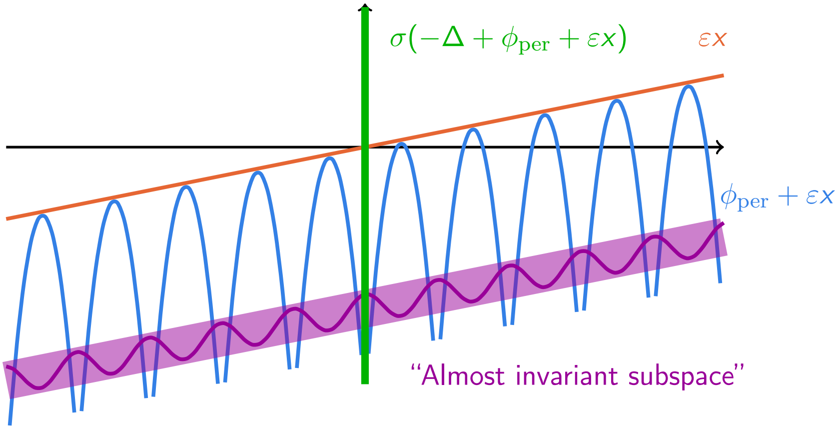
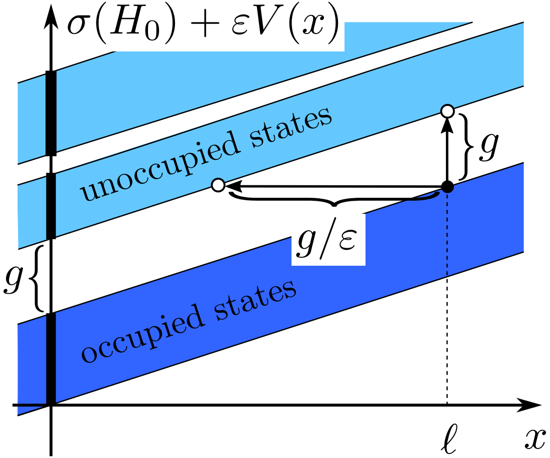
Example: An "Extended Stark Hamiltonian"



A corresponding adiabatic theory was established by **Nenciu, Sordoni** JMP '03 and by **Panati, Spohn, T.** CMP '03, based on techniques from **Helffer, Sjöstrand** MSMF '89.

Physical picture



Response to external driving and adiabatic switching

Modelling the switching process: Let

$$H_\varepsilon(t) := H_0 + \varepsilon\, f(t)\, \Phi$$

with a smooth function $f : \mathbb{R} \to \mathbb{R}$.

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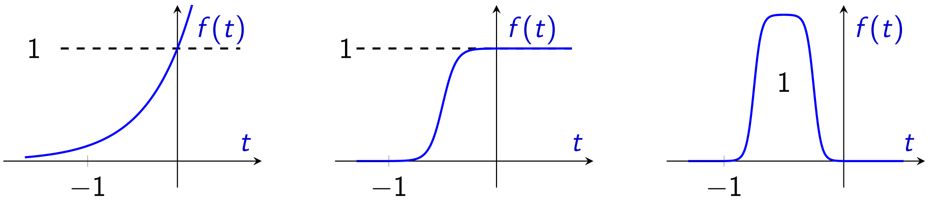
Standard linear response NEASS approach Charge pumping



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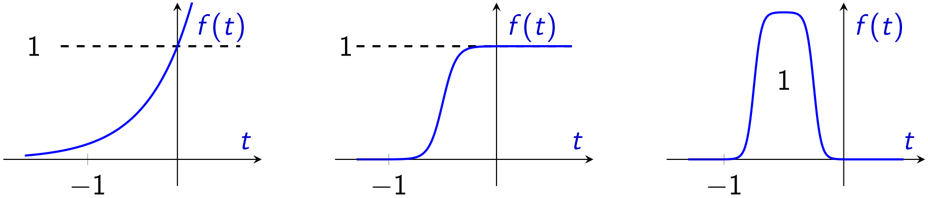

Let $\mathfrak{U}^{\varepsilon,\eta}_{t_0,t}$ denote the dynamics generated by the adiabatically scaled Hamiltonian

$$H_{\varepsilon,\eta}(t) := H_\varepsilon(\eta\, t) \qquad \text{where} \qquad 0 < \eta \ll 1\,.$$

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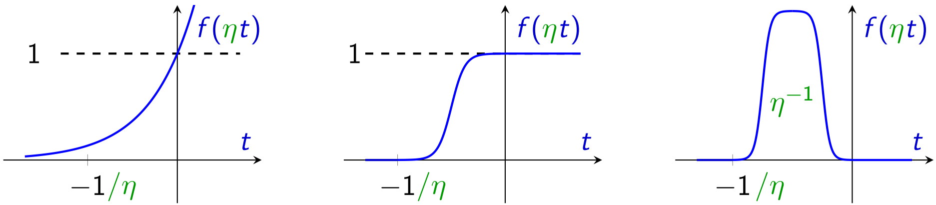

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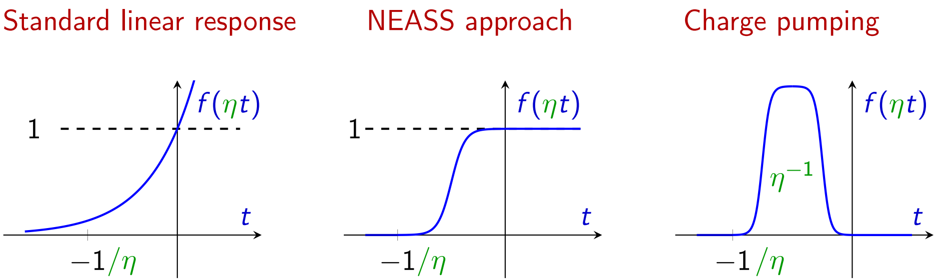

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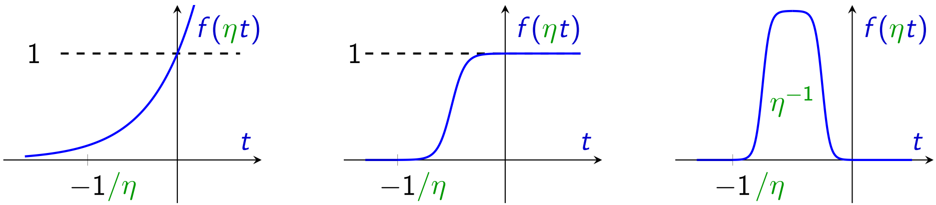
Exp. of linear coefficient of current: $\frac{1}{\varepsilon}\,\omega_0 \circ \mathfrak{U}^{\varepsilon,\eta}_{-\infty,0}(J)$ for $\varepsilon \to 0$ and then $\eta \to 0$.

Response to external driving and adiabatic switching

Standard linear response | NEASS approach | Charge pumping



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Response to external driving and adiabatic switching

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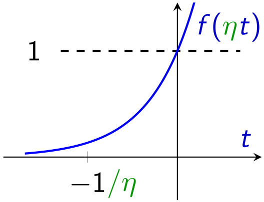
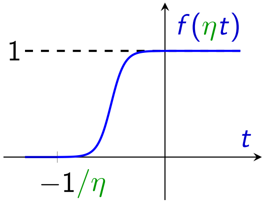
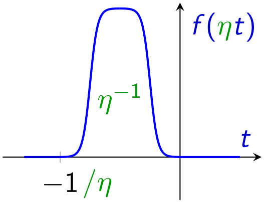


NEASS approach



Charge pumping



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Exp. of transported charge: $\omega_0 \circ \mathfrak{U}^{\varepsilon,\eta}_{-1/\eta,0}(Q) - \omega_0(Q)$ for $\eta = \varepsilon$.

Adiabatic switching and NEASS

Theorem: Becker, T., Wesle '25 (also T. CMP '20 and Henheik, T. FMσ '22)

Assume that

- $H_0 \in B_\infty$, i.e. it is a sum of super-polynomially localized terms, and
- H_0 has a gapped ground state ω_0, and
- Φ is of the form $\Phi = V + X_j$ with $V \in B_\infty$ and X_j the jth component of the position operator.

Then there exists a state ω_ε (NEASS) such that for all $n \in \mathbb{N}$ there is $c_n > 0$ such that for all $t \geq 0$, and $A \in \mathcal{A}_0$

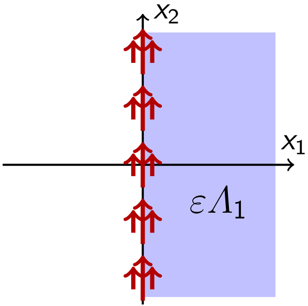
$$\sup_{\eta \in [\varepsilon^n, \varepsilon^{1/n}]} \left| \omega_0 \circ \mathfrak{U}^{\varepsilon,\eta}_{-1/\eta,t}(A) - \omega_\varepsilon(A) \right| \leq c_n\, \varepsilon^n (1 + t^{d+1}) \|A\|_\nu \,.$$

Defining the Hall conductance resp. conductivity

One can consider the current response

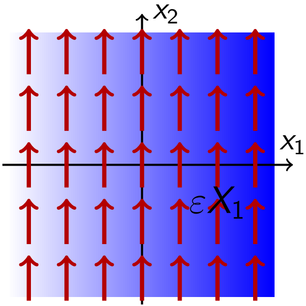
to a **potential step**

$$\Phi = \Lambda_1 := \sum_{x \in \mathbb{N}_0 \times \mathbb{Z}} n_x$$



or to a **potential gradient**

$$\Phi = X_1 := \sum_{x \in \mathbb{Z}^2} x_1 n_x$$



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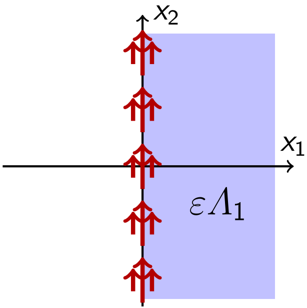

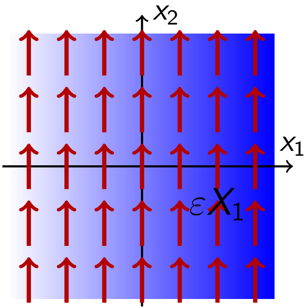

The **response observable** is

the current through any horizontal line the current density into the x_2-direction

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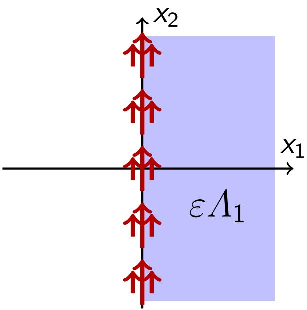

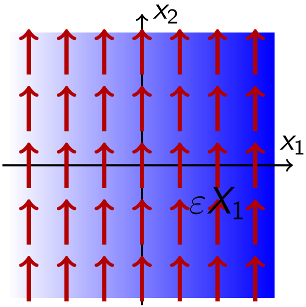

The **response observable** is

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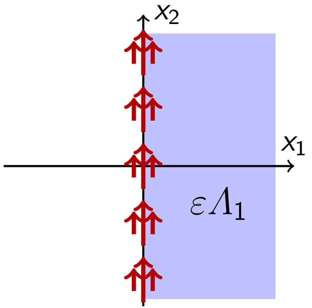

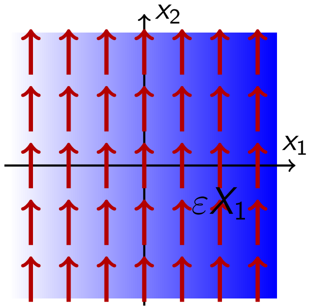

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The perturbation **does not close the gap** and the response is microscopic.

The perturbation **closes the gap** and the response is macroscopic.

Current response of an insulator to a **voltage drop**

Theorem: T., Wesle (2025)

Assume that $H_0 \in B_\infty$ and that H_0 has a gapped ground state ω_0.

Then for the NEASS ω_ε associated with $H_\varepsilon = H_0 + \varepsilon \Lambda_1$ and ω_0 it holds that

$$\omega_\varepsilon(I_2) - \omega_0(I_2) \;=\; -\varepsilon \underbrace{\omega_0\left(\mathrm{i}\,[\Lambda_1^{\mathrm{OD}}, \Lambda_2^{\mathrm{OD}}]\right)}_{=: c_{\mathrm{H}}} \;+\; \mathcal{O}(\varepsilon^\infty)\,,$$

where c_{H} is the "microscopic" Hall conductance.

Here

$$I_2 := \mathrm{i}[H_0, \Lambda_2] = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{\mathrm{i}\mathcal{L}_{H_0} t} \Lambda_2|_{t=0}$$

is the "current flowing into the upper half-plane operator" and

$$\Lambda_k^{\mathrm{OD}} := [H_0, \mathcal{I}_{H_0}(\Lambda_k)]$$

the off-diagonal part of Λ_k with respect to ω_0.

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where c_{H} is the "microscopic" Hall conductance.

Note that no periodicity or homogeneity is assumed and that c_{H} is independent of the choice of the origin and of the orientation of the half-planes.

Moreover, $\mathrm{i}\,[\Lambda_1^{\mathrm{OD}}, \Lambda_2^{\mathrm{OD}}] \in \mathcal{A}_\infty \subset \mathcal{A}$ is a quasi-local observable localized near the origin and can be considered a local Chern marker.

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If ω_ε is a gapped ground state of H_ε, i.e., if the perturbation does not close the gap, then the response is exactly linear:

$$\omega_\varepsilon(I_2) - \omega_0(I_2) \;=\; -\varepsilon\, \omega_0\left(\mathrm{i}\,[\Lambda_1^{\mathrm{OD}}, \Lambda_2^{\mathrm{OD}}]\right).$$

Current response of a periodic insulator to a **constant electric field**

Theorem: Wesle, Marcelli, Miyao, Monaco, T. '24 (CMP 2025)

Assume that $H_0 \in B_\infty$ is periodic and that H_0 has a periodic gapped ground state ω_0.
Then the NEASS ω_ε associated with $H_\varepsilon = H_0 + \varepsilon(V + X_1)$ it holds that

$$\overline{\omega_\varepsilon}(J_{\varepsilon,1}) = \mathcal{O}(\varepsilon^\infty)$$

and

$$\overline{\omega_\varepsilon}(J_{\varepsilon,2}) = -\varepsilon \underbrace{\overline{\omega_0}\left(\mathrm{i}\,[X_1^{\mathrm{OD}}, X_2^{\mathrm{OD}}]\right)}_{=:\sigma_{\mathrm{H}}} + \mathcal{O}(\varepsilon^\infty)\,,$$

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Here $J_{\varepsilon,k} := \mathrm{i}[H_\varepsilon, X_k]$ is the kth component of the current operator,

$$\overline{\omega_\varepsilon}(O) := \lim_{\Lambda \to \mathbb{Z}^2} \frac{1}{|\Lambda|}\,\omega_\varepsilon(O|_\Lambda)$$

denotes the density of an extensive observable, and

$$X_k^{\mathrm{OD}} := [H_0, \mathcal{I}_{H_0}(X_k)]$$

is a periodic interaction.

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where σ_{H} is the Hall conductivity and $\sigma_{\mathrm{H}} = c_{\mathrm{H}}$.

Note that the weakly interacting Hofstadter model satisfies the assumptions of this theorem and the previous one for all magnetic fields.

Some corollaries and consequences

- For the "measured" Hall conductance it follows that

$$c_{\mathrm{H}}^{\mathrm{exp}} := \lim_{L\to\infty} \omega_\varepsilon\left(\frac{J_L}{\Delta U_L}\right) = \sigma_{\mathrm{H}} + \mathcal{O}(\varepsilon^\infty)$$

and that it is deterministic

$$\mathrm{var}(c_{\mathrm{H}}^{\mathrm{exp}}) := \lim_{L\to\infty}\left(\omega_\varepsilon\left(\left(\frac{J_L}{\Delta U_L}\right)^2\right) - \omega_\varepsilon\left(\frac{J_L}{\Delta U_L}\right)^2\right) = 0\,.$$

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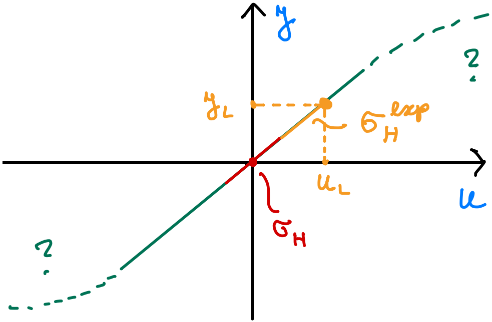
Ohm's law:



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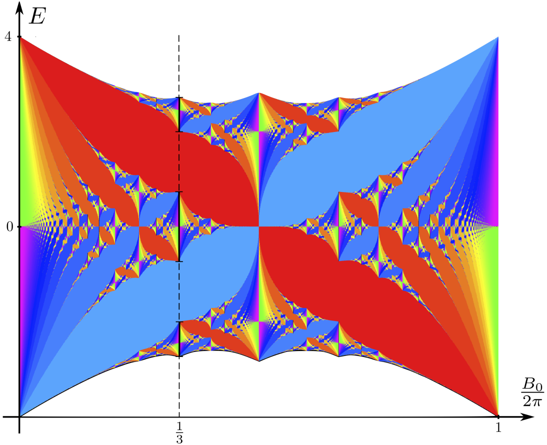
- The Hall conductivity (resp. conductance) σ_{H} is constant within gapped phases defined by automorphic equivalence.
- The Hall conductance is the same for any pair of transversal (generalized) step functions Λ_f and Λ_g.
- The Hall conductivity is the same for any pair of orthogonal directions $a, b \in \mathbb{R}^2$, i.e. one can replace X_1, X_2 by $X_a := a_1 X_1 + a_2 X_2$, $X_b := b_1 X_1 + b_2 X_2$ and the current response $\mathrm{i}[H_\varepsilon, X_b]$ to the driving X_a is again nearly linear with the same linear coefficient σ_{H}.

Gapped phases in the weakly-interacting Hofstadter model

The coloured butterfly: gapped phases of the non-interacting Hofstadter model



Is also the Hall conductivity stable under weak perturbations?

Gapped phases in the weakly-interacting Hofstadter model

Giuliani, Mastropietro, Porta CMP '17, Giuliani '20 prove that the Hall conductivity does not change when adding a sufficiently small periodic perturbation to a gapped periodic non-interacting system. For the Hofstadter model this result applies, however, only for $B \in 2\pi\mathbb{Q}$.

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What about $B \notin 2\pi\mathbb{Q}$?

Problem: Two ground states of the non-interacting Hofstadter model in the same gapped phase but for different magnetic fields are not automorphically equivalent.

Gapped phases in the weakly-interacting Hofstadter model

Giuliani, Mastropietro, Porta CMP '17, Giuliani '20 prove that the Hall conductivity does not change when adding a sufficiently small periodic perturbation to a gapped periodic non-interacting system. For the Hofstadter model this result applies, however, only for $B \in 2\pi\mathbb{Q}$.

What about $B \notin 2\pi\mathbb{Q}$?

Problem: Two ground states of the non-interacting Hofstadter model in the same gapped phase but for different magnetic fields are not automorphically equivalent.

Theorem (Wesle, Marcelli, Miyao, Monaco, T. , in preparation)

For every $\mu_0, B_0 \in \mathbb{R}$ such that $\mu_0 \notin \sigma(h_{B_0})$ there is $\delta > 0$ such that for all $(\mu, B, \lambda) \in \mathbb{R}^3$ with $\|(\mu, B, \lambda) - (\mu_0, B_0, 0)\| < \delta$

$$\sigma_{\mathrm{H}}(\mu, B, \lambda) \;=\; \sigma_{\mathrm{H}}(\mu_0, B_0, 0) \;\in\; \tfrac{1}{2\pi}\mathbb{Z}\,.$$

Macroscopic current response for systems without translation invariance

Theorem (Miyao, Wesle, T., in preparation)

Assume that $H_0 \in B_\infty$ has a gapped ground state ω_0.

Let ω_ε be the corresponding NEASS for the perturbation X_1. Then

$$\overline{\omega_\varepsilon}(J_i) := \lim_{\Lambda \nearrow \Gamma \subset \mathbb{R}^2} \frac{1}{|\Lambda|} \overline{\omega_\varepsilon}(J_i|_\Lambda)$$

exists (up to $\mathcal{O}(\varepsilon^\infty)$) and

$$\begin{aligned} \overline{\omega_\varepsilon}(J_1) &= \mathcal{O}(\varepsilon^\infty) \\ \overline{\omega_\varepsilon}(J_2) &= \varepsilon\, \overline{\omega_0}\left([X_1^{\mathrm{OD}}, X_2^{\mathrm{OD}}]\right) + \mathcal{O}(\varepsilon^\infty)\,. \end{aligned}$$

All properties from the periodic case carry over to the general case.

Course overview

1. **Physics background, mathematical modelling, and questions we would like to answer**
2. Mathematical tools: local approximation of quasi-local operators, the quasi-local inverse of gapped Liouvillians
3. Adiabatic theorem and non-equilibrium almost stationary states (NEASS) as a foundation for understanding (linear) response of gapped systems
4. Approximate Ohm's law: nearly linear Hall current response that is constant in gapped phases (microscopic and macroscopic)
5. Remarks and perspectives: gapped phases, integer quantization, Hall current density in non-periodic systems