Fine structure constant, anomalous magnetic factor and transport coefficients

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- *At the present time I can proudly say that there is no significant difference between experiment and theory! To give you a feeling for the accuracy of these numbers, it comes out something like this: If you were to measure the distance from Los Angeles to New York to this accuracy, it would be exact to the thickness of a human hair. (Feynman)*

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- $a = (g-2)/2 = a_{\mathrm{EW}} + a_{\mathrm{QCD}}$. The a_{EW} is theoretically expressed as series in $\alpha = e^2/\hbar c$ the fine structure constant; the prediction is done truncating the series at a certain order (the constant of weak can be related to α). It is the largest contribution. $a_{\mathrm{EW}} = \sum_n a_n \alpha^n$

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- Of course the validity of the prediction depends on the precision of the value of α which must be determined separately.
- A possibility is consider phenomena which can be also accurately measured and whose theoretical predition is also accurate. A possibility comes from non-relativistic quantum phenomena (condensed matter).

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- Remark: The anomalous magnetic factor regard the property of quantum particle at relativistic energies (Dirac); the transport coefficients regards quantum particles at lower energies (Schroedinger). However they are described by similar mathematical objects (functional integrals) and in certain cases there is an emergent Dirac description even at low energies.

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- Are the low energy phenomena really universal or there are corrections ? Theoretically universality is predicted in the single particle approximation but there are interactions between particles which can give contributions.
- The posed mathematical problems are very similar both for low energy and high energy: for instance the universality has a counterpart in the non-renormalization property. Why g is a series and transport coefficients have exact expressions?

Graphene conductivity

- Graphene optical conductivity; perturbative series starting from the free fermion description; if U is the many body interaction

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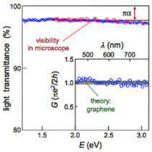
- Experimentally it seems there are No interaction corrections (Geim Nosovelov (2008)



$\sigma = \frac{e^2}{h}\frac{\pi}{2}$. Is this exacly true? Why?

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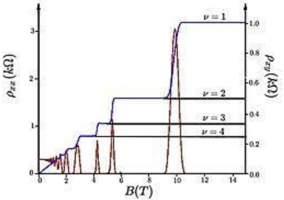
- 

  The quantization of conductivity $\sigma_{xy} = ne^2/h$ holds with very high precision; no sign of corrections due to interaction. Is it exactly true?

$g-2$

- Trunction of perturbation theory $a_{\rm EW} = a_{\rm QED} + a_{\rm WEAK}$ and

$$a_{\rm EW} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n,m}\left(\frac{\alpha}{\pi}\right)^n \lambda^{2m}$$

$\lambda^2 = 4\pi\alpha/\sin\theta$, $\sin^2\theta = 0,223..$, $\cos\theta_W = M_W/M_Z$. Due to perturbative renormalizablity $a_{n,m}$ finite to all orders (quite subtle for chiral gauge theories, Ward Identities, anomaly cancellation etc!).

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- QED contributions. $.{\rm a}_{1,0}=1/2$ Schwinger (1948); $a_{2,0}\approx 0.7658$ Karplus and Knoll in 1950, correction in Sommerfeld (1958); $a_{3,0}$ Laporta (1996); $a_{4,0}$, $a_{5,0}$ Aoyama (2012).

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- Weak contributions. Jackiw Weinberg (1972), Altarelli, Cabibbo and Maiani (1972); Z contribution (M the Z mass)
  $a_{1,Z} = \frac{m^2}{M^2} \frac{1}{4\pi^2} \frac{1-5\kappa^2}{3} \left[1 + (m^2/M^2)\right] = \bar{a}_{1,Z} \left[1 + (m^2/M^2)\right]$, $\kappa = (4\sin^2\theta_W - 1)^{-1}$.
  Proportional to $(m/M)^2$; for electron $(m/M) \approx 10^{-6}$. Largest contribution from QED. For electron $a_{\mathrm{weak}} = 0.0297(5)10^{-12}$, for muon $a_{\mathrm{weak}} = 194.79 \times 10^{-11}$,

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- Experiments show that the exponents are the exactly same in a wide class of system and often coincide with Ising; for instance in $d = 2$ the index β at the ferromagnetic transition is 0.119(8) in Rb_2CoF_4, 0.123(8) in K_2CoF_4, 0.135(3) in Ba_2FeF_6. ($\beta = 1/8$ in Ising); in $d = 3$ $\nu = 0.60.7$ and best numerical results $\nu = 0,6299...$

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- Verified in solvable models like Luttinger or XXZ; even the slightest modification destroy solvabililty

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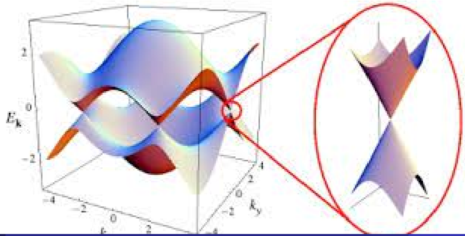
- In the case of QED or electroweak theory one needs a lattice (effective QFT) acting as a cut-off which cannot be removed due to triviality. No dependence on lattice as it has no physical meaning (hence quanities must be independent or with corrections below experimental precision).
- The condensed matter models can be seen as regularized QFT; graphene is an effective QED in $d = 2 + 1$



- The mathematical objects in terms of which such quantities are computed are Euclidean correlations (regularized functional integrals) , which are objects of the form

$$S(\mathbf{x}_1, .., \mathbf{x}_n) = \frac{\int P(d\Phi)e^{-V(\Phi)}\Phi_{\mathbf{x}_1}...\Phi_{\mathbf{x}_n}}{\int P(d\Phi)e^{-V(\Phi)}}$$

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  where x are points in a suitable finite lattice Λ with step a and side L. Averaging over all the field configurations with weight $P(d\Phi)e^{-V(\Phi)}$.
- One has also to consider two cases which can be called bosonic or fermionìc.

Functional integrals

- In the bosonic case $(\Phi = \phi)$ $\phi_x \in \mathbb{R}$, $P(d\phi)$ is a Gaussian measure $(\prod_x d\phi_x e^{-1/2(\phi, A\phi))})$ and $V(\phi)$ is a sum over monomials in ϕ; in the ϕ^4 model $V = \sum_x \frac{\lambda}{4!}\phi_x^4$ (λ is the coupling). Finite dimensional but $O((L/a)^d)$ variables (in the one dimensional case $\sim \int_{-\infty}^{\infty} dx e^{-x^2-\lambda x^4}$)

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- Wick rule in terms of propagator g; The model with a power law decay of g are called critical or massless, with exponential decay massive.
- We will focus on the Grassmann integrals.

Applications

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- In condensed matter conduction electrons in metals are a gas of interacting fermions; interaction can produce dramatic effects like superconductvity, luttinger liquid behavior etc. The equilibrium correlations and transport coefficients can be written as grassmann integrals (Matsubara) Λ is the space- (imaginary) time, side β and L. One is interested in $L \to \infty$ (thermodynamic limit) and zero temperature $\beta \to \infty$) (x_1 discrete and x_0 continuous). Averages with respect to Gibbs factor $e^{-\beta H}$

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- The theory of such functional ntegrals has been originated by some many different physical domanis and the language to describe the property feels such variety.

Dressed and bare quantities

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- In pert theory $\frac{1}{k^2+(m^2+\lambda^2)}$ appears as $\frac{1}{k^2+m^2}\sum_{n=0}^{\infty}(-1)^n(\frac{\lambda^2}{k^2+m^2})^n$

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- In QED or in the SM then $\lim_{\Lambda\to\infty} \tilde{P}_i(\Lambda) < \infty$ finite!; of course one needs to face convergence (typically not convergent, triviality and all that)
- The Renormalization Group is the modern way of implementing renormalization. More than that, it has deeply modified our understanding of physics.

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- Essentially the propagator or covariance $\hat{g}_\Phi(k)$ is written as sum of propagators $\sum_{h=-\infty}^{N} \hat{g}^h_\Phi(k)$ each non vanishing for $c\gamma^{h-1} \leq |k| \leq c\gamma^{h+1}$ for $h \leq N-1$, $\gamma > 1$

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- By the convolution property of Gaussian integrals

$$\int P(d\Phi)e^{-V(\Phi)} = \int P(d\Phi^{<N}) \int P(d\Phi^N)e^{-V(\Phi^{<N}+\Phi^N))} = \int P(d\Phi^{<N})e^{-V^N(\Phi^{<N})}$$

where $P(d\Phi^{<N})$ has propagator $\sum_{h=-\infty}^{N-1} \hat{g}_\Phi^h(k)$, $P(d\Phi^N)$ has propagator $\hat{g}_\Phi^N(k)$ and V^N is sum of monomials of any order in $\phi^{<N}$ integrated over certain kernels.

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- The monomials appearing in V^h of the form $\int dx O_n^{\leq h}$, a local monomial with n fields, behaves essentially as $O_n^{\leq N}$ with a prefactor

$$\gamma^{-(h-N)D_n} \int dx O_n^{\leq N} \quad D_n = d - \frac{(d-\alpha)}{2} n$$

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- In condensed matter typically $N = 0$ fixed. In QFT one wants to take $N \to \infty$; when irrelevant one needs a bare value larger and larger to be $O(1)$ at scale 0 (so outside perturbative accessible regime); when relevant the opposite

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- Most rigorous results regard the case of irrelevant or marginally irrelevant for the ir (or relevant or marginally relevant for the uv). In the relevant or marginally relevant one expects "a non trivial fixed point".
- In the marginally marginal case all coefficients are vanishing. This cannot be proved by perturbative computation and correspond to a line of fixed points. Lines of fixed points, different physical behavior.

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- Starting from the 80's rigorous renormalization was introduced.

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- Fermion series are convergent (good); however bosons have also advantages as can use saddle point arguments (one can decide to write fermions as boson by Hubbard Stratanovich or integrate the bosons considering only fermions).

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- In $d = 4$ effective QFT point of view (a small but finite), triviality of ϕ^4 (Aizenman Duminil-Copin Ann of Math 2020) makes SM probably effective by Higgs.

Rigorous fermionic RG: application to condensed matter

- In the '90 rigorous RG was applied to condensed matter by Benfatto, Gallavotti (JSP 90) and Feldman, Trubowitz (HPA 90); interacting non relativistic fermions in the continuum. Extended singularity in $d \geq 2$; $D = 2 - n/2$, the theory is marginal and in $d \geq 2$ the rcc is function of the angles. In certain directions is marginally relevant (superconductivity). order by order perturbative analysis.

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- There was an attempt to construct non perturbatively the $T = 0$ properties in the continuum at $d = 2$ and proving superconductivity, using also sectors method (Feldman, Magnen, Rivasseau, Trubowitz EPL 1993); several interesting partial results but problem still (very!) open. Difficulty related to marginal relevance and infinitely many couplings.

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- RG and sectors were used to construct the $T = 0$ properties in $d = 2$ of a special lattice model with asymmetric Fermi surface (asymmetry makes rcc effectively irrelevant) (Knoerrer Feldman Trubowitz CMP 2002), Fermi liquid behavior (regularity of counterterm proved order by order Feldman Trubowitz Salmhofer CPAA 1999)

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- Going to lower temperatures is very hard; one can truncate the beta function and see numerically which kind of coupling increases more, see eg Honerkamp Salmhofer (PRB 2002), getting evidence for superconductivity for $d = 2$ attractive Hubbard (connected to the debated problem of high T_c supercondictivity)

- Actually the only cases in which the $T = 0$ properties of condensed matter model have been constructed are when the FS is pointlike; $d = 1$ (Luttinger liquids) and in $d \geq 2$ (graphene, weyl semimetals, Hall systems....); point like Fermi surface.

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- In the first case $D = 2 - n/2$ only 1 rcc; anomalous exponents, marginally marginal theory main difficulty vanishing of beta function.
- In the second $D = 3 - n$ or $D = 4 - 3n/2$; coupling irrelevant main difficulty proving universality of transport coefficients.

Rigorous fermionic RG: Luttinger liquids, vanishing of beta function ad all that

- In $d = 1$ case the first proof of vanishing of beta function was given (Benfatto Gallavotti Mastropietro, PRB 1992, Benfatto Gallavotti Procacci Scoppola CMP 1994) using the exact solution of the Luttinger model by bosonization (Mattis Lieb 1966).

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- This was not (fully) satisfactory as would like to have a self-consistent proof avoiding "exact solutions" which are special (if even in $d = 1$ RG is not enough alone, no hope for future!).
- Later Benfatto Mastropietro (CMP 2002, CMP 2004) developed a nechnique allowing to implement WI in each RG step; control of corrections due cut-off. This allowed the direct proof of vanishing of beta function. Subsequently a better choice of the "reference model" allowed to prove a version of the Adler-Bardeen theorem (Mastropietro JMP 2006) for chiral anomalies.

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- Such methods allowed to rigously prove the Luttinger liquid relations proposed by Haldane (1980) in non solvable models (they were checked before only in solvable ones) (Benfatto Falco Mastropietro CMP 2008, PRL 2008); later this method was applied to interacting dimers (Giuliani Mastropietro Toninelli 2019) (also Pinson Spencer 2008) extending relations in the free case (Kenion Okunkov Sheffield Ann.Math 2002). Regularity+WI

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- The method also allowed to prove the axioms in Thirring and the Coleman equivalence (Benfatto Falco Mastropietro CMP 2007), extended recently (only in the free fermion point) (Bauerschmidt, Webb Eur. Math. Soc 2024)

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- The main difficulty is to compute the trasnport coefficients; physical computaions (Mischenko 2009) showed a strong renormalization, in contrast with experiments. In Giuliani Mastropietro Porta CMP 2010 a theorem was proved establishing the exact universality. Cancellation of irrelevant terms.
- Hubbard interaction; for long range still open! (order by order lines of fixed points)

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- In a gapless case (and no second Melnikov) in the 2D Ising model with quasi periodic disorder (Gallone Mastropietro CMP 2024) the Harris irrelevance conjectured by Luck JSP 1983 was proved.

- Benfatto,Gallavotti; Renormalization Group, Princeton Press (1996)
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General introduction to the methods

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- See also for very related methods
- V. Rivasseau From perturbative to non perturbative renormalization, Princ Press 1991;
- M. Salmhofer Renormalization Springer 1999;
- R Bauerschmidt, DC Brydges, G Slade Introduction to a renormalisation group method Springer 2019;
- Giuliani, Mastropietro, Ryckov, Gentle introduction JHEP 2022

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- Hubbard model on the honeycomb lattice is $H = H_0 + U \sum_{\vec{x}\in\Lambda_A\cup\Lambda_B} \left(n_{\vec{x},\uparrow} - \frac{1}{2}\right)\left(n_{\vec{x},\downarrow} - \frac{1}{2}\right)$ and

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with $\vec{\delta}_1 = (1,0)$, $\vec{\delta}_2 = \frac{1}{2}(-1,\sqrt{3})$, $\vec{\delta}_3 = \frac{1}{2}(-1,-\sqrt{3})$, Λ_A periodic triangular lattice. $a^\pm$ fermionic creation or annihilation operators.

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- $\Pi_{i,i}$ is the FT of $< J_i(\mathbf{x}) J_j(\mathbf{y}) >$ (average respect to $e^{-\beta(H-\mu N)}$) by Kubo formula, J_i current operator $\sigma_i = \lim_{p_0\to 0} \lim_{p\to 0} \lim_{\beta,L\to\infty} \frac{1}{p_0}(\Pi_{i,i}(p) - \Pi_{i,i}(0))$ (need to prove Wick rotation)

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- The order of the limits p_0, p is crucial; reversing a different result is found (zero).

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- An outcome of RG; decomposition of $< J_{i,}(\mathbf{x})J_j(\mathbf{y}) >$ in a faster decaying and slower decaying in x; consequently $\hat{\Pi}_{lm} = \hat{\Pi}^a_{lm} + \hat{\Pi}^b_{lm}$ where:

Graphene

- $\hat{\Pi}^a_{lm}$ contains only marginal terms; it is given by a single graph and is equal to $\frac{Z_l Z_m}{Z^2}\langle \hat{j}_{\mathbf{p},l};\hat{j}_{-\mathbf{p},m}\rangle_{0,v_F}$ where $\langle\cdot\rangle_{0,v_F}$ is the average associated to a non-interacting relativistic system with Fermi velocity $v_F(U)$ and $Z_\mu=\frac{3t}{2}+aU+...$ It is non differentiable in p

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- Similar (simpler as massive) argument for Hall conductivity. Combination of WI and regularity properties. In Hall no multiscale needed is far from the transition lines. Important: difference between massive (non critical) and massless (critical). Reconstruction for real time.

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- A rigorous understanding could be in principle obtained using the effective point of view keeping the energy cut-off Λ fixed.
- All theories should be at the end effective and valid in a certain domain.

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- However in gauge theories renormalizability is subtle; is not simply related to power counting but rely on symmetries which can be broken by (non-perturbative) cut-offs.

Regularized model

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- Euclidean model for the neutral sector of weak forces with a momentum cut-off $\Lambda = 2^{N+1}$ in a volume, $2\lambda \equiv \lambda_+ + \lambda_-$ and $2\lambda\kappa \equiv \lambda_- - \lambda_+$, $e^W =$

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  $\psi^\pm_x = (\psi^\pm_1, \psi^\pm_2)$ Grassmann variables ($s = 1,2$ chiral index), $\mathcal{Z}_\mu$ real field,
- Propagators χ_N cut-off, $Z^\pm_N, m_N$ bare parameters

$$\hat{g}_\psi(k) = \frac{\chi_N(k)}{ik_\mu\gamma^\mu_N + m_N} \qquad \hat{v}^{\mu\nu}(k) = \delta_{\mu\nu}\frac{\chi_N(k)}{k^2 + M^2}$$

$$\gamma^\mu_N \equiv \begin{pmatrix} 0 & Z^+_N\sigma^\mu_+ \\ Z^-_N\sigma^\mu_- & 0 \end{pmatrix},\ m_N \equiv \begin{pmatrix} m^+_N & 0 \\ 0 & m^-_N \end{pmatrix},\ \text{where } \sigma_\pm \equiv (I, \mp i\sigma_1, \mp i\sigma_2, \mp i\sigma_3)$$

Regularized model

- Momentum Cut.-off $\chi_N(k) = \chi_0(2^{-N}k)$ and $\chi_0(k)$ is a rotationally invariant C^∞ function and $\chi_0(t) = 0$ for $|k| \geq 1$ and $\chi_0(t) = 1$ for $|k| \geq 1/2$. (Gevrey class). x in a finite volume with side L and pbc. Gauge fixing term to eliminate the $k_\mu k_\nu / M^2$ term (but invariance broken by cut-off).

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- The 2-point function is $S_{\alpha\beta}(x,y) =< \psi^-_\alpha(x)\psi^{+\prime}_\beta(y) >$ and the vertex function is $(\mathfrak{I}^\mu_L)_{\alpha\beta}(x,y,z) =< (\psi^+(z)\gamma^\mu_N\psi^-(z))\psi^-_\alpha(x)\psi^{+\prime}_\beta(y) >$.
  $\hat\Gamma^\mu(p',p) \equiv [\hat S_L(p')]^{-1} \cdot \hat{\mathfrak{I}}^\mu_L(p',p) \cdot [\hat S_L(p)]^{-1}$ amputated vertex function.

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- The bare parameters have to be chosen depending on the chiral index s.

Observables

- Renormalization conditions One has to choose (if possible) $Z_N^s, Z_N^{J,s}$,m_N^s so that

$$\hat{S}(k) = \frac{1}{ik+m}(1+r(k)) \qquad \hat{\Gamma}^\mu(p',p) = (\gamma_\mu + r_\mu(p,p'))$$

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- The perturbative definition (Minkowski) ofrom the scattering amplitude $\frac{2F(m,m,0)}{[F+G](m,m,0)}\cdot\left(-\frac{1}{2m}\mathbf{S}\cdot\hat{\mathbf{B}}(\mathbf{k})\right)$ where F,B are form factors; hence $g=\frac{2F}{F+G}$ at the mass shell with $p^2=-m^2$.

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- Regularized Euclidean $g-2$. If $\bar{\mathcal{A}}(p) = \lim_{L\to\infty} \frac{\epsilon_{\alpha\mu\sigma\beta}\, p^\sigma (\partial_{p'} - \partial_p)^\alpha [\gamma^5\, \mathcal{C}_{p'}\, \hat{\Gamma}^\mu(p',p)\, \mathcal{C}_p\, \gamma^\beta_{]\ p,p}}{12[\mathcal{C}_p\, p_\mu \hat{\Gamma}^\mu(p,p)]} - 1$ the regularized g is $a^R_Z \equiv \sum_{\ell=0}^K \frac{(im)^\ell}{\ell!}\partial^\ell_{|p|}(0)$. If $K, N \to \infty$ and Wick rotation is performed then it formally coincides with the perturbative definition. The dominant part of the term $m^\ell \partial^\ell \mathcal{A}(0)$ is of order $(m/M)^\ell$ and $m/M << 1$.

Observables

- Our main result is that (Mastropietro PRD 2024,Mas, Bianchessi (prepr)) $a_Z^R = \bar{a}_{1,z}(1 + R_0 + R_1)$ with $|R_0| \leq C_0 \frac{m^2}{M^2}$, $|R_1| \leq \epsilon < 1$ provided that we choose, for $1 > \theta > 0$ $\frac{c_1}{\epsilon} \leq \frac{\Lambda^2}{M^2} \leq \frac{c_2 \epsilon}{(\lambda^2 \log^4(M/m))^{1-\theta}}$ with C_0, C_1, c_1, c_2 are constants.

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- Convergence for Λ/M smaller than $1/\lambda$. Note the *log* dependence over M/m. Hopefully on can replace $1/\lambda$ with exponential $e^{1/\lambda}$.

Renormalization Group

- We integrate the Z_μ; in this way the theory becomes dimensionally non-renormalizable (non local Fermi theory, tentativo 1933) (we will discuss how to use the $1/k^2$)

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.

Renormalization Group

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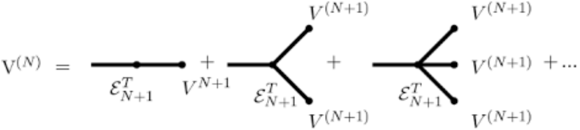

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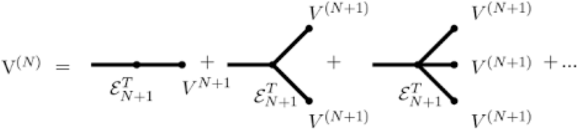

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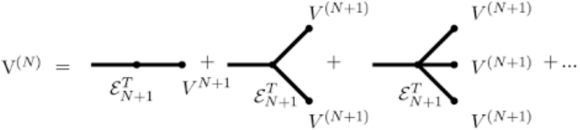

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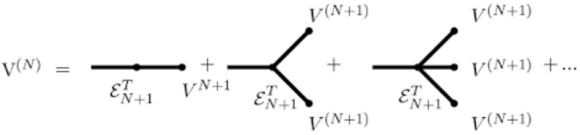

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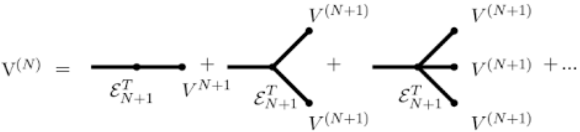

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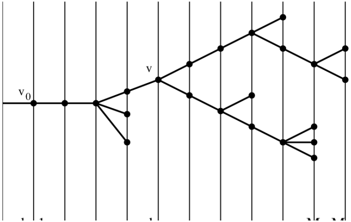
- The terms in $\mathcal{L}V$ correspond to mass and wave function and charge renormalizations; other possible rcc are vanishing by symmetry (unclear with lattice).

Trees

- Iterating the RG one gets a sequence of encapsulated truncated expectation which can be represented as a Gallavotti tree



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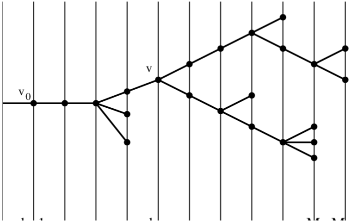

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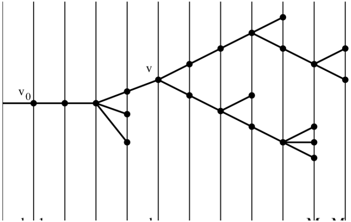

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- One possibility is to use a Battle-Brydges-Federbush formuls for fermionic expectations as sum over trees of propagators times fermionic determinants. f $\tilde{\Psi}(P_1) = \prod_{f \in P} \psi^{\varepsilon(f)}_{\mathbf{x}(f)}$

$$E^T(\tilde{\Psi}(P_1), .., \tilde{\Psi}(P_s)) = \sum_T (\prod_l g_l) \int dP_T(t) \det G_T(t)$$

where T if a set of lines connecting the cluster of points, l is a line of the tree and g_l is a propagator $g(x_l, y_l)$;G^T is a $n - s + 1 \times n - s + 1$ matrix whose elements are $t_{jj'} g(x_{ij}, x_{i'j'})$ formed by elements tot belonging to T; t are parameters between 0 and

The $g-2$

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- In the tree expansion we can distinguish the terms depending only on trees with no irrelevant λ and-points and terms with at least a λ endpoint

$$\hat{\mathfrak{J}}^\mu(p',p)=\hat{\mathfrak{J}}^{1,I}(p',p)+\hat{\mathfrak{J}}^{2,I}(p',p)$$

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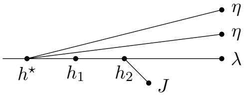
- The exponential is proportional to the minimal dimension of irrelevant term $D\leq -2$ up to an extra log to sum over scales.
- Therefore Γ^2 is given by the sum over n of terms bounded by $n\geq 2$

$$(\lambda^2\Lambda^2/M^2(\log\Lambda/m)^2)^n(m^2/\Lambda^2)=\lambda^2(m^2/M^2)(\lambda^2\Lambda^2/M^2)^{n-1}(\log m/M)^n$$

A careful bound on lowest order shows that there is no log and is close up to a small correction $O((M/\Lambda)^2)$ to the JW result.

Trees



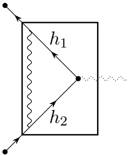


Figure 1: .

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Trees

- Bounded by γ^{-h_1} ($|g_\psi|_1$) times γ^{3h_2} ($|g_\psi|_\infty$) times $1/M^2$ (L_∞) of v times $\gamma^{2(h^*-h_2)}$ (effect of R) so that

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- The log can be improved distinguishing $|k|\leq M$ and $|k\geq M$ and using the $\frac{1}{k^2+M^2}$; constant at the end

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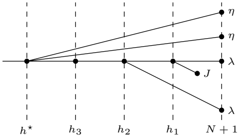

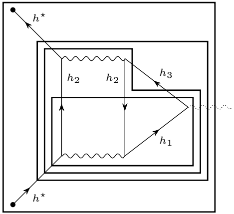

Figure 1: Fourth order diagram with the same structure as [**Karplus_Knoll_1950**], together with corresponding tree.

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Trees

$$|W| \leq \underbrace{\left(\frac{C\lambda^2}{M^2}\right)^2}_{\text{Endpoints}} \cdot \underbrace{\gamma^{-h^\star - h^\star}}_{h^\star \text{ cluster}} \cdot \underbrace{\gamma^{3h_3}}_{h_3 \text{ cluster}} \cdot \underbrace{\gamma^{3h_2 - h_2}}_{h_2 \text{ cluster}} \cdot \underbrace{\gamma^{-h_1}}_{h_1 \text{ cluster}} \cdot \underbrace{\gamma^{2(h^\star - h_3)}}$$

$$|W| \leq \left(\frac{C\lambda^2\Lambda^2}{M^2}\right)^2 \cdot \gamma^{2(h_1-N)+2(h_2-N)-2h^\star} \cdot \gamma^{(h_3-h^\star)\cdot(0-2)+(h_2-h_3)\cdot(-3-0)+(h_1-h_2)\cdot(-3-0)},$$

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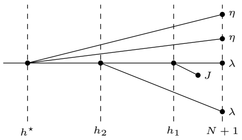
$$|W| \leq \frac{C^2\lambda^4\Lambda^2}{M^2} \cdot \frac{m^2}{M^2} \cdot \gamma^{-2h^\star} \cdot \gamma^{(h_3-h^\star)\cdot 0+(h_2-h_3)\cdot(-1)+(h_1-h_2)\cdot(-1)}.$$

Extracted the good factor m^2/m^2 and the dimensions are all negative so we can sum over scales (the general argument does not use graphs otherwise factorials are generated).

Trees



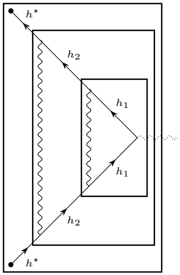


Figure 1: A fourth order diagram with two nested renormalization operators acting nontrivially.

Observables

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(Mastropietro PRD 2024, Mas+Bianchessi prep) $a_R^\Lambda = \bar{a}_{Z,1}(1 + R_{\lambda,N})$ *where, if* $\lambda^2 \leq C_2^{-1}(M^2/\Lambda^2)\log^{-2}(M/m)\log^{-2}(\Lambda/M)$, *then*

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- Physically important have $\log M/m$ instead of M/m in the third
- $g-2$ and Hall or graphene conductivity are used to compute α. One is a series, the other a single expression. Difference between irrelevant or marginal observables. In the latter the contribution from irrelevant term vanish (as in the anomaly).

Higher cut-offs

- Note the $(\lambda^2\Lambda^2/M^2)^n$ typical of a non-renormalizable theory; it produces the condition $\Lambda \sim 1/\lambda$. Perturbative renormalizability suggests that higher exponentially large cut-off in $1/\lambda$ could be reached (by decomposition of boson and fermion). However for gauge theories renormalizability requires properties whose non-perturbative counterpart is unclear. In particular:

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- In non chiral theories, like massive QED, this is achieved with a lattice regularization with Wilson fermions (for instance), but not with a momentum one.
- In the case of chiral theories like EW anomalies breaks the WI; such reduction is expected to happen only if anomalies cancel (only with physical values of lepton and quarks charges) (Bouchiat Hilyopulos Meyer 1972).

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- Such symmetries are however broken by the momentum decomposition introduced by the Wilsonian RG. One has to implement such approximate symmetries and WI in the RG flow.

Grassmann Integrals and Ising models

- Among the simplest places where Grassmann Integrals appear are 2s Dimer or Ising model
- The Ising model with Hamiltonian

$$H = -J \sum_{j=0,..,d} \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j}$$

  where $\Lambda \subset \mathbb{Z}^d$ is a square box of side L with pbc, $\sigma_{\mathbf{x}} = \pm 1$, Λ is a 2D square lattice, $\mathbf{e}_0 = (1, 0, 0, ..)$, $\mathbf{e}_1 = (0, 1, 0, ..)$,...,.
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- Exchange forces give a contribution $+J$ if nearest neighbor spins are in the same direction or $-J$ if they are in different directions. Simplified description of a magnet.
- According to the postulates of statistical mechanics, the thermodynamical averages are defined as, given $A = A(\underline{\sigma})$, $\underline{\sigma} = \{\sigma_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda}$, if $\Omega = \{\pm 1\}^\Lambda$, β inverse temperature

$$< A >_{L,\beta} = \frac{\sum_{\underline{\sigma}\in\Omega} e^{-\beta H(\underline{\sigma})} A}{\sum_{\underline{\sigma}\in\Omega} e^{-\beta H(\underline{\sigma})}}$$

Statistical mechanics, Spins and Dimers

Related to the Ising model is the dimer model. A dimer covering, or perfect matching, is a subset of edges which covers every vertex exactly once, that is every vertex is the end-point of exactly one edge. Partition function $Z_\Lambda = \sum_{M \in \mathcal{M}_\Lambda} \left[\prod_{b \in M} t_b \right]$ where $\mathcal{M}_\Lambda$ is the set of dimer coverings of Λ, b is a vertical or horizontal edge and t_b a weight. If the vertical edge have weight t_V and the horizontal t_o it is solvable, and the solution is very similar to the Ising one (but simpler)

- Then non interacting dimer model has partition function (L box Pbc)

$$Z = \sum_{M \in \mathcal{M}_\Lambda} \Big[\prod_{b \in M} t_b \Big]$$

where $\mathcal{M}_\Lambda$ is the set of dimer coverings of Λ.

- A solvable case corresponds to the case in which the vertical and horizontal weights are the same (more general tilted case in Kenyon et al).
- If $t_v = t_0 = 1$ Z is the number of dimer coverings (hard combinatorics) .
- Kasteleyn (1962): $Z = e^{GL^2/\pi}$, $G = 1 - 3^{-2} + 5^{-2} + ... = 0.915965...$ Catalan's constant, $G = \int_0^{\pi/2} d\mathbf{k}/(\pi)^2 \log 4(\cos^2 k_0 + \cos^2 k_1)$

Gaussian Grassman integrals

- ψ_i set of anticommuting variables $\{\psi_i, \psi_j\} = 0$; a Grassmann integral is

$$\int d\psi\psi = 1 \quad \int d\psi = 0$$

extension to functions by linearity (e.g. $\int d\psi e^{a\psi} = a$).

- If A is a $2n \times 2n$ antisyimmetric matrix

$$PfA = \int \prod_{i=1}^{2n} d\psi_i e^{\frac{1}{2}(\psi, A\psi)} = \frac{1}{2^n n!} \sum_p (-1)^p A_{p_1,p_2} ... A_{2n-1,2n}$$

with $\int \prod_{i=1}^{2n} d\psi_i \psi_1 ... \psi_{2n} = 1$ and zero otherwise. "Gaussian" integral.

- $(PfA)^2 = detA$

- $$Pf\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a \quad Pf\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = af - be + dc$$
- If we consider ψ_i^+, ψ_i^- independent then if B is a square natrix
$$\int \prod_{i=1}^{n} d\psi_i^+ d\psi_i^- e^{(\psi^-, B\psi^+)} = \det B$$

Solvable models and Gaussian Grassman integrals

- The partition function of the dimer model can be written as Gaussian Grassmann integral

$$Z = \int \prod_x d\psi_x e^{\sum_{\mathbf{x}} t_0 \psi_{\mathbf{x}} \partial_0 \psi_{\mathbf{x}} + i t_v \psi_{\mathbf{x}} \partial_i \psi_{\mathbf{x}}}$$

where $\psi_{\mathbf{x}} \partial_0 \psi_{\mathbf{x}} = \psi_{\mathbf{x}} \psi_{\mathbf{x}+e_0}$ (Pfaffian)

- Easy check $L = 2$,

$$\int d\psi_4 d\psi_3 d\psi_2 d\psi_1 (t_o^2 \psi_1 \psi_2 \psi_3 \psi_4 + i^2 t_v^2 \psi_1 \psi_3 \psi_2 \psi_4) = t_v^2 + t_o^2$$

- It is true in general. Kasteleyn lemma JMP1963). Non trivial; correlations are sum of positive objects, Gasmmann integarls are not necesasrely positive
- The gaussian integral can be computed (change of variables using Fourier)

Solvable models and Gaussian Grassman integrals

- We call $P(d\psi) = \int \prod_x d\psi_x e^{\sum_{\mathbf{x}} \psi_{\mathbf{x}}\partial_0\psi_{\mathbf{x}} + i\psi_{\mathbf{x}}\partial_i\psi_{\mathbf{x}}} / Z$ and

$$g(\mathbf{x},\mathbf{y}) = \int P(d\psi)\psi_{\mathbf{x}}\psi_{\mathbf{y}} = \frac{1}{L^2}\sum_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-i\sin k_0 + \sin k_1}$$

  For large distances it decays as $\frac{1}{|\mathbf{x}-\mathbf{y}|}$
- $\sum_{\mathbf{x}} \psi_{\mathbf{x}}\partial_0\psi_{\mathbf{x}} + i\psi_{\mathbf{x}}\partial_i\psi_{\mathbf{x}} = \frac{1}{V}\sum_{\mathbf{k}} \hat\psi_{-\mathbf{k}}\hat\psi_{\mathbf{k}}(e^{-ik_0} + ie^{-ik_1}) = \frac{1}{V}\sum_{\mathbf{k}} \hat\psi_{-\mathbf{k}}\hat\psi_{\mathbf{k}}(-i\sin k_0 + \sin k_1)$
- The dimer correlations $\int P(d\psi)\psi_{\mathbf{x}}\bar\psi_{\mathbf{x}}\psi_{\mathbf{y}}\bar\psi_{\mathbf{y}}$ and the truncated part is are $g(\mathbf{x},\mathbf{y})g(\mathbf{y},\mathbf{x})$; one can see that $g(\mathbf{x},\mathbf{y}) \sim \frac{1}{|\mathbf{x}-\mathbf{y}|}$

Solvable models and Gaussian Grassman integrals

- The Ising model admit a similar representation with

$$g(\mathbf{x},,\mathbf{y}) = \frac{1}{L^2}\sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \begin{pmatrix} i\sin k_0 + \sin k & m(\mathbf{k}) \\ m(\mathbf{k}) & i\sin k_0 - \sin k \end{pmatrix}^{-1}$$

where $m(\mathbf{k}) = C|\beta-\beta_c| + (\cos k_0 + \cos k - 2)$. At $\beta=\beta_c$ the denominator is vanishing at $\mathbf{k}=0$.

- $g(\mathbf{x},\mathbf{y})$ decays as $\frac{e^{-\xi|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}$ with $\xi = O(|\beta-\beta_c|)$: moreover the correlation $<\rho_\mathbf{x} : \rho_\mathbf{y}>_T$ is essentially given by $g(\mathbf{x},\mathbf{y})g(\mathbf{y},\mathbf{x})$.

Grassmann integral representation

- To fix ideas (similar for other systems): Hamiltonian for 1d metals (spinless) is

$$H = -\sum_{x=1}^{L-1}[\frac{1}{2}(a^+_x a^-_{x+1} + a^+_{x+1} a^-_x) - \mu a^+_x a^-_x] + \lambda \sum_{x,y} v(x-y) a^+_x a^-_x a^+_y a^-_y$$

- The correlations can be written in terms of a Grasmann integrals

$$e^{W(A,\phi)} = \int P(d\psi) e^{-V(\psi)+B(A,J,\psi)+\int d\mathbf{x}[\phi^+_\mathbf{x}\psi^-_\mathbf{x}+\psi^-_\mathbf{x}\psi^+_\mathbf{x}]} ,$$

where $\int d\mathbf{x}$ is a shortcut for $\sum_x \int_{-\beta/2}^{\beta/2} dx_0$, $P(d\psi)$ has propagator $\hat{g}(\mathbf{k}) = \frac{1}{-ik_0+\cos k-\mu}$

$$V(\psi) = \lambda \int d\mathbf{x} d\mathbf{y} \tilde{v}(\mathbf{x}-\mathbf{y})\psi^+_\mathbf{x}\psi^+_\mathbf{y}\psi^-_\mathbf{y}\psi^-_\mathbf{x} + \nu_C \int d\mathbf{x}\psi^+_\mathbf{x}\psi^-_\mathbf{x} + \nu \int d\mathbf{x}\psi^+_\mathbf{x}\psi^-_\mathbf{x}$$

with $\mathbf{e_1} = (0,1)$, $\tilde{v}(\mathbf{x}-\mathbf{y}) = \delta(x_0-y_0)v(x-y)$; finally $\nu_C = \lambda v(0)(g(0,0^-) - g(0,0^+))$ and $N = \int d\mathbf{x}\psi^+_\mathbf{x}\psi^-_\mathbf{x}$

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- The ν_C is to adjust the value of the tadpole.

Grassmann integrals

- Two Grassmann variables ψ^+, ψ^-; note that

$$\int d\psi^+ d\psi^- e^{-\psi^+ g^{-1} \psi^-} = \int d\psi^+ d\psi^- (1 - \psi^+ g^{-1} \psi^- + \ldots) = g^{-1}$$

- This is formally similar to, if $z = x + iy$, $dz = \frac{dxdy}{\pi}$

$$\int dz e^{-\bar{z} g^{-1} z} = g$$

Note however that the result is the inverse

- We can define the "gaussian Grassmann integration"

$$P(d\psi) = d\psi^+ d\psi^- e^{-\psi^+ g^{-1} \psi^-} / \int d\psi^+ d\psi^- e^{-\psi^+ g^{-1} \psi^-}$$

then

$$\int P(d\psi) \psi^- \psi^+ = g$$

Grassmann integrals

- If we have a set of variables $\hat{\psi}_{\mathbf{k}}$ where $\mathbf{k}$ are L^d variables then we can define the gaussian Grassmann integration

$$P(d\psi) = \frac{\prod_{\mathbf{k}} d\hat{\psi}^+_{\mathbf{k}} d\hat{\psi}^-_{\mathbf{k}} e^{-\frac{1}{L^d}\sum_{\mathbf{k}} \hat{\psi}^+_{\mathbf{k}} \hat{g}^{-1} \hat{\psi}^-_{\mathbf{k}}}}{\int \prod_{\mathbf{k}} d\hat{\psi}^+_{\mathbf{k}} d\hat{\psi}^-_{\mathbf{k}} e^{-\frac{1}{L^d}\sum_{\mathbf{k}} \hat{\psi}^+_{\mathbf{k}} \hat{g}^{-1} \hat{\psi}^-_{\mathbf{k}}}}$$

  then

$$\int P(d\psi) \hat{\psi}^-_{\mathbf{k}} \hat{\psi}^+_{\mathbf{k}} = L^d \hat{g}(\mathbf{k})$$

- In order to apply to the spin cases we can assume that $\mathbf{k} = (k_0, k_1)$ with $k_i = \frac{2\pi}{L}(n_i + 1/2)$, $n_i = 0, 1, ..., L$. If $\mathbf{x}$ is a point in Λ with periodic boundary conditions we can define $\psi^\pm_{\mathbf{x}} = \frac{1}{L^d}\sum_{\mathbf{k}} e^{\pm i\mathbf{k}\mathbf{x}} \hat{\psi}_{\mathbf{k}}$. Therefore

$$\int P(d\psi) \psi^-_{\mathbf{x}} \psi^+_{\mathbf{x}} = \frac{1}{L^d} \sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} g(\mathbf{k}) \equiv g(\mathbf{x}, \mathbf{y})$$

Grassmann integrals

- Fermionic expectations are expressed by the anticommutative Wick rule

$$E(\psi^-_{\mathbf{x}_1}...\psi^-_{\mathbf{x}_n}\psi^+_{\mathbf{y}_1}...\psi^-_{\mathbf{y}_n}) \equiv \int P(d\psi)\psi^-_{\mathbf{x}_1}...\psi^-_{\mathbf{x}_n}\psi^+_{\mathbf{y}_1}...\psi^-_{\mathbf{y}_n} = detG$$

  where $G_{i,j} = g(\mathbf{x}_i, \mathbf{y}_j)$. More explicitely $\sum_\pi (-1)^{\varepsilon_\pi} \prod_{i=1}^n g(\mathbf{x}_i, \mathbf{y}_{\pi(i)})$
- We want to compute objects like

$$\int P(d\psi)e^V = \sum_{n=0}^\infty \frac{1}{n!}E(V^n)$$

  where V is a non quadratic term like $V = \lambda \sum_{\mathbf{x}} \psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}\psi^+_{\mathbf{x}+e_1}\psi^-_{\mathbf{x}+e_1}$.
- If we use the Wick rule we get that the order n is expressed in terms of Feynman graphs

Grassmann integrals

- Formally the above expressions are very similar to the functional integrals seen for ϕ^4, that is $\int P(d\phi)e^{-\lambda \sum_{\mathbf{x}} \phi_{\mathbf{x}}^4}$. The Feynman graphs are similar up to the sign due to anticommutativiity
- There is however a basic difference. In ϕ^4 a perturbative expansion in λ is surely non convergent; this can be easily understood looking to the simples $d = 0$ example

$$\int_{-\infty}^{\infty} dz d\bar{z} e^{-|z|^2 - \lambda |z|^4}$$

- Instead the Grassmann integral $d = 0$ are analytic in λ

$$\int d\psi^+ d\psi^- e^{-\sum_{\omega=\pm} \psi_\omega^+ \psi_\omega^- - \lambda \psi_+^+ \psi_+^- \psi_-^+ \psi_-^-}$$

- In QFT the Grassmann integrals describe fermions (or spin as in this case) while inetgrals like ϕ^4 describe bosons.

- In order to bound $\int P(d\psi)e^V = \sum_{n=0}^\infty \frac{1}{n!}E(V^n)$ we can write $E(V^n)$ by Wick rule; each term is bounded by $||g||_\infty^{2n}(L^d)^n$, not suitable for taking the $L\to\infty$ limit; moreover the graphs are $n!^2$. Bad bound $\sum_{n=0}^\infty n!(||g||_\infty)^{2n}(L^d)^n$.
- To take care of the bad volume dependence one has to take into account the connection. The truncated expectations are defined in the following way, if X is a monomial in $\psi^\pm$

$$E^T(X;n) = \frac{\partial^n}{\partial\lambda^n}\log\int P(d\psi)e^{\lambda X}|_{\lambda=0}$$

- For instance $E^T(X;2) = E(XX) - E^2(X)$ and so on

Expectations and Gram bounds

- We get therefore the following representation

$$\log \int P(d\psi)e^V = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \log \int P(d\psi)e^{\lambda V}|_{\lambda=0} = \sum_{n=0}^{\infty} \frac{1}{n!} E^T(V;n)$$

- Taking the log one has an expansion in terms of graphs which are bounded by

$$L^d \max(|g|_1, |g|_\infty)^{2n} \frac{1}{n!} |\lambda|^n$$

  as we use the g to perform the sums over coordinates. Note that the L^d is uneffective as is an extensive quantity.
- The graphs divided by the volume are finite if $|g|_1, |g|_\infty$ are finite.
- Two problems:

Expectations and Gram bounds

- This bound for graphs is not finite for critical massless models $|g|_1$ is not finite as (ef fermions in 1d $g = O(1/|\mathbf{x}|)$).
- Outside from criticality (massive) the bound for g is finite. There is however a combinatorial problem. The graphs are $O(n!^2)$ so that the bound is

$$\frac{1}{L^d}|\log \int P(d\psi)e^V| \leq \sum_n n! \max(|g|_1, |g|_\infty)^{2n}|\lambda|^n$$

  which is still bad even outside criticality.
- Lesson: Feynman graphs cannot be really used for achieving convergence.

Expectations and Gram bounds

- The considerations do not use the anticommutativity of Grassmann variables (Pauli principle).
- There is however a key property of Grassmann integrals (first pointed put by Caianiello (1956) and rigorously used by Gawedsky and Kupianen (CMP 1985).
- The first key observation is that the expectation can be bounded by the Gram-Hadamard inequality

Expectations and Gram bounds

- Let us consider a matrix of the form $M_{\alpha,\beta} = (f_\alpha, g_\beta)$ where $\alpha, \beta = 1, .., n$, f_α, g_β are vectors in an Hilbert space, if $(,)$ is the scalar product; then

$$|detM| \leq \prod_{\alpha=1}^{n} ||f_\alpha|| ||g_\alpha||$$

where $||f||^2 = (f, f)$.

- The remarkable point in this bound is that is a C^n bound if the norms are finite, even if the terms is the determinant are $n!$.

Expectations and Gram bounds

- We apply to the expectations using that $E(\psi..) = detG$ with $G_{\alpha,\beta} = g(\mathbf{x}_\alpha; \mathbf{y}_\beta)$. Setting f_α as $g(\mathbf{x}_\alpha;)$ and $g_\alpha = \delta_{\alpha,-}$ one get

$$|E(V, ..V)| \leq L^{dn} \prod_{\alpha=1}^{2n} \sqrt{\sum_\beta |g(\mathbf{x}_\alpha; \mathbf{y}_\beta)|^2} \leq L^{dn} (\sqrt{2n})^{2n} (\sup |g|)^{2n}$$

- Therefore $\int P(d\psi) e^V = \sum_{n=0}^\infty \frac{1}{n!} E(V^n)$ is bounded by $\sum_{n=0}^\infty (||g||_\infty)^{2n} (L^d)^n |\lambda|^n$, there is a similar bound as found by Feynman graphs without $n!$.
- However the radius of convergence shrinks to zero as $L \to \infty$.

Expectations and Gram bounds

- Usually the $g(\mathbf{x}, \mathbf{y})$ can be written as

$$g(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{z}} \bar{A}_{\mathbf{x}}(\mathbf{z}) B_{\mathbf{y}}(\mathbf{z}) = (A, B)$$

with

$$A_{\mathbf{x}}(\mathbf{z}) = \frac{1}{V} \sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{z})} |\hat{g}(\mathbf{k})|^2$$

$$B_{\mathbf{x}}(\mathbf{z}) = \frac{1}{V} \sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{z})} 1/\hat{g}(\mathbf{k})^*$$

- Therefore

$$|E(V^n| \leq L^{dn} |(A, A)|^{2n} |(B, B)|^{2n}$$

so if $|(A, A)|$ is finite then $\int P(d\psi) e^V$ is intere (thanks to the $1/n!$). However in the case we are considering $(A; A)$ is not finite.

Expectations and Gram bounds

- We give some hint on how the bound $|detM| \leq \prod_{\alpha=1}^{n} ||f_\alpha||||g_\alpha||$, if $M_{\alpha,\beta} = (f_\alpha, g_\beta)$ is proved;
- If $g_\beta = e_\beta$ is an orthonormal set then $|detM|^2 \leq \prod_\alpha |f_\alpha|^2$; if the vectors belong to R^n the l.h.s. is the volume of the parallelepiped formed by f_α
- If g are orthogonal we can simply rescale $|detM|^2 \leq \prod_\alpha |f_\alpha|^2 |g_\alpha|^2$
- If g are not orthogonal, by gram-Smidt we can an orthogonal set. We choose $h_1 = g_1$, $h_2 = g_2 - \alpha g_1$ with h_2 orthogonal to h_1; if $\tilde{M}_{\alpha,\beta} = (f_\alpha, h_\beta)$ then $det(\tilde{M}) = detM$ by linearity, as determinant wth 2 equal lines is vanishing.
- $|det(\tilde{M})| \leq \prod_{\alpha=1}^{n} ||f_\alpha||||h_\alpha||$ and $||h_\alpha|| \leq ||g_\alpha||$ since is a projection.

- A convenient representation for the truncated expectation is the Brydges-Battle-Federbush formula if $\tilde{\Psi}(P_1) = \prod_{f \in P} \psi_{\mathbf{x}(f)}^{\varepsilon(f)}$

$$E^T(\tilde{\Psi}(P_1), .., \tilde{\Psi}(P_s)) = \sum_T (\prod_l g_l) \int dP_T(t) \det G_T(t)$$

- T if a set of lines connecting the cluster of points and forming a tree is the clusters are identified in a point
- l is a line of the tree and g_l is a propagator $g(x_l, y_l)$
- G^T is a $n - s + 1 \times n - s + 1$ matrix whose elements are $t_{jj'} g(x_{ij}, x_{i'j'})$ formed by elements tot belonging to T
- t are parameters between 0 and 1 and $dP_T(t)$ is a probability measure.

The BBF formula and the GK bound

- By the above formula one can prove that the partition function is given by a convergent series under suitable conditions on the propagator (Gawedki-Kupianinen, Lesniewski). Note that

$$\frac{1}{L^d}\log\int P(d\psi)^V=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\sum_{\mathbf{x}_1,\dots\mathbf{x}_n}E^T(\psi^+_{\mathbf{x}_1}\psi^-_{\mathbf{x}_1}\psi^+_{\mathbf{x}_1+e_1}\psi^-_{\mathbf{x}_1+e_1},..)$$

  and we use the BBF representation for E^T
- A crucial observation is that the G is also a Gram matrix; even with the t parameters the matrix elements can be written as a scalar product; therefore $|\det G_T(t)|\le\bar{C}^n$ if A, B are bounded by a constant.
- The integral over t is bounded by 1
- The sum $\frac{1}{V}\sum_{\underline{\mathbf{x}}}\prod_l g_l$ is bounded by $(|g|_1)^{n-1}$

The BBF formula and the GK bound

- T is a tree connecting the n clusters.
- The number of unlabeled trees is bounded by C^n
- The connection of clusters with the vertices of the trees produces an extra $n!$
- In conclusion we get a bound $\sum_n C^n \bar{C}^n (|g|_1)^{n-1}|\lambda|^n$ if the norms of he propagators are finite. Avoiding Feynman graph one can prove that the series are convergent
- In the interesting cases the $|g|_1$ is not bounded; we need a multiscale expansion

The BBF formula and the GK bound

- We start from the simple expectation $E(\psi(P_1)...\psi(P_r)) = detG = \int d\psi e^{-Q}$ and $Q = \sum_{j,j'=1}^{r} V_{jj'}$ and $V_{jj'} = \sum_{i\in P_j, i'\in P'_j} \psi^+_{x_{ij}} g(x_{ij}, x_{i'j'}) \psi^-_{x_{i'j'}}$ and G is the matrix with elements the $g(x,y)$
- We introduce set $X_1, ..X_r$ with $X_1 = \{1\}$ and X_{k+1} contained in X_k. We define

$$W(X_1, ..X_r; t_1, .., t_r) = \sum_l \prod_{k=1}^{r} t_k(l) V_l$$

where $t_k(l) = t_k$ if l cross the boundary of X_k and $= 1$ otherwise.

THE BBF FORMULA AND THE GK BOUND

- The idea is to write the simple expectation as a totally connected term which is the truncated expectation plus disconnected terms.
- By definition

$$W(X_1,t_1) = V_{1,1} + t_1\Big(\sum_{k\geq 2}(V_{1,k}+V_{k,1}) + \sum_{k,k'\geq 2} V_{k,k'}$$

and $W(X_1,0) = V_{1,1} + \sum_{k,k'\geq 2} V_{k,k'}$ is not connected; moreover $\partial_1 W(X_1,t_1) = \sum_{k\geq 2} V_{1,k}$ so that

$$e^{-Q} = \int_0^1 dt_1 \partial_1 e^{W_X(X_1,t_1)} + e^{W_X(X_1,0)} =$$

$$\int_0^1 dt_1 \sum_{k\geq 2} V_{1,k} e^{W(X_1,t_1)} + e^{W(X_1,0)}$$

The second term is non connected so is not contributing to E^T; the first term connect

The BBF formula and the GK bound

- We define $X_2 = (1, k)$ (different for each term in the above sum) and for definiteness $k = 2$; then $W(X_1, X_2, t_1, t_2) =$

$$V_{1,1} + V_{2,2} + t_1 t_2 \sum_{k\geq 3} V_{1,k} + t_1 V_{1,2} + t_2 \sum_{k\geq 3} V_{2,k} + \sum_{k,k'\geq 3} V_{k,k'}$$

and

$$W(X_1, X_2, t_1, 1) = W(X_1, t_1)$$

and the non disconnected term is

$$\sum_{l_1,l_2} \int_0^1 dt_1 dt_2 V_{l_1} V_{l_2} t_1(l_2) e^{W_X(X_1,X_2,t_1,t_2)}$$

where $t_1(l) = t_1$ if $l 0 1, k$ with $k \geq 3$; e $t_1(l) = 1$ se $l = 2, k$.

- The third step $X_3=(1,k,k')$ assume is $X_3=(1,2,3)$ and $W(X_1,X_2,X_3 t_1,t_2.t_3)=t_1 t_2 V_{1,3}+t_2 V_{2,3}+$

$$V_{1,1}+V_{2,2}+V_{3,3}+t_1 t_2 t_3\sum_{k\geq 4}V_{1,k}+t_2 t_3\sum_{k\geq 4}V_{2,k}+t_3\sum_{k\geq 4}V_{3,k}+\sum_{k,k'\geq 4}V_{k,k'}$$

  so that

$$\partial_3 W(X_1,X_2,X_3 t_1,t_2.t_3)=t_1 t_2\sum_{k\geq 4}V_{1,k}+t_2\sum_{k\geq 4}V_{2,k}+\sum_{k\geq 4}V_{3,k}$$

- The non-disconnected term is $\sum_{l_1,l_2,l_3}\int_0^1 dt_1 dt_2 dt_3 V_{l_1} V_{l_2} V_{l_3}[t_1(l_2)][t_1(l_3)t_2(l_3)]e^{W(X_1,X_2,X_3 t_1,t_2,t_3)}$
- The product of the t can be written as $t_{n'(l)}...t_{n(l)-1}$ where $n'(l)$ is the minimum k such that l crosses X_k and $n(l)$ the maximum. For instance if $r=4$ $l_2=(1,3)$ one has $n'=1$ ed $n=2$ so t_1; if $l_3=(1,4)$ one has $n'=1,n=3$ hence $t_1 t_2$; if $l_3=(2,4)$ one has $n'=2,n=3$ hence t_2.

The BBF formula and the GK bound

- The lines l form a tree T which connects P_i anchored to P_1. Is a tree and not a line as one can have in the third step $V_{1,j}$ o $V_{2,j}$, $j \geq 3$.
- Iterating and exchanging sums we have that the truncated expectation is

$$\int d\psi \sum_{r=1}^{s} \sum_{T} \sum_{X_1,..,X_{r-1}} \int_0^1 dt_1\, dt_2..dt_{r-1}$$

$$[\prod_{l\in T} V_l t_{n'(l)}...t_{n(l)-1}] e^{W(X_1,X_2,..,X_{r-1},t_1,t_2,..,t_{r-1})}$$

- When we integrate from the V one has the chain of g; the rest gives a determinant involving the fields not in T

The BBF formula and the GK bound

- Finally one can check that

$$\sum_{X_1,..,X_{r-1}} \int_0^1 dt_1\, dt_2..dt_{r-1} \prod_{l\in T} t_{n'(l)}...t_{n(l)-1} = 1$$

- Note that fixed T the ways of choosing X_l is $\prod b_i$ where b_k are the lines exiting from a X_k; if b_k lines are exiting X_{k+1} is defined choosing one of them. On the other hand $\int t^{b-1} = 1/b$.
- For instance if $r = 4$ $l_2 = (1,3)$, $l_3 = (1,4)$ one has $t_1^2 t_2$ and there are 3 lines exiting from X_1 and 2 from X_2; if $l_2 = (1,3)$, $l_3 = (2,4)$ one has $t_1 t_2$ and there are 2 lines exiting from X_1 and 2 from X_2.

Grassmann integral representation

- The source term is

$$B(A,\psi)=\int d\mathbf{x}\Big\{\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}A_0(\mathbf{x})+\frac{1}{2i}[\psi^+_{\mathbf{x}+\mathbf{e}_1}(e^{-iA_1(x)}-1)\psi^-_{\mathbf{x}}+\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}+\mathbf{e}_1}(e^{iA_1(x)}-1)]\Big\}$$

- The correlations are obtained from the derivatives $G^{2,1}_\mu(\mathbf{x},\mathbf{y},\mathbf{z})=\frac{\partial}{\partial A_{\mu,\mathbf{x}}}\frac{\partial^2}{\partial\phi^+_{\mathbf{y}}\partial\phi^-_{\mathbf{z}}}W|_0$, $G^2(\mathbf{y},\mathbf{z})=\frac{\partial^2}{\partial\phi^+_{\mathbf{y}}\partial\phi^-_{\mathbf{z}}}W|_0$, $G^{0,2}_{\mu,\nu}(\mathbf{x},\mathbf{y})=\frac{\partial^2}{\partial A_{\mu,\mathbf{x}}\partial J_{\nu,\mathbf{y}}}W|_0$

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- The derivative with respect to A_1 just produces the current $\psi^+_{\mathbf{x}+e_1}\psi^-_{\mathbf{x}}-\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}+e_1}$

1D metals

- The generic Hamiltonian for 1d metals (spinless) is

$$H = -\sum_{x=1}^{L-1}[\frac{1}{2}(a^+_x a^-_{x+1} + a^+_{x+1} a^-_x) - \mu a^+_x a^-_x] + \lambda \sum_{x,y} v(x-y) a^+_x a^-_x a^+_y a^-_y$$

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- If $v(x-y) = \delta_{x,y+1}$ the model reduces to the XXZ model which is solvable (Yang Yang 1966), in the limited sense that certain thermodynamical quantities can computed (but not the correlations). In particular $\cos\bar{\mu} = \lambda$, then $\bar{\nu} = \frac{\pi}{2\bar{\mu}} = 1 + \frac{2\lambda}{\pi} + O(\lambda^2)$, $K^{-1} = 2(1 - \frac{\bar{\mu}}{\pi})$, $\kappa = [2\pi(\pi/\bar{\mu} - 1)\sin\bar{\mu}]^{-1}$.

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- Several applications: Luttinger liquids (alternative to Fermi liquids), dimers, bulk-edge,vertex models...

Grassmann integral representation

- Grand-canonical correlations, for instance the 2-point function, $\mathbf{x} = (x_0.x)$
  $S_\lambda(\mathbf{x},\mathbf{y}) = \frac{\mathrm{Tr} e^{-\beta H} T a^+_{\mathbf{x}} a^-_{\mathbf{y}}}{\mathrm{Tr} e^{-\beta H}}$

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- If $\lambda = 0$ then $S_0(\mathbf{x}, \mathbf{y}) \equiv g(\mathbf{x}, \mathbf{y})$; if $\mathbf{x} - \mathbf{y} \neq (0, n\beta)$ then

$$g(\mathbf{x}, \mathbf{y}) = \lim_{N \to \infty} \frac{1}{L\beta} \sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{\chi(\gamma^{-N}|k_0|)}{-ik_0 + \cos k - \mu}$$

$(\mu = \cos p_F)$ $\mathbf{k} \in \mathcal{D}_{L,\beta}$ with $\mathcal{D}_{L,\beta} \equiv \mathcal{D}_L \times \mathcal{D}_\beta$,
$\mathcal{D}_L \equiv \{k = 2\pi n/L, n \in \mathbb{Z}, -[L/2] \leq n \leq [(L-1)/2]\}$,
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- $\mathbf{x}-\mathbf{y}\neq(0,n\beta)$ then the l.h.s is $[g(\mathbf{x},\mathbf{y})\to g(x_0-y_0,x-y)]$ $g(0,0^-)$ and in the $\beta,L\to\infty$ is $-p_F/\pi$ while the rhs is $(g(0,0^-)+g(0,0^+))/2$ in the limit $-p_F/\pi+1/2$.

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- For large distances $\cos(k+\omega p_F) - \cos p_F \sim \omega \sin p_F k'$ essentially linear dispersion relation $S_0(\mathbf{x},0) \sim \sum_{\omega=\pm1} \frac{e^{i\omega p_F(x-y)}}{i\omega x + v_F x_o}$ with $v_F = \sin p_F$. The occupation number at $\beta = \infty$ is $\chi(|k| \leq p_F)$ (discontinuous).

Transport coefficients

- If $\rho_{\mathbf{x}} = a^+_{\mathbf{x}} a^-_{\mathbf{x}}$ is the density then $G_{\rho\rho}(\mathbf{x}-\mathbf{y}) = \frac{\mathrm{Tr} e^{-\beta H} T \rho_{\mathbf{x}} \rho_{\mathbf{y}}}{\mathrm{Tr} e^{-\beta H}|_T}$; in the $\lambda = 0$ case it behaves as $(x_0 = 0)$ $\cos(2p_F x) \frac{1}{(2\pi^2) x^{2X_+}} + \frac{1}{(2\pi^2) x^2}$ with $X_+ = 1$ (critical exponent)

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- Is the Kubo response of density to a potential coupled to density

Transport coefficients

- In the same way the current is defined as $j_{\mathbf{x}} = \frac{1}{2i}(a^+_{\mathbf{x}+1}\ a^-_{\mathbf{x}} - a^+_{\mathbf{x}}\ a^-_{\mathbf{x}+1})$ and the diamagnetic current is defined as $j^D_{\mathbf{x}} = -\frac{1}{2i}(a^+_{\mathbf{x}+1}\ a^-_{\mathbf{x}} + a^+_{\mathbf{x}}\ a^-_{\mathbf{x}+1})$; we define

$$G_{jj}(\mathbf{p}) = \int_{-\beta/2}^{\beta/2} \sum_x e^{i\mathbf{p}\mathbf{x}} \frac{\mathrm{Tr} e^{-\beta H}\, T j_{\mathbf{x}} j_0}{\mathrm{Tr} e^{-\beta H}|_T} + \Delta$$

where $\Delta = \int_{-\beta/2}^{\beta/2} \sum_x e^{i\mathbf{p}\mathbf{x}} \frac{\mathrm{Tr} e^{-\beta H}\, T j^D_{\mathbf{x}}}{\mathrm{Tr} e^{-\beta H}|_T}$. The Drude weight is defined as

$$D = \lim_{p_0 \to 0} \lim_{p \to 0} \lim_{L,\beta \to \infty} \hat{G}_{j,j}(p_0, p)$$

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- The Kubo conductivity is related to the Drude weight ($\sim G_{jj}/p_0$ +Wick rotation, see Porta lecture; id $D \neq 0$ metal,infinite conductivity); is the response of current to an external field.

Ward Identities

- Lattice WI (can be derived by conservation fof current; we derive using Grassmann below) says

$$-ip_0 \hat{G}_{\rho,\rho} - i(1-e^{ip})\hat{G}_{j,\rho} = 0 \qquad - ip_0 \hat{G}_{\rho,j} - i(1-e^{ip})\hat{G}_{j,j} = 0$$

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- Therefore if $\hat G$ would be continuous than $D=\kappa=0$ would be vanishing; but they are not continuous by dimensional argument.
- Note that in the case of graphene $\hat G$ is indeed continuus so that the Drude weight is vanishing (but the conductivity is finite)

Anomalous behavior

- **Theorem 1** *For* λ *small enough there exists analytic functions* $p_F(\lambda) = p_F + O(\lambda)$, $\eta(\lambda) = a\lambda^2 + O(\lambda^3)$, $A(\lambda) = O(\lambda)$, $v_F = \sin p_F + O(\lambda)$ *such that*

$$S(\mathbf{x}, 0) = \sum_{\omega=\pm1} \frac{e^{i\omega p_F(\lambda)(x-y)}}{i\omega x + v_F x_o} \frac{(1 + A(\lambda))}{|\mathbf{x} - \mathbf{y}|^\eta} + O(\frac{1}{|\mathbf{x} - \mathbf{y}|^2})$$

Moreover the density and BCS correlations are $X_+ = 1 - \lambda \frac{\hat{v}(0) - \hat{v}(2p_F)}{\pi \sin p_F} + O(\lambda^2)$ *and* $X_- = 1 + \lambda \frac{\hat{v}(0) - \hat{v}(2p_F)}{\pi \sin p_F} + O(\lambda^2)$

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- One effect of the interaction is to modify the Fermi momentum (the position of the singularity)

Anomalous behavior

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$$S(\mathbf{x}, 0) = \sum_{\omega=\pm 1} \frac{e^{i\omega p_F(\lambda)(x-y)}}{i\omega x + v_F x_o} \frac{(1 + A(\lambda))}{|\mathbf{x} - \mathbf{y}|^\eta} + O(\frac{1}{|\mathbf{x} - \mathbf{y}|^2})$$

  Moreover the density and BCS correlations are $X_+ = 1 - \lambda \frac{\hat{v}(0) - \hat{v}(2p_F)}{\pi \sin p_F} + O(\lambda^2)$ *and* $X_- = 1 + \lambda \frac{\hat{v}(0) - \hat{v}(2p_F)}{\pi \sin p_F} + O(\lambda^2)$
- One effect of the interaction is to modify the Fermi momentum (the position of the singularity)
- The most dramatic effect is to produce an anomalous exponent (Luttinger liquid behavior) non trivial function of the coupling. The occupation number becomes continuous. The interacting theory has different properties with respect to the free.

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- The radius of convergence depend on v_F but it is possible to extend the analysis and indeed one can prove that the radius is independent on v_F.

Luttinger liquid behavior

- The exponents are function of all the microscopic detail. It was conjectured (Kadanov 1979, Haldane 1980) that certain relations hold, like $X_+ X_- = 1$; this can be realized at lowest order but to check to all order by the convergent expansion is impossible.

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- **Theorem 2** *Under the same condition as before there exists a function K and v_F such that*

$$X_+ = K, \quad X_- = 1/K, \quad \kappa = \frac{K}{\pi v_F} \quad D = \frac{v_F K}{\pi} \quad 2\eta = K + 1/K - 2$$

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- Recent extension to dimers and at the edge of topological (the transport coefficient depend on details except edge conductance $1/2\pi$), chiral LL.

Grassmann integral representation

- The correlations can be written in terms of a Grasmann integrals

$$e^{W(A,\phi)} = \int P(d\psi) e^{-V(\psi)+\nu_c N + B(A,J,\psi)+\int d\mathbf{x}[\phi^+_\mathbf{x}\psi^-_\mathbf{x}+\psi^-_\mathbf{x}\psi^+_\mathbf{x}]} ,$$

where $\int d\mathbf{x}$ is a shortcut for $\sum_x \int_{-\beta/2}^{\beta/2} dx_0$, $P(d\psi)$ has propagator $g(\mathbf{x}-\mathbf{y})$

$$V(\psi) = \lambda \int d\mathbf{x} d\mathbf{y} \tilde{v}(\mathbf{x}-\mathbf{y}) \psi^+_\mathbf{x}\psi^+_\mathbf{y}\psi^-_\mathbf{y}\psi^-_\mathbf{x} + \nu_C \int d\mathbf{x}\psi^+_\mathbf{x}\psi^-_\mathbf{x} + \nu \int d\mathbf{x}\psi^+_\mathbf{x}\psi^-_\mathbf{x}$$

with $\mathbf{e_1} = (0,1)$, $\tilde{v}(\mathbf{x}-\mathbf{y}) = \delta(x_0-y_0)v(x-y)$; finally $\nu_C = \lambda v(0)(g(0,0^-)-g(0,0^+))$ and $N = \int d\mathbf{x}\psi^+_\mathbf{x}\psi^-_\mathbf{x}$

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- The ν_C is to adjust the value of the tadpole.

Grassmann integral representation

- The source term is

$$B(A,\psi) = \int d\mathbf{x} \Big\{ \psi^+_{\mathbf{x}} \psi^-_{\mathbf{x}} A_0(\mathbf{x}) + \frac{1}{2i} [\psi^+_{\mathbf{x}+\mathbf{e}_1} (e^{-iA_1(x)} - 1)\psi^-_{\mathbf{x}} + \psi^+_{\mathbf{x}} \psi^-_{\mathbf{x}+\mathbf{e}_1} (e^{iA_1(x)} - 1)] \Big\}$$

- The correlations are obtained from the derivatives $G^{2,1}_\mu(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{\partial}{\partial A_{\mu,\mathbf{x}}} \frac{\partial^2}{\partial \phi^+_{\mathbf{y}} \partial \phi^-_{\mathbf{z}}} W|_0$, $G^2(\mathbf{y},\mathbf{z}) = \frac{\partial^2}{\partial \phi^+_{\mathbf{y}} \partial \phi^-_{\mathbf{z}}} W|_0$, $G^{0,2}_{\mu,\nu}(\mathbf{x},\mathbf{y}) = \frac{\partial^2}{\partial A_{\mu,\mathbf{x}} \partial J_{\nu,\mathbf{y}}} W|_0$

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- The derivative with respect to A_1 just produces the current $\psi^+_{\mathbf{x}+e_1}\psi^-_{\mathbf{x}}-\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}+e_1}$

Ward identites

- If we perform the change of variables $\psi^{\pm}_{\mathbf{x}} \to e^{i\alpha_x x}$ we get the identity

$$W(A,\phi) = W(A + d\alpha, \phi e^{-i\alpha})$$

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- This identity generates infinitely many identities between correlations; in particular

$$-ip_0 \hat{G}_0^{2,1}(\mathbf{p},\mathbf{k}) + (1 - e^{ip_1})\hat{G}_1^{2,1}(\mathbf{p},\mathbf{k}) = G^2(\mathbf{k}) - G^2(\mathbf{k}+\mathbf{p})$$

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- Similar identities for the current-current correlations

Renormalization Group analysis

- We replace in the Grassnann integral $\cos p_F$ with $\cos \bar{p}_F - \nu$. We fix $\nu(\lambda, \bar{p}_F)$ so that the singularity is $\bar{p}_F$; then we set $\cos p_F = \cos \bar{p}_F - \nu$ and we get $\bar{p}_F(\lambda, p_F)$.

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- The analysis is done as in the case of graphene;one introduces a partition of the unity using Gevray functions

$$1 = \chi_{uv}(\mathbf{k}) + \chi_{+}(\mathbf{k}) + \chi_{-}(\mathbf{k})$$

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- Noting that g^{uv} decay faster than any power with rate $O(1)$, we integrate it (for technical reasons related to the k_0 unboundeness we perform a multiscale also in uv)

$$\int P(d\psi) e^{V+B} = \int P(d\psi^\pm) e^{V^0(A,\phi,\psi^\pm)}$$

Renormalization Group analysis

- The integration of the $\psi_\pm$ has to be done via a renormalized expansion. We write $g_\pm = \sum_{h=-\infty}^{0} g^h_\pm = g^0 \pm + g^{\leq -1}_\pm$ with $g^h_\pm$ non vanishing in a sector of radies $O(\gamma^h)$ around the singularity

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- The scaling dimension is $2 - n^\psi/2 - n^A$ (and not $3 - n_\psi - n_A$) as for graphene); therefore the dimension of the quartic terms is 0 (not -1); the interaction is marginal.

Renormalization Group analysis

- We have to renormalize then also the quartic terms; $P(d\psi_\pm) = P(d\psi^{\leq -1})P(d\psi^0)$

$$\int P(d\psi^0) e^{V^0} = e^{\sum_{n=0}^{\infty} \frac{1}{n!}\mathcal{E}_0^T(V^0;n)} \equiv e^{V^{-1}}$$

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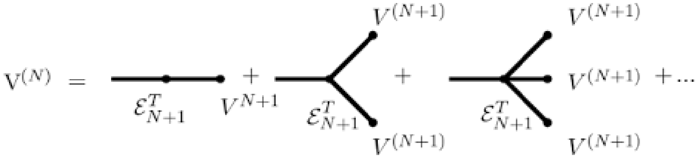
- Graphical representation $(N = -1)$



Renormalization

- We write $V^{-1} = \mathcal{L} V^{-1} + \mathcal{R} V^{-1}$ where $\mathcal{R} = 1 - \mathcal{L}$ and $\mathcal{L}$ act on the kernels $W_{n,m}\psi^n A^m$ as

$$\mathcal{L} W_{4,0}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = W_{4,0}(0,0,0)$$
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and so on.
- For $\phi = 0$

$$\mathcal{L} V^{-1} = l_{-1} \int d\mathbf{x} \psi^+_{+,\mathbf{x}} \psi^-_{+,\mathbf{x}} \psi^+_{-,\mathbf{x}} \psi^-_{-,\mathbf{x}} + z_{-1} \int d\mathbf{x} \psi^+_{\omega,\mathbf{x}} \partial \psi^-_{\omega,\mathbf{x}} +$$
$$n_{-1} \int d\mathbf{x} \psi^+_{\omega,\mathbf{x}} \psi^-_{\omega,\mathbf{x}} + Z^{0)}_{-1} \int A_0 \sum_\omega \psi^+_{\omega,\mathbf{x}} \psi^-_{\omega,\mathbf{x}} + Z^{1)}_{-1} \int A_1 \sum_\omega \omega \psi^+_{\omega,\mathbf{x}} \psi^-_{\omega,\mathbf{x}}$$

Renormalization

- We include the z_{-1} in the free part and we rescale the fields so that

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with

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- $P_{Z_{-1}}$ has propagator essentially given by $g^{\leq -1}/Z_{-1}$
- The new theory has a lower cut-off and different parameters.

Renormalization

- We proceed exactly in the same way writing $P_{Z_{-1}}(d\psi^{\leq -1}) = P_{Z_{-1}}(d\psi^{\leq -2})P_{Z_{-1}}(d\psi^{-1})$ and integrating ψ^{-1}.

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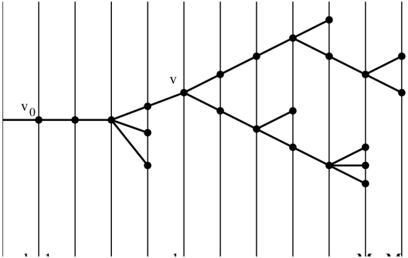
- $\log V^{(-2)} = \sum \frac{1}{n!} \mathcal{E}^T_{-1}(\tilde{\mathcal{L}} V^{-1} + \mathcal{R} V^{-1}; n)$.
- $V^{(-1)}$ is also sum of truncated expectations; hence one gets by linearity terms like $\mathcal{E}^T_{-1}(\tilde{\mathcal{L}} V^0; \mathcal{R}\mathcal{E}^T_0(V, m_1); ...; \mathcal{R}\mathcal{E}^T_0(V^0, m_n)$.

Trees

- Iterating the RG one gets a sequence of encapsulated truncated expectation which can be represented as a Gallavotti trees



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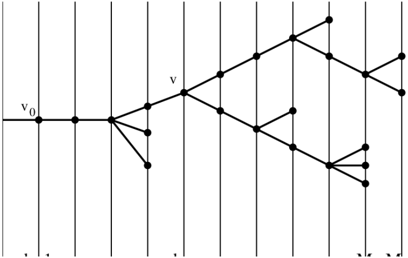

- Trees which can be constructed by joining a point r, the root, with an ordered set of $n \geq 1$ points, the endpoints of the unlabeled tree. The branching points are called non trivial vertices v; the first vertex is called v_0.

Trees

- The labeled trees are obtained associating $h \leq 0$; we introduce a family of vertical lines, labeled by an an integer taking values in $[h, 2]$, and if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the scale of v. The other intersection are the trivial vertices.

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- Cluster representation

Renormalization

- $V^h = \sum_n \sum_\tau V^h(\tau)$; then $V(\tau)$ defined inductively in the following way. $V^h(\tau) = \frac{1}{s!}\mathcal{E}^T_{h+1}(\bar V^{h+1}(\tau_1); ..; \bar V^{h+1}(\tau_1)$, where $\tau_1, .., \tau_s$ are the subtrees with root in v_0, v_0 the first vertex of the tree, and $\bar V^{h+1}(\tau)$ is equal to $\mathcal{R} V^{h+1}(\tau)$ if the subtree is not trivial, otherwise is one of the terms of $\mathcal{L} V^{h+1}$.

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- Iterating the procedure we get that the outcome is a product truncated epxectations summed over sets

$$\sum_{\{\mathbf{P}\}} \prod_v \frac{1}{S_v!} \mathcal{E}^T(P_{v_1}/Q_{v_1};..;P_{v_s}/Q_{v_s})$$

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- Expanding by wick rule the $\mathcal{E}^T$ one gets the renormalized Feynman graphs; one has to avoid this and use the BBF and determinant bounds. The $\mathcal{R}$ depend only on the P.

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- The outcome of this procedue is a series in the $\lambda, \lambda_{-1}, \lambda_{-2}, ..$; it can be expressed as sum over trees with n end-points and root h

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- lemma *If* $|\lambda_h| \leq \varepsilon$ *and* $\frac{Z_{h-1}}{Z_h} \leq e^{|\lambda_h|}$ *than* $||W|| \leq \gamma^{h(2-n_\psi/2-m)} C^n \varepsilon^n$ *for a suitable constant* n

Sketch

- Each truncated expectations is written as the BBF formula; the determinants are bounded by Gram inequality; one get from the integral of propagators and Gram bounds
$$C^n \prod_v \gamma^{-2(S_v-1)h_v} \gamma^{h_v n_v} \prod_v \gamma^{z_v(h_{v'}-h_v)}$$
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- The sum over the P_v is bounded extracting $\gamma^{-a|P_v|}$; one uses that

$$\sum_{\mathbf{P}} \prod_v \gamma^{-\alpha|P_v|} \leq \prod_v \sum_{p_v} \gamma^{-\alpha p_v} \frac{(p_1 + .. + p_s)!}{p_v!((p_1 + .. + p_s - p_v)!} \leq C^n$$

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Flow of the rcc

- Consistency requires that λ_h remain small. The rcc verify recursive equations called beta function; one has (in a simplified form)

$$\lambda_{h-1} = \lambda_h + a_h \lambda_h^2 + b_h \lambda_h^3 + \ldots \quad \frac{Z_{h-1}}{Z_h} = 1 + \tilde{a}_h \lambda_h^2 + \ldots \quad \frac{Z_h^A}{Z_h^A} = 1 + \tilde{b}_h \lambda_h^2 + \ldots$$

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- Suppose that this is true at all orders; then $\lambda_h \to \lambda + O(\lambda^2)$, $Z_h \sim \gamma^{\eta h}$ with $\eta = a\lambda^2 + ..$ and $\frac{Z_h^A}{Z_h} = 1 + O(\lambda)$

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- This would imply the presence of an anomalus dimension $G_2(\mathbf{x}) \sim \sum_h \frac{g^h(\mathbf{x})}{Z_h} \sim \frac{1}{|\mathbf{x}|^{1+\eta}}$

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- The flow of ν_h is controlled by a suitable choice of ν by a fixed point argument.

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where β^h_R contains only end-points λ_h and propagators $g^h_{\omega,R}$; therefore R^h contains an r^k and one can use the short memory property, that is a tree with a non trivial vertex k has an extra $\gamma^{\alpha(h-k)}$. Therefore $|R^h| \leq |\lambda|\gamma^{\theta h}$

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- Short memory follows from the bound $\prod_v \gamma^{\alpha(h_{v'}-h_v)} \leq \prod_v \gamma^{\alpha/2(h_{v'}-h_v)}\gamma^{\alpha/2(h-k)}$

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$$e^{W_N(J,\phi)} = \int P^{[h,N]}(d\psi) e^{V(\psi)+\sum_\omega \int d\mathbf{x} J_\omega \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}+(\phi^+,\psi^-)]}$$

where: $P^{(\leq N)}$ has relativistic propagator with u.v. cutoff $\frac{\chi_{h,N}}{-ik_0+\omega v k}$, and $V = \tilde\lambda \int dxdy v_0(x,y)\psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},+}\psi^+_{\mathbf{y},-}\psi^-_{\mathbf{y},-}$, $\hat v(\mathbf{p})$ exponentially decaying $e^{-(\gamma^{-K}p)^2}$.

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- This model has an extra symmetry $\psi_\omega \to e^{i\alpha_\omega} \psi_\omega$ which is only emerging in the original one
- If we can prove in this reference model that $\tilde{\lambda}_h = \tilde{\lambda} + O(\tilde{\lambda}^2)$ for any h, this implies, by a contradiction argument, that β_R^h is $O(\gamma^h)$; the same is true for the dominant part of the lattice model.

Schwinger-Dyson equation

- The idea is to use SD equations (connect λ_h with λ)

$$G^4(k_1,k_2,k_3,k_4) = \lambda g(k_4)\,G^2(k_3)\,G^{2,1}(\mathbf{k}_1-\mathbf{k}_2;\mathbf{k}_1)\,v(\mathbf{k}_1-\mathbf{k}_2) +$$
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- Let us consider the first term; if $\frac{Z_h^A}{Z_h}=1+O(\lambda_h)$ then $\lambda_h=\lambda+O(\lambda^2)$. But we need cancellarions to say $\eta_1=\eta$; the same more difficult for the second term

A FAMOUS ERROR

- The idea is to combine WI to get cancellations. If one proceeds formally one gets wrong results; the same errors done in different context and different models (Thirring (1958) corrected by Johnson (1961); (Luttinger (1963) corrected by Mattis and Lieb (1965):...); for the historical debate on anomalies see also Adler arxiv 2004. The SD equation for the 2-point function is

$$D_\omega(k) G^{R,2}_\omega(k) = g_\omega(k) + \int dp v(p) G^{R,2,1}_{\omega,-\omega}(p,k)$$

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- Error in exchanging limits! Mathematics is useful after all...

Ward Identies

- Taking into account the cut-offχ one gets after the change of variables, if $D_\omega = -ip_0 + \omega p$

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- $\Delta_{\omega,\omega'}$ is a correction term; it is similar to $G^{2,1}_\omega$ with $\int d\mathbf{k}\, d\mathbf{p}\, C(\mathbf{k},\mathbf{p})\psi^+_{\mathbf{k}+\mathbf{p}}\psi^-_{\mathbf{k}}$ and

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- Is the correction vanishing in the $N \to \infty$ limit? no...

Renormalization Group

- First we have to see to analyze the correlations, and then extend to the correction.
- We want to send $N \to \infty$. If the range of the potential is $O(\gamma^K)$, one can distinguish between scales $\leq K$ (ir) and $\geq K$) (uv). The scaling properties of the propagator are the same but in the second one can use the non locality of the interaction.

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- In the uv region the term $\psi\psi$ is dimensionally bounded by γ^k; instead

$$\| W_\omega^{(0;2)(k)} \| \leq |\lambda_\infty| \| v_K \|_{L^\infty} \| W_{-\omega;\omega}^{(1;2)(k)} \| \sum_{j=k+1}^{N} \| g_\omega^{(j)} \|_{L^1} \leq$$

$$\leq \frac{c_1}{1-\gamma^{-1}} \gamma^{2K} C |\lambda_\infty| \gamma^{-k} \leq C_1 |\lambda_\infty| \gamma^k \gamma^{-2(k-K)}$$

Renormalization Group

- In the same way the temrm $A\psi\psi$ can be written as sum of several terms; for instance

$$|W^{(1;2)(k)}_{a,\omega';\omega}\| \leq |\lambda_\infty|\|v_K\|_{L^\infty}\|W^{(2;2)(k)}_{\omega',-\omega;\omega}\|\sum_{j=k+1}^{N}\|g^{(j)}_\omega\|_{L^1} \leq CC_1|\lambda_\infty|\gamma^{-2(k-K)}\;.$$

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- one uses that $\int g^2_\omega(k) = 0$

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- Decompose the three propagators g_ω into scales and then bound by the L^∞ norm the propagator of lowest scale, while the two others are used to control the integration over the inner space variables through their L^1 norm. Hence we get:

$$|\lambda_\infty|^2 \| v_K \|_{L^\infty} \| v_K \|_{L^1} \| W^{(2;2)(k)} \sum_{k+1 \le i' \le j \le i \le N} \| g_\omega^{(j)} \|_{L^1} \| g_\omega^{(i)} \|_{L^1} \| g_\omega^{(i')} \|_{L^\infty}$$

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- and $\gamma^{2K} \sum_{k+1\leq i'\leq j\leq i\leq N} \gamma^{-k}\gamma^{i'}\gamma^{-i}\gamma^{-j} \leq \gamma^{2K}\gamma^{-k} \sum_{k+1\leq i\leq N} \gamma^{-i}(j-k) \leq C\gamma^{-2(k-K)}$

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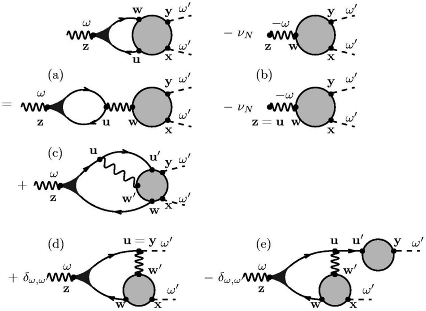
- The integration of the infrared scales can be done as before; the dominant part of the beta function is the same.

Emerging Ward Identities



$$\lim_{N\to\infty} \Delta_\omega = \lambda_\infty / 4\pi D_{-\omega}(\mathbf{p}) G^{2,1}_{-\omega}$$

The correction and, for gains due to the long range interaction, the contrubution of irreducible terms is vanish as $N \to \infty$.

Anomaly

- $$\lambda \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{C_{N;\omega}(\mathbf{k}, \mathbf{k}-\mathbf{p})}{D_{-\omega}(\mathbf{p})} \hat{g}_\omega^{(\le N)}(\mathbf{k}) \hat{g}_\omega^{(\le N)}(\mathbf{k}-\mathbf{p}) =$$
$$= \quad -\lambda \frac{D_\omega(\mathbf{p})}{D_{-\omega}(\mathbf{p})} \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{u_0\big(\gamma^{-N}|\mathbf{k}-\mathbf{p}|\big)\chi_0\big(\gamma^{-N}|\mathbf{k}|\big)}{D_\omega(\mathbf{k}-\mathbf{p})D_\omega(\mathbf{k})} +$$
$$+\lambda \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{\chi_0\big(\gamma^{-N}|\mathbf{k}|\big) - \chi_0\big(\gamma^{-N}|\mathbf{k}-\mathbf{p}|\big)}{D_\omega(\mathbf{k}-\mathbf{p})D_{-\omega}(\mathbf{p})}$$

$u_0 = 1 - \chi$ First contribution in the r.h.s. vanishes by the symmetry $\hat{g}_\omega(\mathbf{k}) = -i\omega \hat{g}_\omega(\mathbf{k}^*)$, $\mathbf{k}^* = (-k_0, k)$.

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- Remaining term
$$-\frac{\lambda}{2} \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{\chi_0'(|\mathbf{k}|)}{|\mathbf{k}|} = -\frac{\lambda}{4\pi} \int_1^\infty d\rho\, \chi_0'(\rho) = \frac{\lambda}{4\pi}\,.$$

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- The term with Δ at k and n need no renormalization;as there is an extra γ^{h-k};
- This implies that Z_h^A/Z_h is essentially h independent. A similar argument for the second term in the SD equation.

Luttinger relations

- We still use the reference model without ir cut-off with parameters to fix the amplitude.
- $$e^{W_N(J,\phi)} = \int P_Z^{[\leq N]}(d\psi) e^{V(\sqrt{Z}\psi)+\sum_{j=1}^2 Z^{(j)}(J^{(j)},\rho^{(j)})+Z[(\psi^+,\phi^-)+(\phi^+,\psi^-)]}$$

  where: $P_Z^{(\leq N)}$ has relativistic propagator with u.v. cutoff, and V is a non-local quartic interaction with kernel $v_0(x,y)$ and coupling $\tilde{\lambda}$.

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- If we call $\tilde{\lambda}_h$ the analogues of λ_h, it is possible to choose $\tilde{\lambda} = \tilde{\lambda}(\lambda) = \lambda + O(\lambda^2)$ so that $\tilde{\lambda}_{-\infty} = \lambda_{-\infty}$ and $|\lambda_h - \tilde{\lambda}_h| \leq C\lambda^2\gamma^h$. This is a consequence of short memory and the fact that $\beta_h^L - \beta_h = O(\lambda^2\gamma^h)$.

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- Similarly we choose $\tilde{Z}$, $\tilde{Z}^{(1)}$ and $\tilde{Z}^{(2)}$ so that $\tilde{Z}_h/\tilde{Z}_{h-1}$ is asymptotically equal to Z_h/Z_{h-1} and so on.

Relation

- As a consequence if $\hat{G}^R_{\rho,\rho}$ is the analogue of $\hat{G}_{\rho,\rho}$ for the reference model then

$$\hat{G}_{\rho,\rho}(p) = \hat{G}^R_{\rho,\rho}(p) + A(p)$$

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- Similar relations for $\hat{G}_{,j}(p)$
- The advantage is that the reference model has two set of Ward Identies associated to the chiral symmetry,

WI for the reference model

- If $D_\omega = -ip_0 + \omega cp$

$$D_\omega(\mathbf{p})\,\hat{G}^R_{\omega,\omega}(\mathbf{p}) - \tau\ \hat{v}_0(\mathbf{p})\,D_{-\omega}(\mathbf{p})\,\hat{G}^R_{-\omega,\omega}(\mathbf{p}) + \frac{1}{4\pi cZ^2}D_{-\omega}(\mathbf{p}) = 0\ ,$$
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- $\hat{v}_0(\mathbf{p}) = 1 + O(\mathbf{p})$, we get: $\hat{G}^R_{\omega,\omega}(\mathbf{p}) = -\frac{1}{Z^2}\frac{1}{4\pi c(1-\tau^2)}\frac{D_{-\omega}(\mathbf{p})}{D_\omega(\mathbf{p})}$ and $\hat{G}_{-\omega,\omega}(\mathbf{p}) = -\frac{1}{Z^2}\frac{\tau}{4\pi c(1-\tau^2)}$ so that

$$\hat{G}^{0,2}_{th,\rho,\rho} = -\frac{1}{4\pi cZ^2}\frac{(Z^{(3)})^2}{1-\tau^2}\left[\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} + \frac{D_+(\mathbf{p})}{D_-(\mathbf{p})} + 2\tau\right] + O(\mathbf{p})\ .$$

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- The $\hat{R}_{\rho,\rho} = \hat{G}^R_{\rho,\rho} + A(\mathbf{p})$; the $A(\mathbf{p})$ is countinuous. The above expression is explicit but depend on $A(0), \tau, Z$, etc which are complicate series of λ.

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- We notice now that the vertex function in reference is equal t the lattice up to smaller term $O(k)$; therefore

$$-ip_0\ \hat{G}^{2,1}_\rho(\mathbf{k}'+\mathbf{p}^\omega_F, \mathbf{k}'+\mathbf{p}+\mathbf{p}^\omega_F) + \omega p\frac{Z^{(3)}}{\tilde{Z}^{(3)}} v_F \hat{G}^{2,1}_j(\mathbf{k}'+\mathbf{p}^\omega_F, \mathbf{k}'+\mathbf{p}+\mathbf{p}^\omega_F) =$$

$$= \frac{Z^{(3)}}{(1-\tau)Z}\left[\hat{G}^2(\mathbf{k}'+\mathbf{p}^\omega_F) - \hat{G}^2(\mathbf{k}'+\mathbf{p}+\mathbf{p}^\omega_F)\right][1+O(\kappa^\theta)] ,$$

so that $\frac{Z^{(3)}}{(1-\tau)Z} = 1$ and $\frac{Z^{(3)}}{\tilde{Z}^{(3)}} v_F = 1$

WI for the reference model

- By comparing with the lattice WI we get, if $K = \frac{1-(\lambda_\infty/4\pi v_F)}{1+(\lambda_\infty/4\pi v_F)}$ then

$$G_{\rho\rho} = \frac{K}{\pi v_F} \frac{v_F^2 p^2}{p_0^2 + v_F^2 p^2} + R(\mathbf{p}) \ ,$$

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- By this $\kappa=Kv_F/\pi$,

Drude weight

- $$\hat{D}(\mathbf{p}) = \frac{-1}{4\pi v_F Z^2} \frac{(\tilde{Z}^{(3)})^2}{1-\tau^2} \left[\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} + \frac{D_+(\mathbf{p})}{D_-(\mathbf{p})} - 2\tau \right] + A_{J,J}(0) + \Delta + O(\mathbf{p})$$

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- $$\hat{D}(\mathbf{p}) = \frac{v_F}{\pi} K \frac{p_0^2}{p_0^2 + v_F^2 p^2} + H(\mathbf{p})$$