Quantum Systems at The Brink: Critical Potentials and Finite Moments

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Outline

- Eigenstates at threshold
- Setting of problem
- Critical potential
- Absence and existence result
- Momentum result
- Examples

Introduction

- Existence of bound states is directly related to stability of the quantum system
- Studied systems are described by

$$H = -\Delta + V_+ - V_-$$

- Eigenvalues converging to the threshold of the essential spectrum can persist or vanish

- Existence of bound states at the essential spectrum threshold is related to the asymptotic behaviour of the potential

Known Results

- Stabilisation of nucleus by Coulombic repulsion barrier (Gamov 1930)
- Existence and absence of bound state at the threshold for H^- (due to Hoffmann-Ostenhof, Hoffmann-Ostenhof and Simon $\sim$1980)
- Existence of bound states for repulsive Coulomb interaction (due to Bolle, Gesztesy, Schweiger 1985)
- Absence result: any continuous potential decaying faster than
$$V_+ \leq \frac{1}{|x|^2}\left(\frac{3}{4} + \frac{1}{\ln(|x|)}\right)$$
in $\mathbb{R}^3$ do not produce zero energy eigenstates (Benguria,Yarur 1990)
- Estimates on Green's functions in $\mathbb{R}^3$ for slowly decaying potentials
$$V > \frac{3}{4|x|^2} \ \text{existence} \quad \text{and} \quad V \leq \frac{3}{4|x|^2} \ \text{absence}$$
(Gridnev, Garcia 2007)

Setup

Setting of Problem

- H is a self-adjoint realization of $P = -\Delta + V$ in $L^2(\mathbb{R}^d)$
- Consider (weak) solutions of the Schrödinger equation

$$-\Delta u(x) + V(x)u(x) = 0$$

Assumptions

- $V \in K_d^{loc}(\mathbb{R}^d)$ i.e. $V\mathbb{I}_{B_R(0)} \in K_d(\mathbb{R}^d)$ for all R
- V_- relatively form small with respect to $-\Delta + V_+$, i.e., there exist $0 \leq a < 1$ and $b \geq 0$ such that

$$\langle \psi, V_- \psi \rangle = \| V_-^{1/2} \psi \|^2 \leq a(\| \nabla \psi \|^2 + \| \sqrt{V_+} \psi \|^2) + b \| \psi \|^2$$

- $\lim_{|x| \to \infty} V = 0$

Goal

- Find sufficient and necessary conditions on V for absence and existence of a bound state at the threshold
- Find sufficient and necessary conditions on V for finiteness of $\langle \psi | |x|^c | \psi \rangle$ for a threshold state

Critical Potential

Critical potential

The potential V is *critical* if the Schrödinger operator H has spectrum $\sigma(H) = \sigma_{ess}(H) = [0, \infty)$ and for all nontrivial compactly supported potentials $W \geq 0$ which are infinitesimally form bounded with respect to $-\Delta + V_+$ the family of operators $H_\lambda = H - \lambda W$ has essential spectrum $\sigma_{ess}(H_\lambda) = [0, \infty)$ and a negative energy bound state for all $\lambda > 0$.

One often says that a Schrödinger operator with a critical potential has a *virtual level* (at zero energy).

- Virtual level is either *a bound state* or
- It is *a zero energy resonance*.

Examples

$d = 1, 2$: $V = 0$ is critical potential
$d = 1, 2$: any nontrivial potential needs to change sign and $\int V > 0$ for $V \in L^1$
$d \geq 3$: Hardy potential $V = -\frac{(d-2)^2}{4} \frac{1}{x^2}$ is critical potential

Absence condition

H is a self-adjoint realization of $P = -\Delta + V$ in $L^2(\mathbb{R}^d)$

Assumptions

- $V \in K_d^{loc}(\mathbb{R}^d)$
- $\lim_{|x|\to\infty} V = 0$
- $V_- = \sup(-V, 0)$ is relatively form small w.r.t. $-\Delta + V_+$

Theorem (Absence of a zero energy ground state)

The potential V is as above and $\sigma(H) = [0, \infty)$. If for some $m \in \mathbb{N}_0$ and $R > e_m$

$$V(x) \leq \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2} \sum_{j=1}^{m} \prod_{k=1}^{j} \ln_k^{-1}(|x|)$$

for all $|x| \geq R$, then zero is not an eigenvalue of the Schrödinger operator H.

Notation: $e_1 = e$, $e_j = e^{e_{j-1}}$, $\ln_1(x) := \ln(|x|)$ and $\ln_{j+1} := \ln(\ln_j(x))$ for $j \in \mathbb{N}_0$.

Existence condition

H is a self-adjoint realization of $P = -\Delta + V$ in $L^2(\mathbb{R}^d)$

Assumptions

- $V \in K_d^{loc}(\mathbb{R}^d)$
- $\lim_{|x|\to\infty} V = 0$
- $V_- = \sup(-V, 0)$ is relatively form small w.r.t. $-\Delta + V_+$
- V is critical

Theorem (Existence of a zero energy ground state for critical potentials)

The potential V is as above. If for some $m \in \mathbb{N}_0$, $\epsilon > 0$, and $R > e_m$

$$V(x) \geq \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2} \sum_{j=1}^{m} \prod_{k=1}^{j} \ln_k^{-1}(|x|) + \frac{\epsilon}{|x|^2} \prod_{k=1}^{m} \ln_k^{-1}(|x|)$$

for all $|x| \geq R$, then zero is an eigenvalue of H.

Notation: $e_1 = e$, $e_j = e^{e_{j-1}}$, $\ln_1(x) := \ln|x|$ and $\ln_{j+1} := \ln(\ln_j(x))$ for $j \in \mathbb{N}_0$.

Low order expansions

Considering $n = 0, 1, 2$ we get

- absence:

$$V_0(x) \leq \frac{d(4-d)}{4|x|^2}$$

$$V_1(x) \leq \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2 \ln|x|}$$

$$V_2(x) \leq \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2 \ln|x|} + \frac{1}{|x|^2 \ln|x| \ln(\ln|x|)}$$

- existence:

$$V_0(x) \geq \frac{d(4-d)+\epsilon}{4|x|^2}$$

$$V_1(x) \geq \frac{d(4-d)}{4|x|^2} + \frac{1+\epsilon}{|x|^2 \ln|x|}$$

$$V_2(x) \geq \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2 \ln|x|} + \frac{1+\epsilon}{|x|^2 \ln|x| \ln(\ln|x|)}$$

for any $\epsilon > 0$

Conditions for Unboudness of $|x|^c$-moments

H is a self-adjoint realization of $P = -\Delta + V$ in $L^2(\mathbb{R}^d)$

Assumptions

- $V \in K_d^{loc}(\mathbb{R}^d)$
- $\lim_{|x|\to\infty} V = 0$
- $V_- = \sup(-V, 0)$ is relatively form small w.r.t. $-\Delta + V_+$

Theorem (Unboudness of $\langle\psi||x|^c|\psi\rangle$)

Let the potential V be as above and $\sigma(H) = [0, \infty)$. Then zero is not an eigenvalue of the Schrödinger operator H with an eigensate such that $\langle\psi||x|^c|\psi\rangle < \infty$ provided that

$$V(x) \leq \frac{d(4-d) + c^2 + 4c}{4|x|^2} + \frac{1 + c/2}{|x|^2 \ln|x|} + \frac{1 + c/2}{|x|^2 \ln|x| \ln(\ln(|x|))}$$

for all $|x|$ large enough.

Conditions for Finiteness of $|x|^c$-moments

H is a self-adjoint realization of $P = -\Delta + V$ in $L^2(\mathbb{R}^d)$

Assumptions

- $V \in K_d^{loc}(\mathbb{R}^d)$
- $\lim_{|x|\to\infty} V = 0$
- $V_- = \sup(-V, 0)$ is relatively form small w.r.t. $-\Delta + V_+$
- V is critical

Theorem (Existence of a zero energy ground state with $\langle\psi||x|^c|\psi\rangle < \infty$)

Let the potential V be as above and $\epsilon, c > 0$. Then zero is an eigenvalue of H such that its eigenfunction satisfy $\langle\psi||x|^c|\psi\rangle < \infty$ provided that one of these conditions hold

$$V_0(x) \geq V_0(x) \geq \frac{d(4-d)+c^2+4c}{4|x|^2} + \frac{\epsilon(2c+4)}{4|x|^2}$$

$$V_1(x) \geq \frac{d(4-d)+c^2+4c}{4|x|^2} + \frac{1+c/2}{|x|^2 \ln|x|} + \frac{\epsilon(2c+4)}{4|x|^2 \ln|x|}$$

$$V_2(x) \geq \frac{d(4-d)+c^2+4c}{4|x|^2} + \frac{1+c/2}{|x|^2 \ln|x|} + \frac{1+c/2+\epsilon(c/2+1)}{|x|^2 \ln|x| \ln(\ln(|x|))}$$

for all $|x|$ large enough.

Critical potential

Lemma

Let H be as above. Moreover let H have a ground state with the eigenvalue $E_0 = 0$. Then V must be a critical potential.

The converse does not hold.

Counter example

The potential $-\frac{(d-2)^2}{4|x|^2}$ in $\mathbb{R}^d$ with $d \geq 2$. To see this consider the well-known Hardy inequality

$$\frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} \mathrm{d}x \leq \int_{\mathbb{R}^d} |\nabla \psi(x)|^2 \mathrm{d}x$$

where $\psi \in H_0^1(\mathbb{R}^d)$.

Note that the Hardy inequality is sharp and $-\Delta - \frac{(d-2)^2}{4|x|^2}$ does not have a zero energy ground state. In particular this reproves that there is no L^2-minimizer for the Hardy inequality.

Model Potential

$$V_{\alpha,d}(x) := \frac{4\alpha^2 - (d-2)^2}{4\big(1+|x|^2\big)} + \frac{1-(\alpha+d/2)^2}{\big(1+|x|^2\big)^2}$$

Theorem

Let $d \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and $H_{\alpha,d} = -\Delta + V_{\alpha,d}$ be the self-adjoint Schrödinger operator with potential $V_{\alpha,d}$ given above. Then

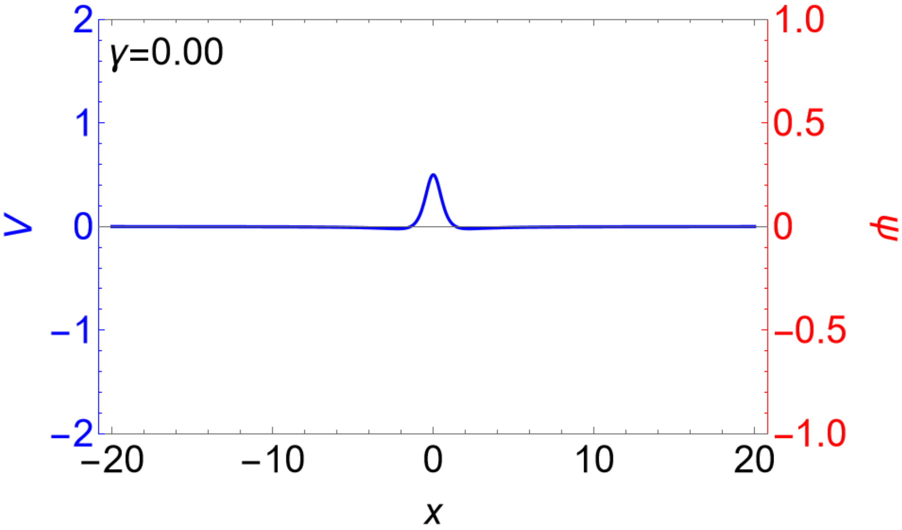
a) $\sigma(H_{\alpha,d}) = \sigma_{ess}(H_{\alpha,d}) = [0,\infty)$.

b) *For all $\alpha \geq 0$ the potential $V_{\alpha,d}$ is critical, that is, the Schrödinger operator $H_{\alpha,d}$ has a virtual level.*

c) *Zero is not an eigenvalue of $H_{\alpha,d}$ when $0 \leq \alpha \leq 1$. For $\alpha > 1$ zero is an eigenvalue. The zero energy resonance for $0 \leq \alpha \leq 1$, respectively ground state for $\alpha > 1$, is given by $\psi_{\alpha,d}(x) = (1+|x|^2)^{(2-d)/4-\alpha/2}$.*

d) *For $\alpha < 0$, the potential $V_{\alpha,d}$ is subcritical, and zero is neither an eigenvalue nor a resonance.*

Model Potential 1D



Concluding Remarks

- Calculation of asymptotic expansion for critical potential
- Phase transition for $d = 4$
- Absence result/unboundness of $|x|^c$

$$V_0(x) \leq \frac{d(4-d)+c^2+4c}{4|x|^2}$$
$$V_1(x) \leq \frac{d(4-d)+c^2+4c}{4|x|^2} + \frac{1+c/2}{|x|^2 \ln|x|}$$
$$V_2(x) \leq \frac{d(4-d)+c^2+4c}{4|x|^2} + \frac{1+c/2}{|x|^2 \ln|x|} + \frac{1+c/2}{|x|^2 \ln|x| \ln(\ln(|x|))}$$
$$\vdots$$

- Existence/boundness of $|x|^c$

$$V_0(x) \geq \frac{d(4-d)+c^2+4c+\epsilon}{4|x|^2}$$
$$V_1(x) \geq \frac{d(4-d)+c^2+4c}{4|x|^2} + \frac{1+c/2+\epsilon}{|x|^2 \ln|x|}$$
$$V_2(x) \geq \frac{d(4-d)+c^2+4c}{4|x|^2} + \frac{1+c/2}{|x|^2 \ln|x|} + \frac{1+c/2+\epsilon}{|x|^2 \ln|x| \ln(\ln(|x|))}$$
$$\vdots \quad \text{for any } \epsilon > 0$$

- Hundertmark, Jex, Lange: Quantum systems at the brink: existence of bound states, critical potentials, and dimensionality (2023)

Thank you for your attention