INTEGER QUANTUM HALL EFFECT & QUASI-PERIODIC PERTURBATIONS

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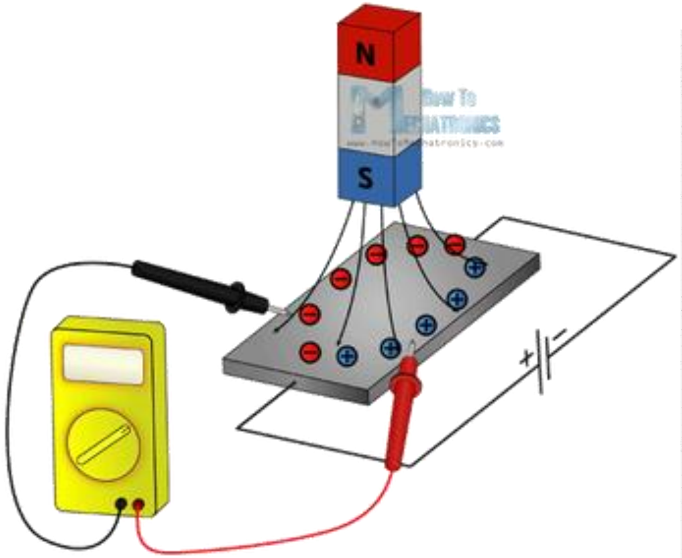
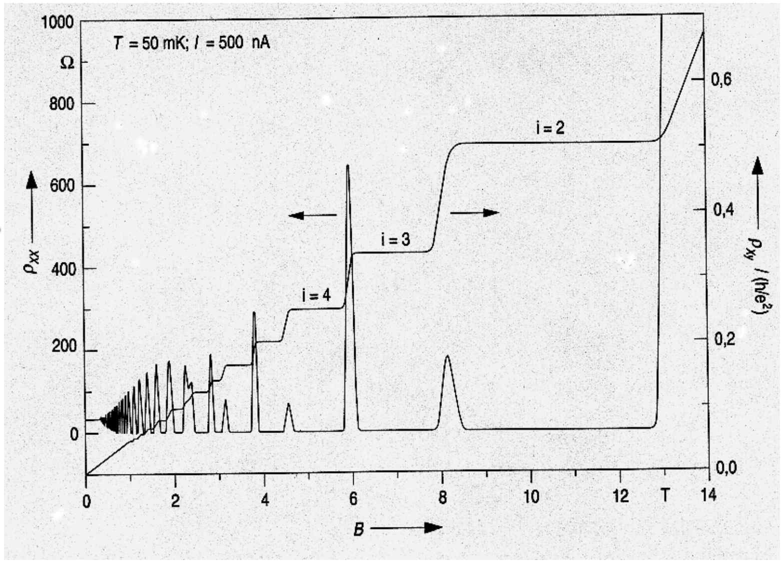
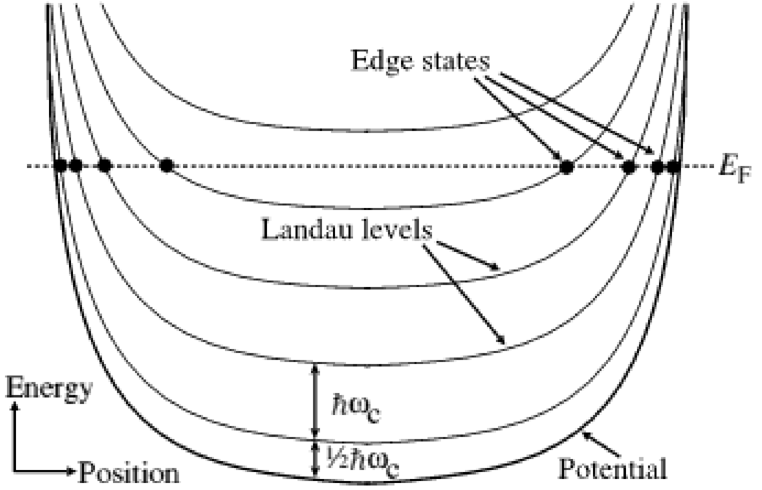
CONTEXT: QUANTUM HALL EFFECT







$$H_{\text{Landau}} = -\frac{\hbar}{2m_e}\partial_x^2 + \frac{1}{2m_e}\left(\hbar\partial_y - Bx\right)^2,$$

$$\text{on } L^2(\mathbb{R}^2)$$

$$eE_2 = -e\frac{v_1}{c}B_3$$

$$\sigma_{\text{Hall}} = n\frac{e^2}{h}, \quad n \in \mathbb{Z}$$

$$\sigma\left(H_{\text{Landau}}\right) = \left\{\kappa\left(n + \frac{1}{2}\right) \,\middle|\, n \in \mathbb{N}\right\}.$$

Classical Hall Effect

(Integer) Quantum Hall Effect

Bulk-boundary correspondence

THE UNPERTURBED SYSTEM

$$\mathcal{H} = \sum_{\vec{x},\vec{y}\in\Lambda_L} a^*_{\vec{x}} H(\vec{x},\vec{y}) a_{\vec{y}}$$

The ***Hamiltonian*** is defined on the fermionic Fock space on a lattice cylinder (with one-edge for simplicity of exposition)

$$\mathcal{H}\colon \mathcal{F}_L \to \mathcal{F}_L, \quad \mathcal{F}_L = \bigoplus_{n\geq 0} \left(\ell^2(\Lambda_L)\right)^{\wedge n}$$

The ***state*** we consider is a ***Gibbs state*** at inverse temperature β and chemical potential (fixing the average particle number) μ is

$$\rho_{\beta,\mu,L} = \frac{\exp\left(-\beta(\mathcal{H}-\mu\mathcal{N})\right)}{\mathsf{Z}}, \quad \mathrm{Tr}\, \rho_{\beta,\mu,L} = 1$$

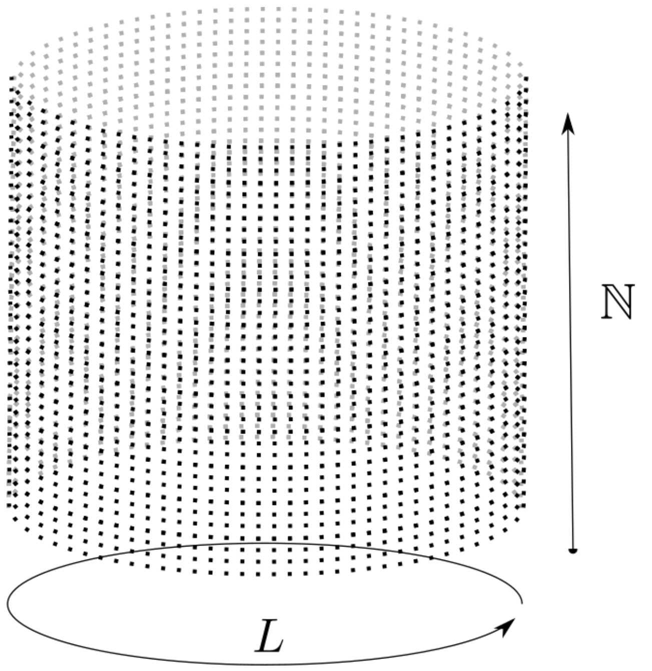
A fundamental object is the ***2-point function***

$$S(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) := \lim_{\beta\to\infty} \lim_{L_1,L_1\to\infty} \mathrm{Tr}\left(\mathbf{T} a_{\vec{\boldsymbol{x}}} a^*_{\vec{\boldsymbol{y}}} \rho_{\beta,\mu,\vec{L}}\right)$$

$$a^{\#}_{\vec{\boldsymbol{x}}} = \mathrm{e}^{x_0\mathcal{H}} a^{\#}_{\vec{x}} \mathrm{e}^{-x_0\mathcal{H}}, \quad \vec{\boldsymbol{x}} \equiv (x_0, x_1, x_2) \equiv (\boldsymbol{x}, x_2)$$

Interesting *per se* and is related to higher correlations by the Wick formula.

$$\Lambda_L \simeq (\mathbb{Z}/L\mathbb{Z}) \times \mathbb{N}$$



THE UNPERTURBED SPECTRUM

$$\mathcal{H} = \sum_{\vec{x},\vec{y}\in\Lambda_L} a^*_{\vec{x}} H(x_1 - y_1; x_2, y_2) a_{\vec{y}}$$

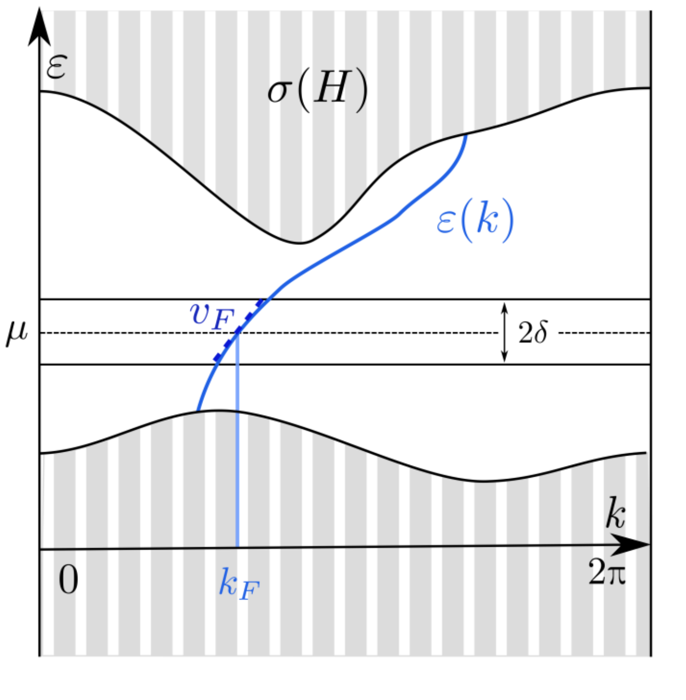
- H is **translation-invariant** in x_1, finite range; this allows to implement a Bloch-Floquet decomposition, namely partial Fourier transform in the x_1 variable.

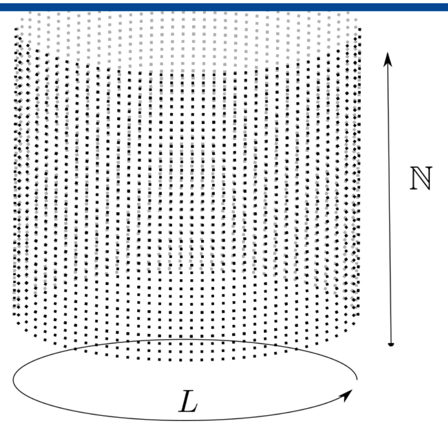
$$\mathcal{H} = \bigoplus_{k_1} \hat{\mathcal{H}}(k_1), \quad \hat{\mathcal{H}}(k_1) = \sum_{x_2,y_2} \hat{a}^*_{k_1;x_2} \hat{H}(k; x_2, y_2) \hat{a}_{k_1;y_2}$$

- There exists an interval [μ-δ, μ+δ] such that the only spectrum is given by one ***family*** of eigenvalues with exponentially decaying eigenfunctions (***edge-mode***)

$$\hat{\mathcal{H}}(k_1)\xi(k_1) = \varepsilon(k_1)\xi(k_1), \quad |\xi(k_1, x_2)| \leq \mathrm{e}^{-cx_2}.$$

- The magnetic field is already included in H.
- An example is given by the Haldane model.





THE PERTURBED HAMILTONIAN

$$\mathcal{H}(\lambda) = \sum_{\vec{x},\vec{y}\in\Lambda_L} a^*_{\vec{x}} H(\vec{x},\vec{y}) a_{\vec{y}} + \lambda \sum_{\vec{x}\in\Lambda_L} \varphi(\alpha x_1, x_2) a^*_{\vec{x}} a_{\vec{x}}$$

- $\varphi(\alpha x_1, x_2) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}n\alpha x_1} \hat{\varphi}_n(x_2)$ is analytic (coefficients decay exponentially in *n*) and quasi-periodic;
- the frequency is (approximately) ***Diophantine***: $|n\alpha|_{\mathbb{T}} \geq \dfrac{c}{|n|^2}$
- PROBLEM: the Hamiltonian is not diagonal in Fourier space (for simplicity in 1-dim)

$$\hat{H}_\lambda(k;p) := \frac{1}{L}\sum_{x,y} \mathrm{e}^{\mathrm{i}(xk-py)} \big(H(x,y) + \lambda\delta_{x,y}\varphi(x)\big) = -\delta_{k,p}\hat{H}(k) + \lambda\delta_{p,k+n\alpha}\hat{\varphi}_n.$$

We can express, at least heuristically, the resolvent as a power series

$$(H_\lambda - \mu)^{-1} = \sum_N (-\lambda)^N \Big(\big((H-\mu)^{-1}\varphi\big)^N (H-\mu)^{-1}\Big)$$

SOME HEURISTICS

$$(H_\lambda - \mu)^{-1} = \sum_N (-\lambda)^N \left(\left((H-\mu)^{-1}\varphi \right)^N (H-\mu)^{-1} \right)$$

Our expression in Fourier space becomes a multiple convolution

$$(\hat{H}_\lambda - \mu)^{-1}(k, k+n\alpha)$$
$$= \sum_N (-\lambda)^N \sum_{n_1,\ldots,n_N \geq 1} (\hat{H}(k)-\mu)^{-1}\, \hat{\varphi}_{n_1}\, (\hat{H}(k+n_1\alpha)-\mu)^{-1} \hat{\varphi}_{n_2} \cdots \hat{\varphi}_{n_N} \left(\hat{H}\left(k+(n_1+\ldots+n_N)\alpha\right)-\mu\right)^{-1}$$

Let us consider a «first order» term

$$(\hat{H}(k)-\mu)^{-1} \hat{\varphi}_{n_1} (\hat{H}(k+n_1\alpha)-\mu)^{-1}$$

- It is most singular when both k and $k+n_1\alpha$ are close to k_F, say around k_F+2^{-M}.
- Then $n_1\alpha$ is around 2^{-M+1}.
- By the Diophantine condition $|n\alpha|_{\mathbb{T}} \geq c|n|^{-2}$, n_1 must be *big:* $|n_1| \geq c2^{\frac{M+1}{2}}$.
- Then we can use the exponential decay of φ in n.
- **BUT** this only works in general for terms with nonzero $n_1+n_2+\ldots+n_N$ (need of treating them differently).

Note: it is crucial to have *one* Fermi point.

RESULT (1)

[ARXIV.ORG/ABS/2410.03946]

For $|\lambda| < \lambda_0$,

$$S_2(\vec{\boldsymbol{x}};\vec{\boldsymbol{y}}) = \mathrm{e}^{\mathrm{i}k_F(\lambda)(x_1-y_1)} Z(\vec{x}) g_{\mathrm{s}}(\boldsymbol{x}-\boldsymbol{y}) \overline{Z(\vec{y})} + R(\vec{\boldsymbol{x}};\vec{\boldsymbol{y}})$$

where g_s is a *renormalized* 2-point function of 1-dimensional, relativistic fermions

$$g_{\mathrm{s}}(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{\beta L_1} \sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}_F(\lambda))\cdot(\boldsymbol{x}-\boldsymbol{y})} \frac{\chi\big(\boldsymbol{k}-\boldsymbol{k}_F(\lambda)\big)}{\mathrm{i}v_0(\lambda)k_0 + v_1(\lambda)\big(k_1 - k_F(\lambda)\big)}$$

with $k_F(0) = k_F,\ v_0(0) = 1,\ v_1(0) = v_F,$ and Z is a *quasi-periodic modulation*

$$Z(\vec{x}) = \sum_{n\in\mathbb{N}} Z_n(x_2)\mathrm{e}^{-\mathrm{i}n\alpha x_1} = \xi(k_F, x_2) + \mathrm{O}(\lambda)$$

The remainder is again subleading $|R(\vec{\boldsymbol{x}};\vec{\boldsymbol{y}})| \leq C_\theta \mathrm{e}^{-c|x_2-y_2|}\big(1+\|\boldsymbol{x}-\boldsymbol{y}\|^{1+\theta}\big)^{-1}$

The result is obtained through an inductive multi-scale analysis (Constructive Renormalization Group).
Note also that higher correlation functions are determined by Wick rule.

KUBO EDGE TRANSPORT COEFFICIENTS

The ***Kubo transport coefficient*** express the *linear response* to adiabatic perturbations.

We consider pertubation of the density near the edge.

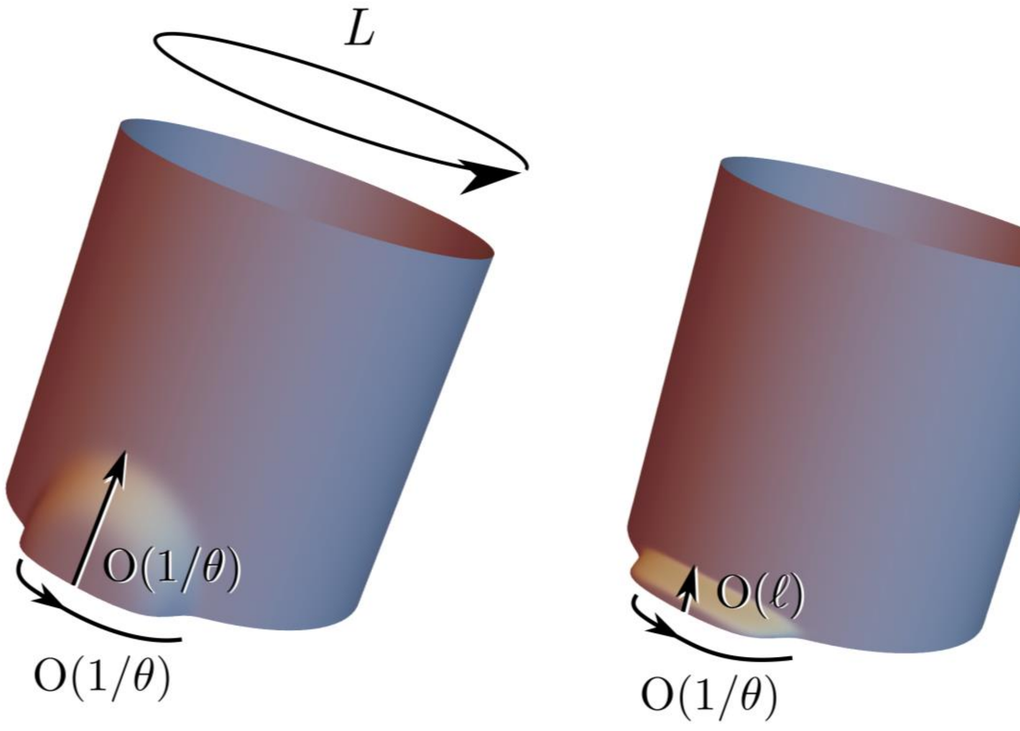
$$\mathcal{H}(\eta t) = \mathcal{H} - \varepsilon\, \mathrm{e}^{\eta t} \mathcal{P}\ , \qquad \mathcal{P} = \textstyle\sum_{\vec{x}\in\Lambda} \mu(\theta \vec{x}) a^*_{\vec{x}} a_{\vec{x}}$$

and we are interested in the (smeared) ***density*** *n* or ***current operators*** *j* expectation values.

The ***Kubo Edge Conductance*** is the defined by the ***Kubo formula***

$$\chi^1_{\mathrm{edge}} = \overset{\star}{\lim}\, \mathrm{i} \int_{-\infty}^{0} \mathrm{d}t\, \mathrm{e}^{\eta t}\, \mathrm{Tr}\left(\left[j_1(\phi_{\theta,\ell}), n_t(\mu_\theta) \right] \rho_{\beta,\mu,\vec{L}} \right)$$

$$\overset{\star}{\lim} = \lim_{\ell\to\infty} \lim_{\theta\to 0} \lim_{\eta\to 0} \lim_{\beta\to\infty} \lim_{L\to\infty}$$



Depiction of j and n cut-offs.

The same expression with $n=j_0$ in place of j_1 defines the ***Kubo Edge Susceptibility*** χ^0_{edge}

RESULT (2)

[ARXIV.ORG/ABS/2410.03946]

The edge conductance is still **quantized** and the edge susceptibility is not

$$\frac{\chi^1_{\text{edge}}}{\langle \mu, \phi \rangle_{\text{edge}}} = \frac{\operatorname{sgn}\left(v_1(\lambda)\right)}{2\pi}$$

$$\frac{\chi^0_{\text{edge}}}{\langle \mu, \phi \rangle_{\text{edge}}} = \frac{1}{2\pi \left| v_1(\lambda) / v_0(\lambda) \right|}$$

BONUS: there exist explicit relations between the renormalized velocities and the modulation Z

$$v_0(\lambda) = \sum_{n, x_2} Z_n(x_2) \overline{Z_n(x_2)},$$

$$v_1(\lambda) = \sum_{n, x_2, y_2} \overline{Z_n(x_2)} \partial_{k_1} \hat{H}\left(k_F(\lambda) - n\alpha; x_2, y_2\right) Z_n(y_2)$$

The relation follows from the multi-scale analysis and the continuity equation (Ward identities)

$$\partial_t n_{t,\vec{x}} + \sum_{i=1,2} \mathrm{d}_{x_i} j_{i,t,\vec{x}} = 0$$

CONCLUSIONS & PERSPECTIVES

- We computed quite explicitly the effect of quasi-periodicity on a non-interacting, traslation-invariant fermionic system with a single edge-mode both at finite and infinite volume and inverse temperature.
- The effect amounts to a renormalization of some parameters: Fermi point, Fermi velocities.
- In the infinite volume limit we can compute also edge transport coefficients.

- Justification of linear response (also with **H. Singh**).
- Multiple-edge modes (work in progress)

Thank you for your attention!