About the exact solutions of the 2D classical and 1D quantum Ising models

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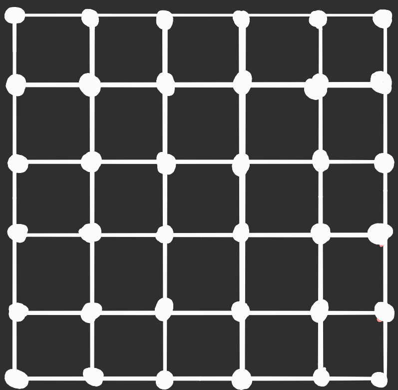
Cergy 2025 - Quantissima Sur Oise
September 8th, 2025

Joint work with Daniel Ueltschi

Ising model: 1D model studied by Ising (1925)



$\Lambda \subset \mathbb{Z}^d$, finite

$\mathcal{E}_\Lambda$: set of nearest-neighbour edges

Spin configurations: $(\sigma_x)_{x\in\Lambda}$, $\sigma_x = \pm 1$

Hamiltonian: $H_\Lambda(\sigma) = -\sum_{xy\in\mathcal{E}_\Lambda} \sigma_x \sigma_y$

Free energy: $f_\Lambda(\beta) = -\frac{1}{\beta|\Lambda|} \log \sum_{(\sigma_x)} e^{-\beta H_\Lambda(\sigma)}$

$$f(\beta) = \lim_{\Lambda \uparrow \mathbb{Z}^d} f_\Lambda(\beta)$$

$$\boxed{\text{Question: Is } f(\beta) \text{ analytic in } \beta\text{?}}$$

The non-analyticity indicates phase transition

2D Ising:

$f(\beta)$ calculated by Onsager ('44) on the square lattice (algebraically)

Further work by Kaufman ('49)

Formula for the triang. lattice: Houtappel ('50)

Combinat. way: Kac, Ward ('52) (sq. lattice)

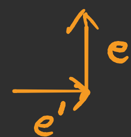
Potts ('54) (triang. lattice)

Kac-Ward method (combinatorial)

$(V, \mathcal{E}_V)$: planar graph, $\vec{\mathcal{E}}_V$: set of directed edges

$K, W \in \vec{\mathcal{E}}_V \times \vec{\mathcal{E}}_V$

$$K_{e',e} := \mathbb{1}_{e' \triangleright e} \exp\left(\frac{i}{2} \measuredangle(e',e)\right)$$



$$W_{e,e'} := \delta_{e,e'} \tanh(J_e)$$

Theorem: (Kac-Ward formula)

$$Z_V^{\text{Ising}}(\beta) = \sqrt{\det(1-KW)}\left[2^{|V|}\prod_{xy \in \mathcal{E}_V}\cosh(\beta J_{xy})\right]$$

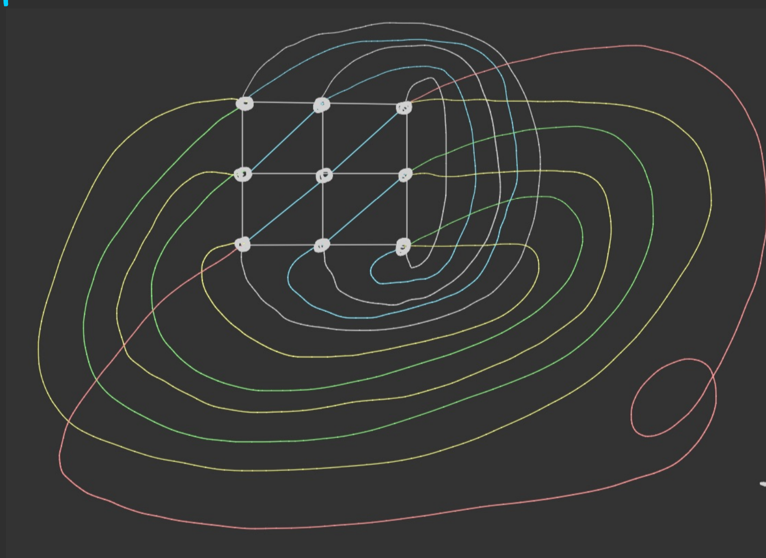
Kac-Ward determinant free b.c $\longleftrightarrow$ Ising free b.c

$\updownarrow$

Kac-Ward determinant periodic b.c

Ising periodic b.c

A faithful projection of the torus on the plane



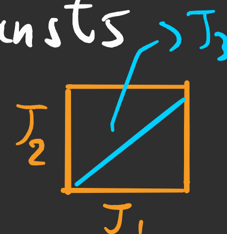
Rigorous results:

- Kager, Lis, Meester ('14): controlling expans. for $\beta < \beta_c$
- Aizenman, Warzel ('18): looking for certain expect.

  This holds for nonnegative couplings $J_e \geq 0$

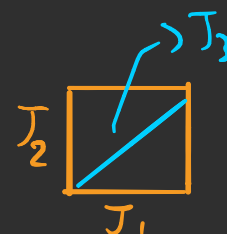
Triang. lattice with arbitary constansts

A, Ueltschi ('24): signed measures



Free energy on the triang. lattice:

$$\boxed{-\beta f(\beta J_1, \beta J_2, \beta J_3) = \log 2 + \frac{1}{8\pi^2} \int_{[-\pi,\pi]^2} dk_1\, dk_2 \, \log\left[\prod_{i=1}^{3} \cosh 2\beta J_i + \prod_{i=1}^{3} \sinh 2\beta J_i - \sum_{i=1}^{3} \sinh 2\beta J_i \cos k_i\right]}$$

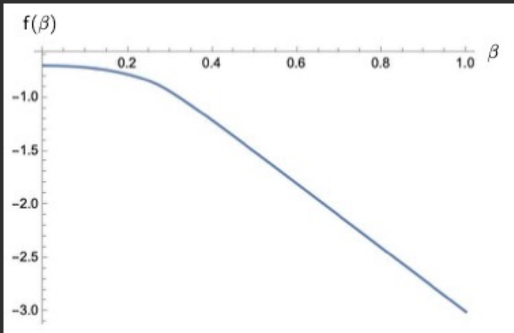
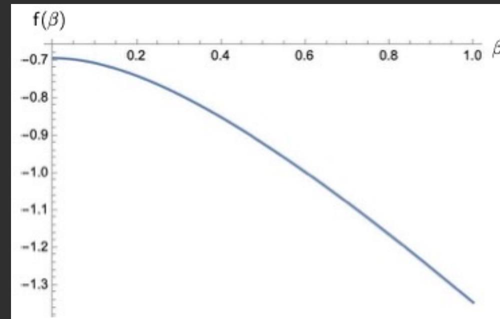
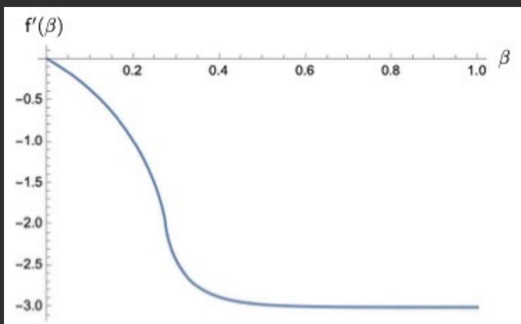
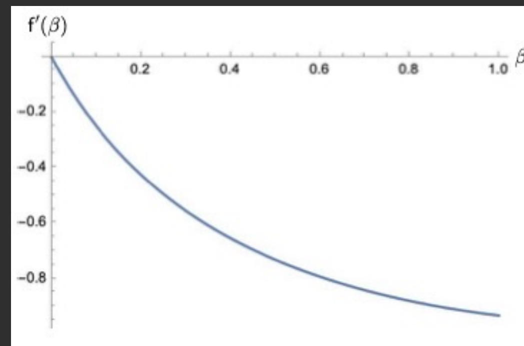
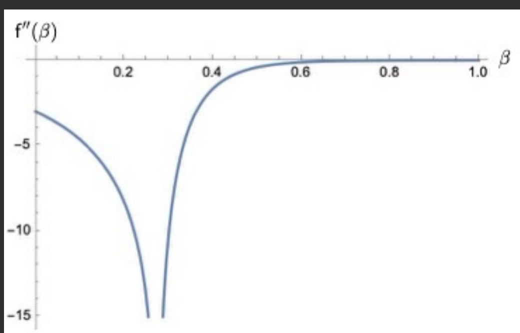
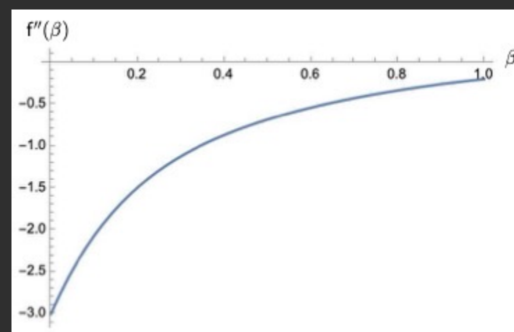


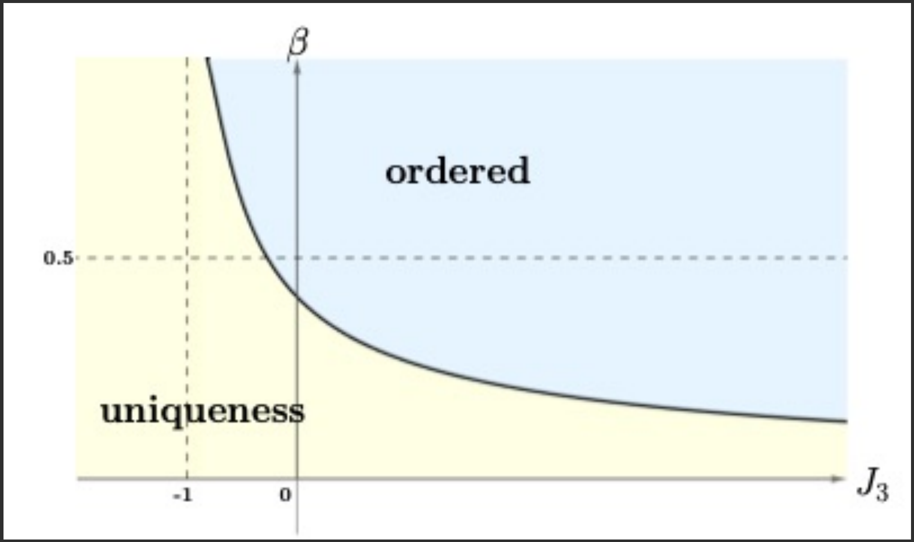
<u>Remark:</u> If $J_1 = J_2 \geq 0$ and $J_3 > -J_1$

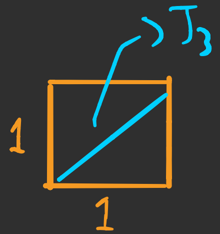
not analytic at β_c

2^{nd} derivative not cont. at β_c

(log divergence)

ferromagnetic $J_e \equiv 1$		antiferromagnetic, $J_e \equiv -1$
	$f(\beta)$	
	$\frac{d}{d\beta} f(\beta)$	
	$\frac{d^2}{d\beta^2} f(\beta)$	





Phase diagram with $J_1 = J_2 = 1$

The free energy lacks analyticity at the line that separates the "ordered" and "uniqueness" phases

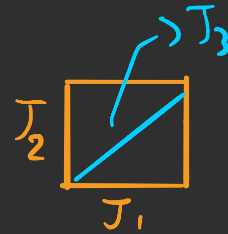
Cimasoni – Duminil-Copin – Li equation for the critical temperature of the Ising model

They provide β_c in a general bi-periodic planar lattice as a unique solution of an algebraic equation

Triangular lattice: $1+x^3 = 3x+3x^2$, $x=\tanh(\beta J)$, $J>0$

$\boxed{\text{Equation proved for nonnegative coupling constants}}$

It also holds for $J_1=J_2\geq 0$ and $J_3\in\mathbb{R}$



It does not hold for $J_1=J_2<0$ and $J_3\in\mathbb{R}$

1D Quantum Ising model

$\Lambda = \{1, \dots, L\}$

State space: $\mathcal{H}_\Lambda = \mathrm{span}_{\mathbb{C}} \{ \{-1,1\}^\Lambda \}$

Spin operators: $S_i^{(1)}, S_i^{(2)}, S_i^{(3)}$ $i \in \Lambda$

$S_i^{(1)} |w\rangle := \frac{1}{2} |w^{(i)}\rangle$ $w^{(i)}$ flip of w at i

Hamiltonian: $H_\Lambda = -\sum_{i=1}^{L} S_i^{(3)} S_{i+1}^{(3)} - h \sum_{i=1}^{L} S_i^{(1)}$

Free energy: $f(\beta) = \lim_{\Lambda \uparrow \mathbb{Z}} -\frac{1}{\beta |\Lambda|} \log \mathrm{Tr}(e^{-\beta H})$

1D q. spin systems are very interesting in the ground state energy

power law decay $\rightsquigarrow$ exponential decay

In some cases,

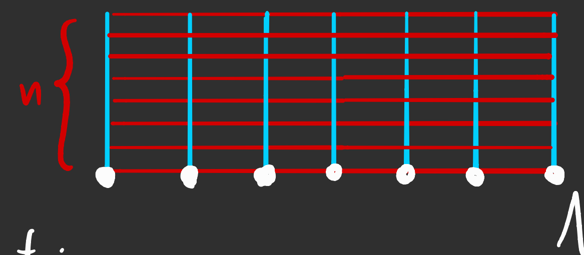
1D quantum spin systems $\rightsquigarrow$ 2D classical spin syst.

<u>1D Q. Ising ~ 2D cl. Ising</u> (Trotter formula)

$$Z^{Q.Is}_{\Lambda} = \lim_{n \to \infty} Z^{Cl.Is.}_{\Lambda,n}\left(J_1 = \frac{\beta}{n},\ J_2 = -\log\left(\frac{2\beta h}{n}\right)\right)$$

Ising spin at vertices

$\Lambda \times \{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$

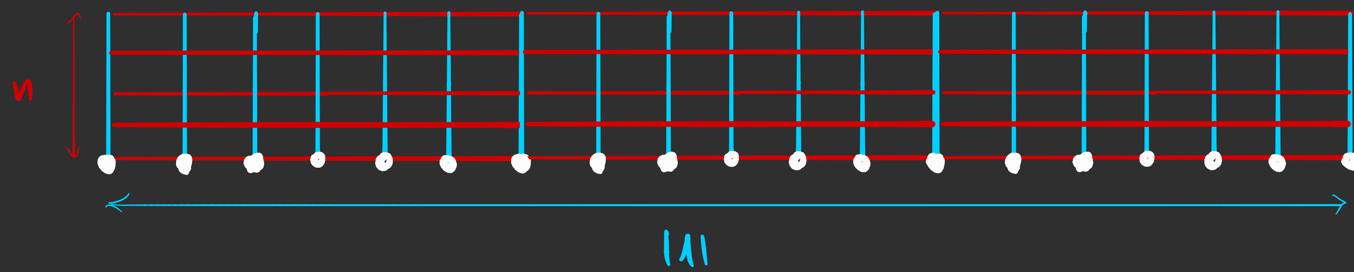


The extra dim. converges to continuum

$$f(\beta) = \lim_{\Lambda \Uparrow \mathbb{Z}} \lim_{n \to \infty} -\frac{1}{\beta |\Lambda|} \log Z^{\text{cl. Is.}}_{\Lambda, n} \left(J_1(n), J_2(n) \right)$$

$$= \lim_{n \to \infty} \lim_{\Lambda \Uparrow \mathbb{Z}} -\frac{1}{\beta |\Lambda|} \log Z^{\text{cl. Is.}}_{\Lambda, n} \left(J_1(n), J_2(n) \right)$$

The expansion of the free energy of the classical Ising at infinite cylinder applies.



$$\boxed{f(\beta, h) = \log(2) + \frac{1}{\pi} \int_0^{\pi} dk \log\left[\cosh\left(\frac{\beta}{2} \sqrt{\left(\frac{1}{4} + h^2 + h \cos k \right)} \right) \right]}$$

Thank you!