Repeated Interaction Quantum Systems

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August 31, 2007

Abstract

We consider a quantum system $\mathcal{S}$ which interacts in a successive way with elements $\mathcal{E}_k$ of a chain of independent quantum subsystems. We will consider two situations: either identical interactions or random ones. In both cases, we show that, under suitable assumptions, the system approaches a *repeated interaction asymptotic state* in the limit of large times. In the case of identical interactions, we also show that if the reference state is chosen so that $\mathcal{S}$ and $\mathcal{E}$ are individually in equilibrium at positive temperatures, then the repeated interaction asymptotic state satisfies an average second law of thermodynamics. Our method is based on the analysis of products of effective operators modelling the effects of each interaction. In the random situation we obtain results on the infinite products of independant identically distributed random matrices. These results also apply to e.g. inhomogeneous Markov chains (products of random stochastic matrices).

Keywords: Open quantum systems, Non-equilibrium quantum theory, Time dependent interactions, Random matrices.
AMS classification: 37A30, 60H25, 82C10.

1 Introduction

A repeated interaction quantum system consists of a subsystem $\mathcal{S}$ which interacts successively with the elements $\mathcal{E}_m$ of a chain $\mathcal{C} = \mathcal{E}_1 + \mathcal{E}_2 + \cdots$ of independent quantum systems. At each moment in time, $\mathcal{S}$ interacts precisely with one $\mathcal{E}_m$ (m increases as time does), while the other elements in the chain evolve freely according to their intrinsic (uncoupled) dynamics. The complete evolution is described by the intrinsic dynamics of $\mathcal{S}$ and of all the $\mathcal{E}_m$, plus an interaction between $\mathcal{S}$ and $\mathcal{E}_m$, for each m. The latter consists of an interaction time $\tau_m > 0$, and an interaction operator V_m (acting on $\mathcal{S}$ and $\mathcal{E}_m$); during the time interval $[\tau_1 + \cdots + \tau_{m-1}, \tau_1 + \cdots + \tau_m)$, $\mathcal{S}$ is coupled to $\mathcal{E}_m$ via V_m. The system $\mathcal{S} + \mathcal{C}$ is called a repeated interaction quantum system. One

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may think of $\mathcal{S}$ as being the system of interest, like a particle enclosed in a container, and of $\mathcal{C}$ as a chain of measuring apparatuses $\mathcal{E}_k$ that are brought into contact with the particle in a sequential manner.

We will consider two situations. In the first one, all the elements $\mathcal{E}_m$ are identical, as well as the interaction times τ_m and the interaction operators V_m. Our goal is to study the large time behaviour of repeated interaction quantum systems, and in particular to describe the effect of the repeated interactions on the system $\mathcal{S}$. One of our main results is the construction of the time–asymptotic state, which we call a *repeated interaction asymptotic state* (RIAS). We will also analyse the thermodynamical properties of this asymptotic state: energy variation, entropy production and their relation.

The theoretical and practical importance of repeated interaction quantum systems is exemplified by systems of radiation-matter coupling, where atoms interact with modes of the quantized electromagnetic field. In this setting, the system $\mathcal{S}$ describes one or several modes of the field in a cavity and the chain $\mathcal{C}$ represents a beam of atoms $\mathcal{E}_m$ that are injected into the cavity. So-called "One-Atom Masers", where the beam is tuned in such a way that at each given moment a *single atom* is inside a microwave cavity have been experimentally realized in laboratories [8, 10]. In an idealized model for this process, one assumes that all the elements $\mathcal{E}_m$ represent a copy of the same, fixed quantum system, and that the interaction is given by a fixed interaction time and a fixed interaction operator (the same for all m). Such *idealized repeated interaction systems* have been analyzed mathematically in [11, 3]. However, it is clear that in actual experiments, neither the interaction time τ_m (or V_m) nor the elements $\mathcal{E}_m$ can be exactly the same for all m! Typically, the interaction time will be *random*, given e.g. by a Gaussian or by a uniform distribution around a mean value, and the state of the incoming atoms will be random as well, for instance determined by a temperature that fluctuates slightly around a mean temperature. (In experiments, the atoms are ejected from an atom oven, then they are cooled down to a wanted temperature before entering the cavity.) It is therefore important to develop a theory that allows for *random repeated interactions*, which is the second situation we shall consider. Our approach of random interactions reduces the problem to the study of infinite products of a certain class of random matrices. The convergence results we obtain on such products also apply to other situations like inhomogeneous Markov chains where at each step the transition matrice is chosen at random.

In this paper, we shall only outline some of the arguments. All the results presented here come from [3, 4] where more detailed proofs are available. For more references, we also refer the reader to [3, 4]. Results on the convergence (of some kind) of products of random matrices are numerous. We refer to [4], and references therein, concerning this part of our work.

2 Identical repeated interactions

To describe the system, we use the convenient framework of quantum statistical mechanics. For an introduction to this framework, we refer to e.g. [1, 2]. In order to make the presentation as simple as possible, we first restrict to identical interactions. Random interactions will be considered in the next section.

2.1 Description of the model

In the case of identical interactions, the model consists of a system $\mathcal{S}$ which is coupled to a chain $\mathcal{C} = \mathcal{E} + \mathcal{E} + \cdots$ of identical elements $\mathcal{E}$. We describe $\mathcal{S}$ and $\mathcal{E}$ as W^*–dynamical systems $(\mathfrak{M}_\mathcal{S}, \alpha_\mathcal{S}^t)$ and $(\mathfrak{M}_\mathcal{E}, \alpha_\mathcal{E}^t)$, where $\mathfrak{M}_\mathcal{S}$, $\mathfrak{M}_\mathcal{E}$ are von Neumann algebras "of observables" acting on the Hilbert spaces $\mathcal{H}_\mathcal{S}$, $\mathcal{H}_\mathcal{E}$, respectively, and where $\alpha_\mathcal{S}^t$ and $\alpha_\mathcal{E}^t$ are (σ–weakly continuous) groups of $*$automorphisms describing the Heisenberg dynamics. In this paper, we consider the situation $\dim \mathcal{H}_\mathcal{S} < \infty$ and $\dim \mathcal{H}_\mathcal{E} \leq \infty$.

We assume that there are distinguished vectors $\psi_\mathcal{S} \in \mathcal{H}_\mathcal{S}$ and $\psi_\mathcal{E} \in \mathcal{H}_\mathcal{E}$, determining states on $\mathfrak{M}_\mathcal{S}$ and $\mathfrak{M}_\mathcal{E}$ which are invariant w.r.t. $\alpha_\mathcal{S}^t$ and $\alpha_\mathcal{E}^t$, respectively, and we assume that $\psi_\mathcal{S}$ and $\psi_\mathcal{E}$ are cyclic and separating for $\mathfrak{M}_\mathcal{S}$ and $\mathfrak{M}_\mathcal{E}$, respectively. One may typically think of these distinguished vectors as being KMS vectors.

The Hilbert space of the chain $\mathcal{C}$ is defined to be the infinite tensor product

$$\mathcal{H}_\mathcal{C} = \otimes_{m \geq 1} \mathcal{H}_\mathcal{E}$$

w.r.t. the reference vector

$$\psi_\mathcal{C} = \psi_\mathcal{E} \otimes \psi_\mathcal{E} \cdots . \tag{2.1}$$

In other words, $\mathcal{H}_\mathcal{C}$ is obtained by taking the completion of the vector space of finite linear combinations of vectors $\otimes_{m \geq 1} \phi_m$, where $\phi_m \in \mathcal{H}_\mathcal{E}$, $\phi_m = \psi_\mathcal{E}$ except for finitely many indices, in the norm induced by the inner product

$$\langle \otimes_m \phi_m, \otimes_m \chi_m \rangle = \prod_m \langle \phi_m, \chi_m \rangle_{\mathcal{H}_\mathcal{E}} .$$

We introduce the von Neumann algebra $\mathfrak{M}_\mathcal{C} = \otimes_{m \geq 1} \mathfrak{M}_\mathcal{E}$ acting on $\otimes_{m \geq 1} \mathcal{H}_\mathcal{E}$, which is obtained by taking the weak closure of finite linear combinations of operators $\otimes_{m \geq 1} A_m$, where $A_m \in \mathfrak{M}_\mathcal{E}$ and $A_m = \mathbb{1}_{\mathcal{H}_\mathcal{E}}$ except for finitely many indices.

The operator algebra containing the observables of the total system is the von Neumann algebra $\mathfrak{M} = \mathfrak{M}_\mathcal{S} \otimes \mathfrak{M}_\mathcal{C}$ which acts on the Hilbert space $\mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{C}$.

The *repeated interaction dynamics* of observables in $\mathfrak{M}$ is characterized by an interaction time $0 < \tau < \infty$ and a selfadjoint interaction operator

$$V \in \mathfrak{M}_\mathcal{S} \otimes \mathfrak{M}_\mathcal{E}.$$

For times $t \in [\tau(m-1), \tau m)$, $\mathcal{S}$ interacts with the m–th element of the chain, while all other elements of the chain evolve freely (each one according to the dynamics $\alpha_\mathcal{E}^t$). The interaction of $\mathcal{S}$ with every element in the chain is the same (given by V).

Let $L_\mathcal{S}$ and $L_\mathcal{E}$ be the *standard Liouvilleans* ("positive temperature Hamiltonians", c.f. references of [6, 9]), uniquely characterized by the following properties: $L_\#$ (where $\# = \mathcal{S}, \mathcal{E}$) are selfadjoint operators on $\mathcal{H}_\#$ which implement the dynamics $\alpha_\#^t$,

$$\alpha_\#^t(A) = \mathrm{e}^{\mathrm{i} t L_\#} A \mathrm{e}^{-\mathrm{i} t L_\#}, \quad \forall A \in \mathfrak{M}_\#,$$

and
$$L_{\#}\psi_{\#} = 0.$$

We define the selfadjoint operator

$$L = L_{\mathcal{S}} + L_{\mathcal{E}} + V, \tag{2.2}$$

omitting trivial factors $\mathbb{1}_{\mathcal{S}}$ or $\mathbb{1}_{\mathcal{E}}$ (by $L_{\mathcal{S}}$ in (2.2) we really mean $L_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{E}}$, etc). L generates the automorphism group $\mathrm{e}^{\mathrm{i}tL} \cdot \mathrm{e}^{-\mathrm{i}tL}$ of $\mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}}$, the interacting dynamics between $\mathcal{S}$ and an element $\mathcal{E}$ of the chain $\mathcal{C}$. The explicit form of the operator V is dictated by the underlying physics, we give an example in Section 4.

For $m \geq 1$ let us denote by

$$\widetilde{L}_m = L_m + \sum_{k \neq m} L_{\mathcal{E},k} \tag{2.3}$$

the generator of the total dynamics during the interval $[(m-1)\tau, m\tau)$. We have introduced L_m, the operator on $\mathcal{H}$ that acts trivially on all elements of the chain except for the m–th one. On the remaining part of the space (which is isomorphic to $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}}$), L_m acts as L, (2.2). We have also set $L_{\mathcal{E},k}$ to be the operator on $\mathcal{H}$ that acts nontrivially only on the k–th element of the chain, on which it equals $L_{\mathcal{E}}$. Of course, the infinite sum in (2.3) must be interpreted in the strong sense on $\mathcal{H}$.

Decompose $t \in \mathbb{R}_+$ as
$$t = m(t)\tau + s(t), \tag{2.4}$$

where $m(t)$ is the integer measuring the number of complete interactions of duration τ the system $\mathcal{S}$ has undergone at time t, and where $0 \leq s(t) < \tau$. The repeated interaction dynamics of an operator A on $\mathcal{H}$ is defined by

$$\alpha^t(A) = U_{\mathrm{RI}}(t)^* \, A \, U_{\mathrm{RI}}(t) \tag{2.5}$$

where

$$U_{\mathrm{RI}}(t) = \mathrm{e}^{-\mathrm{i}s(t)\widetilde{L}_{m(t)+1}} \mathrm{e}^{-\mathrm{i}\tau\widetilde{L}_{m(t)}} \cdots \mathrm{e}^{-\mathrm{i}\tau\widetilde{L}_1} \tag{2.6}$$

defines the Schrödinger dynamics on $\mathcal{H}$. According to this dynamics $\mathcal{S}$ interacts in succession, for a fixed duration τ and a fixed interaction V, with the first $m(t)$ elements of the chain, and for the remaining duration $s(t)$ with the $(m(t)+1)$–th element of the chain. Being the propagator of a "time-dependent Hamiltonian" (which is piecewise constant), $U_{\mathrm{RI}}(t)$ does not have the group property in t.

2.2   The reduced dynamics operator

Our goal is to examine the large time behaviour of expectation values of certain observables in normal states ϱ on $\mathfrak{M}$ (states given by a density matrix on $\mathcal{H}$). The system $\mathcal{S}$ feels an *effective dynamics* induced by the interaction with the chain $\mathcal{C}$. Under a suitable *ergodicity assumption* on this effective dynamics the small system is driven to an asymptotic state, as time increases.

The effects of an interaction on the system $\mathcal{S}$ can be described as follows. We introduce

$$P := \mathbb{1}_{\mathcal{H}_\mathcal{S}} \otimes |\psi_\mathcal{C}\rangle\langle\psi_\mathcal{C}|, \tag{2.7}$$

the orthogonal projection onto $\mathcal{H}_\mathcal{S} \otimes \mathbb{C}\psi_\mathcal{C} \cong \mathcal{H}_\mathcal{S}$, where $\psi_\mathcal{C}$ is given in (2.1). If B is an operator acting on $\mathcal{H}$ then we identify PBP as an operator acting on $\mathcal{H}_\mathcal{S}$. Let $\phi \in \mathcal{H}_\mathcal{S}$. Since $\psi_\mathcal{S}$ is cyclic for $\mathfrak{M}_\mathcal{S}$ and $\dim\mathcal{H}_\mathcal{S} < +\infty$, there exists $A \in \mathfrak{M}_\mathcal{S}$ such that $\phi = A\psi_\mathcal{S}$. The reduced dynamics operator M is then defined by

$$M\phi := P\alpha^\tau(A \otimes \mathbb{1}_\mathcal{C})\psi_\mathcal{S} \otimes \psi_\mathcal{C}. \tag{2.8}$$

Proposition 2.1 *The operator M is a contraction on $\mathcal{H}_\mathcal{S}$ endowed with the norm $|||\phi||| = |||A\psi_\mathcal{S}||| := \|A\|_{\mathcal{B}(\mathcal{H}_\mathcal{S})}$. In particular, $\mathrm{spec}(M) \subset \{z \in \mathbb{C} \mid |z| \leq 1\}$ and all eigenvalues lying on the unit circle are semisimple. Moreover 1 is an eigenvalue for M with corresponding eigenvector $\psi_\mathcal{S}$.*

Our main assumption (E) is an ergodicity assumption on the discrete dynamics generated by M

(E) The spectrum of M on the complex unit circle consists of the single eigenvalue $\{1\}$. This eigenvalue is simple (with corresponding eigenvector $\psi_\mathcal{S}$).

Assumption (E) guarantees that the adjoint operator M^* has a unique invariant vector, called $\psi_\mathcal{S}^*$ (normalized as $\langle\psi_\mathcal{S}^*, \psi_\mathcal{S}\rangle = 1$), and that

$$\lim_{m\to\infty} M^m = \pi := |\psi_\mathcal{S}\rangle\langle\psi_\mathcal{S}^*|, \tag{2.9}$$

in the operator sense, where π is the rank one projection which projects onto $\mathbb{C}\psi_\mathcal{S}$ along $(\mathbb{C}\psi_\mathcal{S}^*)^\perp$. In fact, we have the following easy estimate (valid for any matrix M with spectrum inside the unit disk and satisfying (E))

Proposition 2.2 *For any $\epsilon > 0$ there exists a constant C_ϵ s.t.*

$$\|M^m - \pi\| \leq C_\epsilon \mathrm{e}^{-m(\gamma-\epsilon)},$$

for all $m \geq 0$, where $\gamma := \min_{z\in\mathrm{spec}(M)\setminus\{1\}} |\log|z|| > 0$.

The parameter γ measures the speed of convergence.
Remark. If all eigenvalues of M are semisimple then in Proposition 2.2 we have $\|M^m - \pi\| \leq C\mathrm{e}^{-m\gamma}$ for some constant C and all $m \geq 0$.

Before stating our results, we finally discuss the kinds of observables we consider. One such class is $\mathfrak{M}_\mathcal{S} \subset \mathfrak{M}$ which consists of observables on the system $\mathcal{S}$ only. There are other observables of interest. We may think of the system $\mathcal{S}$ as being fixed in space and of the chain as passing by $\mathcal{S}$ so that at the moment t, the $(m(t)+1)$–th element $\mathcal{E}$ is located near $\mathcal{S}$, c.f. (2.4). A detector placed in the vicinity of $\mathcal{S}$ can measure at this moment in time observables of $\mathcal{S}$ and those of the $(m(t)+1)$–th element in the chain, i.e., an "instantaneous observable" of the form $A_\mathcal{S} \otimes \vartheta_{m(t)+1}(A_\mathcal{E})$, where $A_\mathcal{S} \in \mathfrak{M}_\mathcal{S}$, $A_\mathcal{E} \in \mathfrak{M}_\mathcal{E}$, and $\vartheta_m : \mathfrak{M}_\mathcal{E} \to \mathfrak{M}_\mathcal{C}$ is defined by

$$\vartheta_m(A_\mathcal{E}) = \mathbb{1}_\mathcal{E} \otimes \cdots \otimes \mathbb{1}_\mathcal{E} \otimes A_\mathcal{E} \otimes \mathbb{1}_\mathcal{E} \cdots \tag{2.10}$$

where the $A_\mathcal{E}$ on the right side of (2.10) acts on the m–th factor in the chain. We call these observables *instantaneous observables*. An example of such an observable is the energy flux (variation) of the system $\mathcal{S}$ (see Section 2.4). One may also consider more general observables, [3, 5].

2.3 Asymptotic state

We consider the large time limit of expectations of instantaneous observables

$$E(t) := \varrho\big(\alpha^t(A_\mathcal{S} \otimes \vartheta_{m(t)+1}(A_\mathcal{E}))\big)$$

for normal initial states ϱ on $\mathfrak{M}$. Define the state ϱ_+ on $\mathfrak{M}_\mathcal{S}$ by

$$\varrho_+(A_\mathcal{S}) := \langle \psi_\mathcal{S}^*, A_\mathcal{S}\psi_\mathcal{S} \rangle, \tag{2.11}$$

where $\psi_\mathcal{S}^*$ is defined in (2.9).

Theorem 2.3 *([3]) Suppose that assumption (E) is satisfied. Let ϱ be a normal state on $\mathfrak{M}$. Then, for any $\epsilon > 0$, there exists C_ϵ s.t. for all $t \geq 0$*

$$|E(t) - E_+(t)| \leq C_\epsilon e^{-(\gamma-\epsilon)t/\tau}, \tag{2.12}$$

where $\gamma > 0$ is given in Proposition 2.2, and where E_+ is the τ-periodic function

$$E_+(t) = \varrho_+ \Big(P\alpha^{s(t)}\Big(A_\mathcal{S} \otimes A_\mathcal{E}\Big)P\Big).$$

In particular, $E_+(n\tau) = \varrho_+(A_\mathcal{S})\langle \psi_\mathcal{E}, A_\mathcal{E}\psi_\mathcal{E}\rangle$.

Remark. Using (2.12) and the uniqueness of the limit, one can see that the state ϱ_+ does *not* depend on the choice of the reference state $\psi_\mathcal{S}$.

Idea of the proof: In order not to muddle the essence of the argument, we take the initial state of the entire system to be given by the vector $\psi_0 = \psi_\mathcal{S} \otimes \psi_\mathcal{C}$. Recall that

$$\alpha^t(A_\mathcal{S} \otimes \vartheta_{m(t)+1}(A_\mathcal{E})) \tag{2.13}$$
$$= e^{i\tau\widetilde{L}_1} \cdots e^{i\tau\widetilde{L}_{m(t)}} e^{is(t)\widetilde{L}_{m(t)+1}} (A_\mathcal{S} \otimes \vartheta_{m(t)+1}(A_\mathcal{E})) e^{-is(t)\widetilde{L}_{m(t)+1}} e^{-i\tau\widetilde{L}_{m(t)}} \cdots e^{-i\tau\widetilde{L}_1}.$$

The first step consists in the following decomposition which serves to isolate the dynamics of the elements $\mathcal{E}$ which do not interact at a given time. Taking into account (2.3) we can write

$$e^{-is\widetilde{L}_{m+1}} e^{-i\tau\widetilde{L}_m} \cdots e^{-i\tau\widetilde{L}_1} = U_m^- e^{-isL_{m+1}} e^{-i\tau L_m} \cdots e^{-i\tau L_1} U_m^+,$$

where

$$
\begin{aligned}
U_m^- &= \exp\left[-i\sum_{j=1}^m [(m-j)\tau + s]L_{\mathcal{E},j}\right], \\
U_m^+ &= \exp\left[-i\sum_{j=2}^{m+1}(j-1)\tau L_{\mathcal{E},j} - i(m\tau + s)\sum_{j\geq m+2} L_{\mathcal{E},j}\right].
\end{aligned}
$$

One easily sees that $U_m^+ \psi_0 = \psi_0$ and that U_m^- commutes with $A_{\mathcal{S}} \otimes \vartheta_{m+1}(A_{\mathcal{E}})$, so that (2.13) rewrites

$$
\begin{aligned}
E(t) \quad = \quad & \langle \psi_0, \mathrm{e}^{\mathrm{i}\tau L_1} \cdots \mathrm{e}^{\mathrm{i}\tau L_{m(t)}} \mathrm{e}^{\mathrm{i}s(t)L_{m(t)+1}} (A_{\mathcal{S}} \otimes \vartheta_{m(t)+1}(A_{\mathcal{E}})) & (2.14) \\
& \mathrm{e}^{-\mathrm{i}s(t)L_{m(t)+1}} \mathrm{e}^{-\mathrm{i}\tau L_{m(t)}} \cdots \mathrm{e}^{-\mathrm{i}\tau L_1} \psi_0 \rangle.
\end{aligned}
$$

The second step is to replace, for all m, the Liouvillean L_m by another (non selfadjoint) generator K_m of the interacting dynamics, called a C-Liouville operator, which satisfy the following additional property:

$$
K_m \psi_{\mathcal{S}} \otimes \psi_{\mathcal{E}} = 0, \tag{2.15}
$$

i.e. it "kills" the reference vector. The C-Liouville operator has been introduced in [6] to study non-equilibrium steady states (NESS).

Remark. For the existence of such a generator, we refer the reader to e.g. [6, 9]. One can also get an explicit expression for it in terms of the Liouvillean and the modular data of the pair $(\mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}}, \psi_{\mathcal{S}} \otimes \psi_{\mathcal{E}})$ [1, 2, 6].

Since the operators K_m are also generators of the dynamics, and using (2.15), (2.14) becomes

$$
E(t) = \langle \psi_0, \mathrm{e}^{\mathrm{i}\tau K_1} \cdots \mathrm{e}^{\mathrm{i}\tau K_{m(t)}} \mathrm{e}^{\mathrm{i}s(t)K_{m(t)+1}} (A_{\mathcal{S}} \otimes \vartheta_{m(t)+1}(A_{\mathcal{E}})) \psi_0 \rangle. \tag{2.16}
$$

The next step is to use the independance of the various elements of the chain and to rewrite (2.16) in terms of the reduced dynamics operator M. Recall that P is the orthogonal projection onto $\mathcal{H}_{\mathcal{S}} \otimes \mathbb{C}\psi_{\mathcal{C}}$. Note that $P\psi_0 = \psi_0$. Thus we have

$$
E(t) = \langle \psi_0, P\mathrm{e}^{\mathrm{i}\tau K_1} \cdots \mathrm{e}^{\mathrm{i}\tau K_{m(t)}} \mathrm{e}^{\mathrm{i}s(t)K_{m(t)+1}} (A_{\mathcal{S}} \otimes \vartheta_{m(t)+1}(A_{\mathcal{E}})) P\psi_0 \rangle. \tag{2.17}
$$

We also denote $P_{\psi_{\mathcal{E}}} := |\psi_{\mathcal{E}}\rangle\langle\psi_{\mathcal{E}}|$ so that $P = \mathbb{1}_{\mathcal{S}} \otimes P_{\psi_{\mathcal{E}}} \otimes P_{\psi_{\mathcal{E}}} \otimes \cdots$. We have then

$$
P\mathrm{e}^{\mathrm{i}\tau K_1} = P\mathrm{e}^{\mathrm{i}\tau K_1} (\mathbb{1}_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{E}} \otimes P_{\psi_{\mathcal{E}}} \otimes P_{\psi_{\mathcal{E}}} \otimes \cdots),
$$

and

$$
\begin{aligned}
& \mathrm{e}^{\mathrm{i}\tau K_2} \cdots \mathrm{e}^{\mathrm{i}\tau K_{m(t)}} \mathrm{e}^{\mathrm{i}s(t)K_{m(t)+1}} (A_{\mathcal{S}} \otimes \vartheta_{m(t)+1}(A_{\mathcal{E}})) P \\
& = (\mathbb{1}_{\mathcal{S}} \otimes P_{\psi_{\mathcal{E}}} \otimes \mathbb{1}_{\mathcal{E}} \otimes \mathbb{1}_{\mathcal{E}} \otimes \cdots) \mathrm{e}^{\mathrm{i}\tau K_2} \cdots \mathrm{e}^{\mathrm{i}\tau K_{m(t)}} \mathrm{e}^{\mathrm{i}s(t)K_{m(t)+1}} (A_{\mathcal{S}} \otimes \vartheta_{m(t)+1}(A_{\mathcal{E}})) P.
\end{aligned}
$$

Combining these two, we get

$$
\begin{aligned}
& P\mathrm{e}^{\mathrm{i}\tau K_1} \cdots \mathrm{e}^{\mathrm{i}\tau K_{m(t)}} \mathrm{e}^{\mathrm{i}s(t)K_{m(t)+1}} (A_{\mathcal{S}} \otimes \vartheta_{m(t)+1}(A_{\mathcal{E}})) P \\
& = P\mathrm{e}^{\mathrm{i}\tau K_1} P\mathrm{e}^{\mathrm{i}\tau K_2} \cdots \mathrm{e}^{\mathrm{i}\tau K_{m(t)}} \mathrm{e}^{\mathrm{i}s(t)K_{m(t)+1}} (A_{\mathcal{S}} \otimes \vartheta_{m(t)+1}(A_{\mathcal{E}})) P,
\end{aligned}
$$

and by induction

$$
\begin{aligned}
& P\mathrm{e}^{\mathrm{i}\tau K_1} \cdots \mathrm{e}^{\mathrm{i}\tau K_{m(t)}} \mathrm{e}^{\mathrm{i}s(t)K_{m(t)+1}} (A_{\mathcal{S}} \otimes \vartheta_{m(t)+1}(A_{\mathcal{E}})) P \\
& = (P\mathrm{e}^{\mathrm{i}\tau K_1} P)(P\mathrm{e}^{\mathrm{i}\tau K_2} P) \cdots (P\mathrm{e}^{\mathrm{i}\tau K_{m(t)}} P)(P\mathrm{e}^{\mathrm{i}s(t)K_{m(t)+1}} (A_{\mathcal{S}} \otimes \vartheta_{m(t)+1}(A_{\mathcal{E}})) P).
\end{aligned}
$$

Since all the interactions are identical, using (2.8),(2.15), together with the identification $\mathcal{H}_\mathcal{S} \otimes \mathbb{C}\psi_\mathcal{C} \cong \mathcal{H}_\mathcal{S}$, we have $Pe^{i\tau K_m}P \cong M$ for all k. Hence

$$E(t) = \langle \psi_\mathcal{S}, M^{m(t)}Pe^{is(t)K}(A_\mathcal{S} \otimes A_\mathcal{E})P\psi_\mathcal{S}\rangle.$$

We now use Proposition 2.2 to conclude that

$$\left| E(t) - \langle \psi_\mathcal{S}^*, Pe^{is(t)K}(A_\mathcal{S} \otimes A_\mathcal{E})P\psi_\mathcal{S}\rangle \right| \leq C_\epsilon e^{-(\gamma-\epsilon)t/\tau}.$$

It remains to note that

$$\langle \psi_\mathcal{S}^*, Pe^{is(t)K}(A_\mathcal{S} \otimes A_\mathcal{E})P\psi_\mathcal{S}\rangle$$
$$= \langle \psi_\mathcal{S}^*, Pe^{is(t)K}(A_\mathcal{S} \otimes A_\mathcal{E})e^{-is(t)K}P\psi_\mathcal{S}\rangle = \langle \psi_\mathcal{S}^*, P\alpha^{s(t)}(A_\mathcal{S} \otimes A_\mathcal{E})P\psi_\mathcal{S}\rangle.$$

2.4 Energy and Entropy

It may not be meaningful to speak about the total energy of the system, because it may have to be considered as being infinite, e.g. if the elements $\mathcal{E}$ of the chain are infinitely extended quantum systems with non-vanishing energy density. However, we can define the time variation of the total energy of the system and link it to its entropy variation, giving us an average 2nd law of thermodynamics for RIAS.

2.4.1 Energy

During each interaction, i.e. each time interval of the form $[m\tau, (m+1)\tau)$, $m \geq 0$, the full system is autonomous. Therefore there is no energy variation in it. On the other hand, there might be an energy jump when the system $\mathcal{S}$ shifts from one element of the chain to the other, i.e. as time passes the moments $m\tau$.

In order to quantify these energy jumps, let us assume for a moment that the various components, i.e. $\mathcal{S}$ and the elements $\mathcal{E}$, are described via the usual Hamiltonian framework. We denote by $\mathfrak{h}_\mathcal{S}$ and $h_\mathcal{S}$ the Hilbert space and Hamiltonian describing the system $\mathcal{S}$, by $\mathfrak{h}_\mathcal{E}$ and $h_\mathcal{E}$ those for an element $\mathcal{E}$, and let v be a selfadjoint operator on $\mathfrak{h}_\mathcal{S} \otimes \mathfrak{h}_\mathcal{E}$ describing the interaction between $\mathcal{S}$ and $\mathcal{E}$. In the same way as in Section 2.1, during each time interval $[(m-1)\tau, m\tau)$, the hamiltonian of the total system writes

$$h_m = h_\mathcal{S} + \sum_k h_{\mathcal{E},k} + v_m,$$

where $h_{\mathcal{E},k}$ acts non trivially on the k-th element of the chain on which it equals $h_\mathcal{E}$, and $v_m = v$ acting on $\mathcal{S}$ and the m-th element of the chain. It is then clear that energy change from time t_1 to time t_2 writes

$$\Delta E(t_2, t_1) = U(t_2)^* h_{m(t_2)+1} U(t_2) - U(t_1)^* h_{m(t_1)+1} U(t_1), \tag{2.18}$$

where

$$U(t) = e^{-is(t)h_{m(t)+1}}e^{-i\tau h_{m(t)}} \cdots e^{-i\tau h_1},$$

and where we decomposed t_1 and t_2 as in (2.4). Now, for $(k-1)\tau \leq t_1 < k\tau \leq t_2 < (k+1)\tau$, it is easy to see that (2.18) simplifies to

$$\Delta E(t_2, t_1) = U(k\tau)^*(v_{k+1} - v_k)U(k\tau) =: \delta E(k\tau), \tag{2.19}$$

which we interpret as the energy jump observable as time passes the moment $k\tau$.

Starting from a Hamiltonian description of the system and given reference states $\varrho_{\mathcal{S}}$ (resp. $\varrho_{\mathcal{E}}$) of $\mathcal{S}$ (resp. $\mathcal{E}$), to get the algebraic description of Section 2.1 one then performs the GNS representation $(\mathcal{H}_{\mathcal{S}}, \pi_{\mathcal{S}}, \psi_{\mathcal{S}})$ of $(\mathcal{B}(\mathfrak{h}_{\mathcal{S}}), \varrho_{\mathcal{S}})$, and similarly for $\mathcal{E}$ (see e.g. [2] and Section 4 for a concrete example). The interaction operator V is then given by $V = \pi_{\mathcal{S}} \otimes \pi_{\mathcal{E}}(v)$.

Now, for any observable $O \in \mathcal{B}(\mathfrak{h}_{\mathcal{S}}) \otimes_{m \geq 1} \mathcal{B}(\mathfrak{h}_{\mathcal{E}})$, one has $\pi(U(t)^*OU(t)) = \alpha^t(\pi(O))$ where

$$\pi(O_{\mathcal{S}} \otimes_{m \geq 1} O_{\mathcal{E},m}) := \pi_{\mathcal{S}}(O_{\mathcal{S}}) \otimes_{m \geq 1} \pi_{\mathcal{E}}(O_{\mathcal{E},m}),$$

so that

$$j(k) := \pi(\delta E(k\tau)) = \alpha^{k\tau}(V_{k+1} - V_k), \tag{2.20}$$

where we set (see (2.10))

$$V_k = [\mathbb{1}_{\mathfrak{M}_{\mathcal{S}}} \otimes \vartheta_k](V). \tag{2.21}$$

In view of the above (formal) discussion, we therefore define the energy jump observable of the repeated interaction system defined in Section 2.1 as time passes moment $k\tau$ by $j(k) := \alpha^{k\tau}(V_{k+1} - V_k)$.

Theorem 2.3 then tells us that for any normal state ϱ on $\mathfrak{M}$ and for any $\epsilon > 0$, there is a constant C_ϵ s.t.

$$\left|\varrho(j(k)) - \varrho_+\left(j_+\right)\right| \leq C_\epsilon e^{-k(\gamma - \epsilon)}, \tag{2.22}$$

where

$$j_+ = PVP - P\alpha^\tau(V)P = -i \int_0^\tau P\alpha^s([L_{\mathcal{S}} + L_{\mathcal{E}}, V])P \, ds.$$

Relation (2.22) and the fact that the energy is piecewise constant shows that $\varrho_+(j_+)$ is the change of energy in any interval of length τ, in the large time limit. We thus call

$$dE_+ = \frac{1}{\tau}\varrho_+(j_+) \tag{2.23}$$

the *asymptotic energy production*. The asymptotic energy production does not depend on the initial state of the system.

Remark. It is not hard to see that the expectation of the energy jump is constant in the state $\varrho_+ \otimes \varrho_{\mathcal{C}}$, where $\varrho_{\mathcal{C}}$ is the vector state on $\mathfrak{M}_{\mathcal{C}}$ determined by $\psi_{\mathcal{C}}$, (2.1):

$$\varrho_+ \otimes \varrho_{\mathcal{C}}(j(k)) = \varrho_+(j_+), \quad \forall k \geq 1. \tag{2.24}$$

We introduce the variation of the total energy, $\Delta E(t)$, between the instants $t = m(t)\tau + s(t)$ and $t = 0$. It is the sum of the energy jumps,

$$\Delta E(t) = \sum_{k=1}^{m(t)} j(k), \quad \text{for } t \geq \tau, \tag{2.25}$$

and $\Delta E(t) = 0$ if $0 \le t < \tau$. Estimate (2.22) shows that for any normal state ϱ on $\mathfrak{M}$ there is a constant C s.t.

$$\left| \frac{\varrho(\Delta E(t))}{t} - \mathrm{d}E_+ \right| \le \frac{C}{t}, \tag{2.26}$$

for all $t > 0$. The energy grows asymptotically linearly in time.

2.4.2 Entropy, average 2$^{\text{nd}}$ law of thermodynamics

Let ϱ and ϱ_0 be two normal states on $\mathfrak{M}$. The relative entropy of ϱ with respect to ϱ_0 is denoted by $\mathrm{Ent}(\varrho|\varrho_0)$, where our definition of relative entropy differs from that one given in [2] by a sign, so that in our case, $\mathrm{Ent}(\varrho|\varrho_0) \ge 0$.

   For a thermodynamic interpretation of the entropy and its relation to the energy variation, we assume in this section that $\psi_{\mathcal{S}}$ is a $(\beta_{\mathcal{S}}, \alpha_{\mathcal{S}}^t)$–KMS state on $\mathfrak{M}_{\mathcal{S}}$, and that $\psi_{\mathcal{E}}$ is a $(\beta_{\mathcal{E}}, \alpha_{\mathcal{E}}^t)$–KMS state on $\mathfrak{M}_{\mathcal{E}}$, where $\beta_{\mathcal{S}}, \beta_{\mathcal{E}}$ are inverse temperatures. Let ϱ_0 be the state on $\mathfrak{M}$ determined by the vector $\psi_{\mathcal{S}} \otimes \psi_{\mathcal{C}}$ (c.f. (2.1)).

   We are interested in the change of relative entropy of the repeated interaction system as time evolves.

Proposition 2.4 *([3]) Let ϱ be any normal state on $\mathfrak{M}$. Then we have*

$$\mathrm{Ent}\big(\varrho \circ \alpha^t | \varrho_0\big) - \mathrm{Ent}(\varrho|\varrho_0) = \varrho\Big(\beta_{\mathcal{E}} \Delta E(t) - \alpha^t(X(t)) + X(0)\Big), \tag{2.27}$$

where $\Delta E(t)$ is the variation of the total energy (see (2.25)), and where

$$X(t) = \beta_{\mathcal{E}} V_{m(t)+1} + (\beta_{\mathcal{E}} - \beta_{\mathcal{S}})L_{\mathcal{S}},$$

with V_k given by (2.21).

Idea of the proof: The proof of (2.27) is based on the *entropy production formula* [7]: given any normal state ϱ of $\mathfrak{M}$ and any unitary U on $\mathcal{H}$ we have (our definition of entropy differs from the one in [7] by a sign)

$$\mathrm{Ent}\big(\varrho(U^* \cdot U)|\varrho_0\big) - \mathrm{Ent}(\varrho|\varrho_0) = \varrho\Big(U^*\Big[\beta_{\mathcal{E}} \sum_k L_{\mathcal{E},k} + \beta_{\mathcal{S}} L_{\mathcal{S}}\Big]U - \beta_{\mathcal{E}} \sum_k L_{\mathcal{E},k} - \beta_{\mathcal{S}} L_{\mathcal{S}}\Big). \tag{2.28}$$

Taking $U = U_{\mathrm{RI}}(t)$, the argument of ϱ in (2.28) can be written as

$$\beta_{\mathcal{E}}\Big\{\alpha^t\Big(\sum_k L_{\mathcal{E},k} + L_{\mathcal{S}}\Big) - \sum_k L_{\mathcal{E},k} - L_{\mathcal{S}}\Big\} - (\beta_{\mathcal{E}} - \beta_{\mathcal{S}})\Big(\alpha^t(L_{\mathcal{S}}) - L_{\mathcal{S}}\Big),$$

so it suffices to prove (2.27) for $\beta_{\mathcal{S}} = \beta_{\mathcal{E}} = \beta$, i.e. to show that

$$\alpha^t\Big(\sum_k L_{\mathcal{E},k} + L_{\mathcal{S}}\Big) - \sum_k L_{\mathcal{E},k} - L_{\mathcal{S}} = \Delta E(t) - \alpha^t(V_{m(t)+1}) + V_1. \tag{2.29}$$

Eq. (2.29) follows from the identity

$$\alpha^{m\tau+s}(\widetilde{L}_{m+1}) - \alpha^{m\tau}(\widetilde{L}_{m+1}) = 0, \quad 0 \le s < \tau,$$

which reflects the conservation of energy during the time intervall $[m\tau, (m+1)\tau)$, and using (2.20) and (2.25) (see [3] for more details). □

If $\text{Ent}(\varrho|\varrho_0) < \infty$ then all terms in (2.27) are bounded uniformly in t, except possibly $\text{Ent}(\varrho \circ \alpha^t|\varrho_0)$ and $\varrho(\beta_\mathcal{E} \Delta E(t))$. Hence (2.23) and (2.26) show that for any normal state ϱ on $\mathfrak{M}$ there is a constant C s.t.

$$\left| \frac{\text{Ent}(\varrho \circ \alpha^t|\varrho_0)}{t} - \frac{\beta_\mathcal{E}}{\tau} \varrho_+(j_+) \right| \le \frac{C}{t}, \tag{2.30}$$

for all $t > 0$. The entropy grows linearly in time, for large times.

The relative entropy is non–negative, so (2.30) shows that

$$\varrho_+(j_+) \ge 0.$$

We show in [3] that $\varrho_+(j_+)$ is strictly positive for concrete systems. It follows from (2.30) also that

$$\sup_{t \ge 0} \left| \text{Ent}(\varrho \circ \alpha^t|\varrho_0) \right| < \infty \iff \varrho_+(j_+) = 0.$$

Since $\varrho_+(j_+)$ is independent of ϱ it follows that for a given interaction (V, τ) the relative entropy either diverges for all initial states ϱ, as $t \to \infty$, or it stays bounded for all initial states ϱ. *In particular, if $\varrho_+(j_+) > 0$ then there does not exist any normal state ϱ on $\mathfrak{M}$ which is invariant under α^t (i.e., such that $\varrho \circ \alpha^t = \varrho$, for all $t \ge 0$).*

Proposition 2.5 *([3]) We have*

$$\lim_{t \to \infty} \left[\text{Ent}(\varrho \circ \alpha^{t+\tau}|\varrho_0) - \text{Ent}(\varrho \circ \alpha^t|\varrho_0) \right] = \beta_\mathcal{E} \, \varrho_+(j_+). \tag{2.31}$$

The change of entropy during an interval of duration τ, for $t \to \infty$, is thus given by $\beta_\mathcal{E} \, \varrho_+(j_+) \ge 0$. We call

$$\mathrm{d}S_+ = \frac{\beta_\mathcal{E}}{\tau} \varrho_+(j_+) \tag{2.32}$$

the (average) *asymptotic entropy production*. The quantity $\mathrm{d}S_+$ represents the increase in entropy per unit time, in the limit of large times. It does not depend on the initial state of the system.

Remark. One sees easily that the expectation of $\mathrm{d}S_+$ is constant in the state $\varrho_+ \otimes \varrho_\mathcal{C}$ (see also (2.24)).

Relations (2.23) and (2.32) lead us to the *average* 2nd *law of thermodynamics*,

$$\mathrm{d}E_+ = T_\mathcal{E} \mathrm{d}S_+, \quad T_\mathcal{E} = 1/\beta_\mathcal{E}.$$

3 Random repeated interactions

We now turn to the situation where the interactions are taken at random. This randomness may have various origins: either the interaction time may vary, or the reference state of the elements in the chain (via their temperature for example), or even the elements themselves may differ (one can imagine that some impurities occur from time to time). In a first part (Section 3.1) we show how to generalize the analysis of Section 2 to the situation of non identical interactions, and reduce the problem to a matrix product. In Section 3.2, we then study infinite products of iid random matrices.

3.1   From repeated interactions to matrices products

The system $\mathcal{S}$ now interacts with a chain $\mathcal{C} = \mathcal{E}_1 + \mathcal{E}_2 + \cdots$ of independent quantum systems, but the elements $\mathcal{E}_m$ may be different. Each element is now described via a Hilbert space $\mathcal{H}_{\mathcal{E}_m}$, a Von Neumann algebra $\mathfrak{M}_{\mathcal{E}_m} \subset \mathcal{B}(\mathcal{H}_{\mathcal{E}_m})$ of observables. The uncoupled dynamics of $\mathcal{E}_m$ is given by a group of $*$-automorphism $\mathbb{R} \ni t \mapsto \alpha_{\mathcal{E}_m}^t(\cdot)$. We also introduce reference vectors $\psi_{\mathcal{E}_m} \in \mathcal{H}_{\mathcal{E}_m}$ which are cyclic and separating for $\mathfrak{M}_{\mathcal{E}_m}$. The Hilbert space of states of the total system is then the tensor product $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{C}}$, where $\mathcal{H}_{\mathcal{C}} = \bigotimes_{m \geq 1} \mathcal{H}_{\mathcal{E}_m}$, and where the infinite product is taken with respect to $\psi_{\mathcal{C}} = \bigotimes_{m \geq 1} \psi_{\mathcal{E}_m}$. The non-interacting dynamics is the product of the individual dynamics, defined on the algebra $\mathfrak{M}_{\mathcal{S}} \bigotimes_{m \geq 1} \mathfrak{M}_{\mathcal{E}_m}$ by $\alpha_{\mathcal{S}}^t \bigotimes_{m \geq 1} \alpha_{\mathcal{E}_m}^t$. The interaction times are now given by $\tau_1, \tau_2, \cdots$ and the interaction operators by $V_1, V_2, \cdots$

As in Section 2, we introduce the Liouville operators $L_{\#}$ which satisfy

$$\alpha_{\#}^t(A_{\#}) = \mathrm{e}^{\mathrm{i}tL_{\#}} A_{\#} \mathrm{e}^{-\mathrm{i}tL_{\#}}, \ t \in \mathbb{R}, \ \text{ and } L_{\#}\psi_{\#} = 0,$$

for any $A_{\#} \in \mathfrak{M}_{\#}$, and where $\#$ stands for either $\mathcal{S}$ or $\mathcal{E}_m$. We then define the (discrete) repeated interaction Schrödinger dynamics of a state vector $\phi \in \mathcal{H}$, for $m \geq 0$, by

$$U(m)\phi := \mathrm{e}^{-\mathrm{i}\tau_m \widetilde{L}_m} \cdots \mathrm{e}^{-\mathrm{i}\tau_2 \widetilde{L}_2} \mathrm{e}^{-\mathrm{i}\tau_1 \widetilde{L}_1} \phi,$$

where

$$\widetilde{L}_k = L_k + \sum_{n \neq k} L_{\mathcal{E}_n} \tag{3.1}$$

describes the dynamics of the system during the time interval $[\tau_1 + \cdots + \tau_{k-1}, \tau_1 + \cdots + \tau_k)$, which corresponds to the time-step k of our discrete process. Hence L_k is

$$L_k = L_{\mathcal{S}} + L_{\mathcal{E}_k} + V_k, \tag{3.2}$$

acting on $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}_k}$. Of course, we understand that the operator $L_{\mathcal{E}_n}$ in (3.1) acts nontrivially only on the n-th factor of the Hilbert space $\mathcal{H}_{\mathcal{C}}$ of the chain.

Our goal is to understand the large-time asymptotics $(m \to \infty)$ of expectations

$$\varrho\left(U(m)^* O U(m)\right) = \varrho(\alpha^m(O)),$$

for normal states ϱ and certain classes of observables O. In this section, for simplicity, we only consider observables on the system $\mathcal{S}$, i.e. $O = A_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{C}}$. For more general observables, e.g. instantaneous observables, we refer to [5].

We now apply the same strategy as in the identical interactions case. We introduce the C-Liouvilleans K_m, denote by $P := \mathbb{1}_{\mathcal{S}} \otimes |\psi_{\mathcal{C}}\rangle\langle\psi_{\mathcal{C}}|$ the orthogonal projection on the system $\mathcal{S}$, and define the reduced operators

$$M_m := P\mathrm{e}^{\mathrm{i}\tau_m K_m} P, \tag{3.3}$$

which are identified with operators on $\mathcal{H}_{\mathcal{S}}$. In the same way as in the previous section, one can show that

$$\langle \psi_0, \alpha^m(A_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{C}})\psi_0 \rangle = \langle \psi_{\mathcal{S}}, M_1 \cdots M_m A_{\mathcal{S}} \psi_{\mathcal{S}} \rangle. \tag{3.4}$$

In order to study the long time behaviour ($m \to \infty$) of the system we thus have to understand the limit

$$\lim_{m\to\infty} M_1 \cdots M_m. \tag{3.5}$$

Finally, note the two following properties (see Proposition 2.1) of the operators M_k which will be crucial in our approach:

1. For all k, M_k is a contraction of $\mathcal{H}_\mathcal{S}$ endowed with the norm $||| \cdot |||$.

2. For all k, $M_k \psi_\mathcal{S} = \psi_\mathcal{S}$.

3.2 Product of random matrices

In this section, we study the convergence of products of matrices of the form (3.5), where $(M_k(\omega))$ is a sequence of independant identically distributed random matrices satisfying properties 1. and 2. above. Note that the norm $||| \cdot |||$ and the vector $\psi_\mathcal{S}$ depend neither on k nor on ω. However, it appears that the particular choice of the norm $||| \cdot |||$ and of the vector $\psi_\mathcal{S}$ are of no importance. The only relevant point is that such quantities exist. We therefore introduce the following class of random matrices.

Let $M(\omega)$ be a random matrix on $\mathbb{C}^d$, with probability space $(\Omega, \mathcal{F}, \mathrm{p})$. We say that $M(\omega)$ is a *random reduced dynamics operator* (RRDO) if

(1) There exists a norm $||| \cdot |||$ on $\mathbb{C}^d$ such that, for all ω, $M(\omega)$ is a contraction on $\mathbb{C}^d$ endowed with the norm $||| \cdot |||$.

(2) There is a vector $\psi_\mathcal{S}$, constant in ω, such that $M(\omega)\psi_\mathcal{S} = \psi_\mathcal{S}$, for all ω.

We normalize $\psi_\mathcal{S}$ such that $\|\psi_\mathcal{S}\| = 1$ where $\| \cdot \|$ denotes the euclidean norm. To an RRDO $M(\omega)$, we associate the (iid) *random reduced dynamics process* (RRDP)

$$\Psi_n(\overline{\omega}) := M(\omega_1) \cdots M(\omega_n), \qquad \overline{\omega} = (\omega_n)_n \in \Omega^{\mathbb{N}^*},$$

and we define the probability measure $\mathrm{d}\mathbb{P}$ on $\Omega^{\mathbb{N}^*}$ in a standard fashion by

$$\mathrm{d}\mathbb{P} = \Pi_{j \geq 1}\mathrm{d}\mathrm{p}_j, \quad \text{where} \quad \mathrm{d}\mathrm{p}_j \equiv \mathrm{d}\mathrm{p}, \quad \forall j \in \mathbb{N}^*.$$

We will denote by $\mathbb{E}(f)$ the expectation value of a random variable $f(\omega)$. Finally, throughout the paper, *a.s.* stands for "almost surely".

Remark: Another important class of systems falling into this category are inhomogeneous Markov chains with *random transition matrices*, i.e. products of *random stochastic matrices*. Stochastic matrices indeed satisfy (1) and (2) with the $\| \cdot \|_\infty$ norm and $\psi_\mathcal{S} = \frac{1}{\sqrt{d}}(1, \cdots, 1)^T$.

As noted in Proposition 2.1, the spectrum of any RDO lies inside the complex unit disk, and 1 is always an eigenvalue with *common* eigenvector $\psi_\mathcal{S}$.

Definition 3.1 *Let $\mathcal{M}_{(E)}$ be the set of RRDO's M which satisfy the ergodic assumption (E) (with invariant vector $\psi_\mathcal{S}$).*

The following result on the product of an iid sequence of RRDO's is the main result of [4].

Theorem 3.2 *([4]) Let $M(\omega)$ be a random reduced dynamics operator. Suppose that* $\mathrm{p}(M(\omega) \in \mathcal{M}_{(E)}) > 0$. *Then* $\mathbb{E}(M) \in \mathcal{M}_{(E)}$, *and we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} M(\omega_1)\cdots M(\omega_n) = P_{1,\mathbb{E}(M)} = |\psi_{\mathcal{S}}\rangle\langle\theta|, \quad \mathbb{P}-a.s., \qquad (3.6)$$

where $P_{1,\mathbb{E}(M)}$ is the (rank one) spectral projection of $\mathbb{E}(M)$ corresponding to the eigenvalue 1, *and*

$$\theta = P_{1,\mathbb{E}(M)}^* \psi_{\mathcal{S}}. \qquad (3.7)$$

Remarks.   1. If moreover the adjoint operators $M^*(\omega)$ have a common invariant vector $\psi_{\mathcal{S}}^*$, then we have the following stronger convergence (see [4]):

$$\lim_{n\to\infty} M(\omega_1)\cdots M(\omega_n) = P_{1,\mathbb{E}(M)} = \frac{|\psi_{\mathcal{S}}\rangle\langle\psi_{\mathcal{S}}^*|}{\langle\psi_{\mathcal{S}}^*|\psi_{\mathcal{S}}\rangle}, \quad \mathbb{P}-a.s.,$$

and the convergence is exponentially fast in n. This applies in particular to products of bistochastic matrices with $\psi_{\mathcal{S}}^* = \psi_{\mathcal{S}} = \frac{1}{\sqrt{d}}(1,\cdots,1)^T$. (See also Theorem 4.3).

2. The Perron-Frobenius Theorem asserts that any stochastic matrix whose entries are all nonzero belongs to $\mathcal{M}_{(E)}$.

When applied to quantum repeated interactions, the above theorem shows

Theorem 3.3 *[4]   Let $\alpha^{n,\overline{\omega}}$ be the random repeated interaction dynamics determined by an RRDO $M(\omega)$. Suppose that* $\mathrm{p}(M(\omega) \in \mathcal{M}_{(E)}) > 0$. *Then for any normal state ϱ and any $A_{\mathcal{S}} \in \mathfrak{M}_{\mathcal{S}}$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \varrho\left(\alpha^{n,\overline{\omega}}(A_{\mathcal{S}})\right) = \langle\theta, A_{\mathcal{S}}\psi_{\mathcal{S}}\rangle =: \varrho_+(A_{\mathcal{S}}), \quad \mathbb{P}-a.s. \qquad (3.8)$$

where θ is given by (3.7).

Remark.   Assume that $M(\omega) \in \mathcal{M}_{(E)}$ for all ω, and denote by $\varrho_{+,\omega}$ the asymptotic state of the *identical* repeated interaction system with RDO $M(\omega)$ (see (2.11)). Then in general we do not have $\varrho_+ = \mathbb{E}(\varrho_{+,\omega})$. In other words, the asymptotic state of a random repeated interaction system *is not* the average of the asymptotic states of the corresponding identical repeated interaction systems.

4   An example: spin systems

We consider the application of our general results to the situation where both $\mathcal{S}$ and $\mathcal{E}$ are two-level "atoms", and where the interaction induces an energy exchange process. This is a particular case of the third example in [3]. We present the results without proofs, a complete analysis of this (and more general) examples are given in [3, 5].

4.1 Description of the model

The observable algebra for $\mathcal{S}$ and for $\mathcal{E}$ is $\mathfrak{A}_\mathcal{S} = \mathfrak{A}_\mathcal{E} = M_2(\mathbb{C})$. Let $E_\mathcal{S}, E_\mathcal{E} > 0$ be the "excited" energy level of $\mathcal{S}$ and of $\mathcal{E}$, respectively. Accordingly, the Hamiltonians are given by $h_\mathcal{S} = \begin{bmatrix} 0 & 0 \\ 0 & E_\mathcal{S} \end{bmatrix}$ and $h_\mathcal{E} = \begin{bmatrix} 0 & 0 \\ 0 & E_\mathcal{E} \end{bmatrix}$ The dynamics are given by $\alpha_\mathcal{S}^t(A) = \mathrm{e}^{ith_\mathcal{S}} A \mathrm{e}^{-ith_\mathcal{S}}$ and $\alpha_\mathcal{E}^t(A) = \mathrm{e}^{ith_\mathcal{E}} A \mathrm{e}^{-ith_\mathcal{E}}$. We choose the reference state of $\mathcal{E}$ to be the Gibbs state at inverse temperature β, i.e.,

$$\varrho_{\beta,\mathcal{E}}(A) = \frac{\mathrm{Tr}(\mathrm{e}^{-\beta h_\mathcal{E}} A)}{\mathrm{Tr}(\mathrm{e}^{-\beta h_\mathcal{E}})}, \tag{4.1}$$

and we choose the reference state for $\mathcal{S}$ to be the tracial state, $\varrho_{0,\mathcal{S}}(A) = \frac{1}{2}\mathrm{Tr}(A)$.

To find a Hilbert space description of the system, one performs the GNS construction of $(\mathfrak{A}_\mathcal{S}, \varrho_{0,\mathcal{S}})$ and $(\mathfrak{A}_\mathcal{E}, \varrho_{\beta,\mathcal{E}})$, see e.g. [2, 3]. In this representation, the Hilbert spaces are $\mathcal{H}_\mathcal{S} = \mathcal{H}_\mathcal{E} = \mathbb{C}^2 \otimes \mathbb{C}^2$, and the vectors representing $\varrho_{0,\mathcal{S}}$ and $\varrho_{\beta,\mathcal{E}}$ are

$$\psi_\mathcal{S} = \frac{1}{\sqrt{2}} \sum_{j=1,2} \varphi_j \otimes \varphi_j \quad \text{and} \quad \psi_{\beta,\mathcal{E}} = \frac{1}{\sqrt{\mathrm{Tr}\ \mathrm{e}^{-\beta h_\mathcal{E}}}} \left[\varphi_1 \otimes \varphi_1 + \mathrm{e}^{-\beta E_\mathcal{E}/2} \varphi_2 \otimes \varphi_2 \right],$$

respectively, i.e., we have $\varrho_{0,\mathcal{S}}(A) = \langle \psi_\mathcal{S}, (A \otimes \mathbb{1})\psi_\mathcal{S} \rangle$ and $\varrho_{\beta,\mathcal{E}}(A) = \langle \psi_{\beta,\mathcal{E}}, (A \otimes \mathbb{1})\psi_{\beta,\mathcal{E}} \rangle$. Here, we have set $\varphi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\varphi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The Liouville operators are then $L_\# = h_\# \otimes \mathbb{1}_{\mathbb{C}^2} - \mathbb{1}_{\mathbb{C}^2} \otimes h_\#$, $\# = \mathcal{S}, \mathcal{E}$. The interaction operator is defined by

$$V := a_\mathcal{S} \otimes \mathbb{1}_{\mathbb{C}^2} \otimes a_\mathcal{E}^* \otimes \mathbb{1}_{\mathbb{C}^2} + a_\mathcal{S}^* \otimes \mathbb{1}_{\mathbb{C}^2} \otimes a_\mathcal{E} \otimes \mathbb{1}_{\mathbb{C}^2},$$

where $a_\# = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $a_\#^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ are the annihilation and creation operators respectively, of $\# = \mathcal{S}, \mathcal{E}$. Finally, the interacting Liouvillean is $L = L_\mathcal{S} + L_\mathcal{E} + \lambda V$ where λ is a coupling constant.

4.2 Results

In general, the main issue is the spectral analysis of the reduced operator M. However, it turns out that in this particular example one can compute explicitly the spectrum of M (see [3, 5] for more general situations).

Lemma 4.1 *The eigenvalues of M are $1, e_0, e_-, e_+$ where*

$$e_0 = \left| \frac{\left(E_\mathcal{S} - E_\mathcal{E} - \sqrt{(E_\mathcal{S} - E_\mathcal{E})^2 + 4\lambda^2} \right)^2 + 4\lambda^2 \mathrm{e}^{i\tau\sqrt{(E_\mathcal{S} - E_\mathcal{E})^2 + 4\lambda^2}}}{\left(E_\mathcal{S} - E_\mathcal{E} - \sqrt{(E_\mathcal{S} - E_\mathcal{E})^2 + 4\lambda^2} \right)^2 + 4\lambda^2} \right|^2,$$

$$e_- = \mathrm{e}^{i\tau \frac{E_\mathcal{S} + E_\mathcal{E} - \sqrt{(E_\mathcal{S} - E_\mathcal{E})^2 + 4\lambda^2}}{2}}$$
$$\times \frac{\left(E_\mathcal{S} - E_\mathcal{E} - \sqrt{(E_\mathcal{S} - E_\mathcal{E})^2 + 4\lambda^2} \right)^2 + 4\lambda^2 \mathrm{e}^{i\tau\sqrt{(E_\mathcal{S} - E_\mathcal{E})^2 + 4\lambda^2}}}{\left(E_\mathcal{S} - E_\mathcal{E} - \sqrt{(E_\mathcal{S} - E_\mathcal{E})^2 + 4\lambda^2} \right)^2 + 4\lambda^2},$$

$$e_+ = e^{-i\tau \frac{E_\mathcal{S} + E_\mathcal{E} - \sqrt{(E_\mathcal{S} - E_\mathcal{E})^2 + 4\lambda^2}}{2}}$$

$$\times \frac{\left(E_\mathcal{S} - E_\mathcal{E} - \sqrt{(E_\mathcal{S} - E_\mathcal{E})^2 + 4\lambda^2}\right)^2 + 4\lambda^2 e^{-i\tau \sqrt{(E_\mathcal{S} - E_\mathcal{E})^2 + 4\lambda^2}}}{\left(E_\mathcal{S} - E_\mathcal{E} - \sqrt{(E_\mathcal{S} - E_\mathcal{E})^2 + 4\lambda^2}\right)^2 + 4\lambda^2}.$$

As a consequence, M satisfies (E) if and only if $\tau\sqrt{(E_\mathcal{S} - E_\mathcal{E})^2 + 4\lambda^2} \notin 2\pi\mathbb{Z}$.

We first consider this system with identical interactions. We thus have the following

Theorem 4.2 *Suppose that $\tau\sqrt{(E_\mathcal{S} - E_\mathcal{E})^2 + 4\lambda^2} \notin 2\pi\mathbb{Z}$, then the spin-spin system satisfies Theorem 2.3. Moreover, the asymptotic state ϱ_+ is the Gibbs state for the system $\mathcal{S}$ at inverse temperature $\beta' := \frac{E_\mathcal{E}}{E_\mathcal{S}}\beta$.*

The situation where the interaction time τ is random is particular in the sense that the adjoint operators $M^*(\omega)$ then have a common invariant vector, so that we don't need to consider ergodic means (see Remark 1 after Theorem 3.2).

Theorem 4.3 *Let $\Omega \ni \omega \to \tau(\omega) \in \mathbb{R}_+^*$ be a random variable. We assume that $\mathrm{p}\left(\tau(\omega)\sqrt{(E_\mathcal{S} - E_\mathcal{E})^2 + 4\lambda^2} \notin 2\pi\mathbb{Z}\right) > 0$. Then there exists $\gamma > 0$ such that for any initial state ϱ, and for any $A_\mathcal{S} \in \mathfrak{M}_\mathcal{S}$,*

$$\varrho(\alpha^{n,\overline{\omega}}(A_\mathcal{S} \otimes \mathbb{1}_\mathcal{C})) = \varrho_{\beta',\mathcal{S}}(A_\mathcal{S}) + O(e^{-\gamma n}), \quad \mathbb{P} - a.s.,$$

where $\varrho_{\beta',\mathcal{S}}$ is the Gibbs state for the system $\mathcal{S}$ at inverse temperature $\beta' := \frac{E_\mathcal{E}}{E_\mathcal{S}}\beta$.

Remark: Physically, it is reasonable to assume that the interaction time $\tau(\omega)$ takes values in an interval of uncertainty, since during an experiment, the interaction time cannot be controlled exactly, but rather fluctuates around some mean value. Hence, the assumption of Theorem 4.3 is certainly satisfied.

Finally, we consider the case where the atoms injected in the cavity vary, which we model via their "excited" energy level.

Theorem 4.4 *Let $\Omega \ni \omega \to E_\mathcal{E}(\omega) \in \mathbb{R}_+^*$ be a random variable. We assume that*

$$\mathrm{p}\left(\tau\sqrt{(E_\mathcal{S} - E_\mathcal{E}(\omega))^2 + 4\lambda^2} \notin 2\pi\mathbb{Z}\right) > 0. \tag{4.2}$$

Then for any initial state ϱ and for any $A_\mathcal{S} \in \mathfrak{M}_\mathcal{S}$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \varrho(\alpha^{n,\overline{\omega}}(A_\mathcal{S} \otimes \mathbb{1}_\mathcal{C})) = \varrho_{\overline{\beta},\mathcal{S}}(A_\mathcal{S}), \quad \mathbb{P} - a.s.,$$

where $\varrho_{\overline{\beta},\mathcal{S}}$ is the Gibbs state (4.1) of the small system for the inverse temperature

$$\overline{\beta} := -\frac{1}{E_\mathcal{S}} \log\left(2\left[1 - (1 - \mathbb{E}(e_0(\omega)))^{-1}\,\mathbb{E}\left((1 - e_0(\omega))(1 - 2Z_{\beta'(\omega),\mathcal{S}}^{-1})\right)\right]^{-1} - 1\right), \tag{4.3}$$

where $Z_{\beta',\mathcal{S}} = \mathrm{Tr}(e^{-\beta' h_\mathcal{S}})$.

Remarks. 1. The expression (4.3) for the "asymptotic temperature" is of course also valid in the case of Theorem 4.3. However, in that case, $\beta'(\omega) \equiv \beta'$ and $\overline{\beta} = \beta'$.

2. In general, $\overline{\beta} \neq \mathbb{E}(\beta'(\omega))$ which reflects the fact that we usually have $\varrho_+ \neq \mathbb{E}(\varrho_{+,\omega})$, as mentioned in the remark after Theorem 3.3.

3. Physically, one can imagine that different kinds of atoms are injected in the cavity, or that from time to time some "impurity" occurs. It is therefore reasonable to consider a finite probability space Ω, so that assumption (4.2) is satisfied provided at least one atom has excited energy $E_{\mathcal{E}}$ satisfying $\tau\sqrt{(E_{\mathcal{S}} - E_{\mathcal{E}})^2 + 4\lambda^2} \notin 2\pi\mathbb{Z}$.

4. Of course, one can combine Theorems 4.3 and 4.4 to obtain a result for systems where the interaction times, excitation energies (and other parameters, like temperatures of the "incoming atoms") fluctuate (see [5]).

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