Mathematical Physics/Functional Analysis

The ground state problem for a quantum Hamiltonian describing friction

Existence d'un état fondamental pour un hamiltonien quantique décrivant le frottement

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Abstract

In this note, we consider the quantum version of a hamiltonian model describing friction. This model consists of a particle which interacts with a bosonic reservoir representing a homogeneous medium through which the particle moves. We show that if the particle is confined, then the Hamiltonian admits a ground state if and only if a suitable infrared condition is satisfied. The latter is violated in the case of linear friction, but satisfied when the friction force is proportional to a higher power of the particle speed.

Résumé

Dans cette note, on considère la version quantique d'un modèle hamiltonien décrivant le phénomène de frottement. Ce modèle consiste en une particule en interaction avec un réservoir de bosons représentant un milieu homogène dans lequel la particule se déplace. On montre que si la particule est confinée, alors le hamiltonien admet un état fondamental si et seulement si une condition infrarouge adaptée est satisfaite. Cette dernière est violée dans le cas d'un frottement linéaire, mais satisfaite lorsque la force de frottement est proportionnelle à une puissance plus élevée de la vitesse de la particule.

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Version française abrégée

Dans cette note, on considère la version quantique du hamiltonien classique (1) introduit dans [3]. Le système étudié décrit une particule, d-dimensionnelle, se déplaçant dans un environnement dissipatif homogène (représenté par un champ scalaire) de telle façon que celle-ci ressente une force de frottement effective. Cette dernière dépend de la vitesse de la particule en fonction d'un certain paramètre n entier (≥ 3). En particulier, la force de frottement est proportionnelle à la vitesse si et seulement si $n = 3$. On parle alors de frottement linéaire.

Nous considérons ici le cas où la particule est confinée à l'aide d'un potentiel extérieur V. L'espace de Hilbert $\mathcal{H}$ décrivant l'état du système s'écrit $L^2(\mathbb{R}^d, \mathrm{d}q) \otimes \mathcal{F}$ où $\mathcal{F}$ est l'espace de Fock bosonique sur $L^2(\mathbb{R}^{d+n}, \mathrm{d}x\,\mathrm{d}k)$. On s'intéresse d'abord au caractère auto-adjoint du hamiltonien H (Proposition 1). Cette question est en effet équivalente à celle de l'existence de solutions pour l'équation de Schrödinger correspondante : $i\partial_t \psi_t = H\psi_t$, où $\psi_t \in \mathcal{H}$ représente l'état du système au temps t.

Une fois le hamiltonien H bien défini (en tant qu'opérateur auto-adjoint), il est aisé de voir que celui-ci est borné inférieurement. Une question naturelle est alors de savoir si la borne inférieure du spectre de H est une valeur propre. On parle alors d'existence d'un état fondamental. Cette borne inférieure n'étant pas isolée du spectre essentiel, le fait que celle-ci soit une valeur propre n'est pas évident. On peut montrer qu'un état fondamental existe pourvu qu'un certaine condition infrarouge (IR) soit vérifée (Théorème 3). Réciproquement, si cette condition n'est pas satisfaite, on montre qu'il n'y a pas d'état fondamental sous l'hypothèse supplémentaire que la particule ait une charge non nulle (Proposition 4).

Finalement, on peut mettre en relation l'existence, ou non, d'un état fondamental avec le type de frottement décrit. En effet, lorsque celui-ci est non-linéaire, il y a un état fondamental, tandis que lorsque le frottement est linéaire, il n'y a génériquement pas (si la charge de la particule n'est pas nulle) d'état fondamental.

1. Introduction

Open systems are models in which a small system (particle) interacts with a large one (environment). They are used to describe dissipation phenomena from a Hamiltonian point of view. In [3], together with S. De Bièvre, we introduced a classical Hamiltonian model of a particle moving through a homogeneous dissipative medium at zero temperature in such a way that the particle experiences an effective *linear* friction force proportional to its velocity. We describe briefly this model. The particle moves in $\mathbb{R}^d$, we denote by q its position and by p its momentum. The mass of the particle is set to one, it plays no role in our study. The medium consists at each point in the physical space $\mathbb{R}^d$ of a vibration field modelling an obstacle with which the particle exchanges energy and momentum. Namely, for all $x \in \mathbb{R}^d$, this obstacle is described by a scalar field $\mathbb{R}^n \ni y \mapsto \phi(x, y) \in \mathbb{R}$. We would like to emphasize that the variable y should not be interpreted geometrically, it indexes the degrees of freedom of the obstacle. We will denote by $\pi(x, y)$ the field conjugated to ϕ. The phase space of the system is then $\mathcal{E} := \mathbb{R}^d \times \mathbb{R}^d \times E \times L^2(\mathbb{R}^{d+n})$, where E is the completion of $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^n)$ with the norm $\|\phi\| = \|\nabla_y \phi\|_{L^2}$, and the Hamiltonian is

$$H(q,p,\phi,\pi) = \frac{p^2}{2} + V(q) + \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}y \, (c^2 |\nabla_y \phi(x,y)|^2 + |\pi(x,y)|^2) + \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}y \, \rho_1(x-q)\rho_2(y)\phi(x,y). \quad (1)$$

This Hamiltonian consists of three parts. The first one, $\frac{p^2}{2} + V(q)$, is the energy of the particle (V is an external potential). The second one, $\frac{1}{2}\int (c^2|\nabla_y \phi|^2 + |\pi|^2)$, is the energy of the field and c is the speed of the wave propagation. Finally, the last term is the interaction term. The functions ρ_1 and ρ_2 determine the coupling between the particle and the field and are smooth radial functions with compact support. One can think of the function ρ_1 as describing the charge distribution of the particle. The "total" charge of the particle is then $\int \rho_1(x)\mathrm{d}x = \hat{\rho}_1(0)$.

Another way of describing this model is the following. If we perform a Fourier transform of the fields ϕ and π with respect to the variable y, the Hamiltonian becomes

$$H = \frac{p^2}{2} + V(q) + \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}k \, (c^2|k|^2|\hat{\phi}(x,k)|^2 + |\hat{\pi}(x,k)|^2) + \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}k \, \rho_1(x-q)\hat{\rho}_2(k)\hat{\phi}(x,k).$$

In this representation, the obstacles are described by reservoirs of oscillators, indexed by $k \in \mathbb{R}^n$, and of frequency $\omega(k) = c|k|$. The quantity $\hat{\phi}(x,k)$ is then the amplitude of the oscillator located at point x and of index k. Now, the parameter n can be related to the number of oscillators present at the point x through the volume element $\mathrm{d}k = |k|^{n-1}\mathrm{d}|k|\mathrm{d}\Omega$, where $\mathrm{d}\Omega$ is the volume element of the unit sphere S^{n-1}. It is therefore not surprising that the kind of friction this model describes depends on the parameter n as we will see.

We studied the asymptotic behaviour ($t \to +\infty$) of the particle motion for two categories of potentials: linear ones (constant external force) and confining ones. We proved that under suitable assumptions (on the initial conditions), for c sufficiently large and, most importantly, $n = 3$, the particle behaves asymptotically as if its motion was governed by the effective equation

$$\ddot{q}(t) + \gamma\dot{q}(t) = -\nabla V(q(t)),$$

where the friction coefficient γ is non negative and is explicit in terms of the parameters of the model:

$$\gamma := \frac{\pi}{c^3}|\hat{\rho}_2(0)|^2 \int_{\mathbb{R}^n} \mathrm{d}\xi \int_{\mathbb{R}^{d-1}} \mathrm{d}\eta \, |\hat{\rho}_1(|\xi|,\eta)|^2. \tag{2}$$

For instance, if $V = -F\cdot q$, which means that we apply a constant external force F, the particle reaches (exponentially fast) an asymptotic velocity $v(F) = \frac{F}{\gamma}$ which is proportional to the applied force. This is, in particular, at the origin of Ohm's law. If now V is confining, the particle stops at a critical point of the potential (still exponentially if this point is a minimum).

In [3], we mostly concentrated on linear friction, which is the reason why the condition $n = 3$ was required. However, for other values of n (> 3), our model still describes friction. Indeed, the reaction force of the environment on a particle moving with velocity v takes the form $-\gamma|v|^{n-3}v$ (at least for small v and where γ is defined in (2)). One can therefore see that we have linear friction when $n = 3$, and otherwise a friction force which is proportional to some other power of the velocity of the particle.

Our goal is now to begin the study of the quantum version of the model (1). We will consider here the case where V is confining, and look at the question of existence of a ground state. Since the speed of the wave propagation will not play a role, we take it equal to 1.

2. The quantum model

The Hilbert space of the system will be $\mathcal{H} := L^2(\mathbb{R}^d, \mathrm{d}q) \otimes \mathcal{F}$, where $L^2(\mathbb{R}^d, \mathrm{d}q)$ represents the space of the particle and $\mathcal{F}$ is the bosonic Fock space over $\mathfrak{h} = L^2(\mathbb{R}^{d+n}, \mathrm{d}x\,\mathrm{d}k)$ and represents the space of the field:

$$\mathcal{F} := \oplus_{m=0}^{\infty}\mathfrak{h}_m,$$

where $\mathfrak{h}_m = \otimes_s^m \mathfrak{h}$ denotes the m-fold symmetric tensor product of $\mathfrak{h}$ and $\mathfrak{h}_0 = \mathbf{C}$.

The free Hamiltonian H_0 (*i.e.* without interaction) of the system will be

$$H_0 = (-\frac{1}{2}\Delta + V) \otimes 1 + 1 \otimes \int_{\mathbf{R}^{d+n}} \mathrm{d}x\, \mathrm{d}k\, \omega(x,k) a^*(x,k) a(x,k) = H_p \otimes 1 + 1 \otimes H_f, \tag{3}$$

where $\omega(x,k) = \omega(k) = |k|$ is the dispersion relation of the bosons and $a(x,k), a^*(x,k)$ are the distributional annihilation and creation operators on $\mathcal{F}$ (see *e.g.* [8]). They satisfy the usual canonical commutation relations:

$$[a(x,k), a^*(x',k')] = \delta(x-x')\delta(k-k'), \quad [a(x,k), a(x',k')] = [a^*(x,k), a^*(x',k')] = 0. \tag{4}$$

The interaction term is given by

$$H_I := \int \mathrm{d}x\, \mathrm{d}k\ \rho_1(x-Q)\frac{\hat{\rho}_2(k)}{\sqrt{2\omega(k)}} \otimes a^*(x,k) + \rho_1(x-Q)\frac{\overline{\hat{\rho}}_2(k)}{\sqrt{2\omega(k)}} \otimes a(x,k), \tag{5}$$

where ρ_1 and ρ_2 are two smooth radial functions with compact support, and $\rho_1(x-Q)$ is the multiplication operator on $L^2(\mathbf{R}^d, \mathrm{d}q)$ by the function $\rho_1(x-\cdot)$. Finally, the full Hamiltonian of the interacting system is therefore

$$H := H_0 + H_I.$$

3. Self-adjointness and existence of a ground state

From now, we will suppose that $n \geq 3$.

The first step in the study of a quantum Hamiltonian is to prove that it is a well defined self-adjoint operator on some dense domain of $\mathcal{H}$. We assume that the potential V satisfies

(C) $\quad V \in L^2_{loc}(\mathbf{R}^d), \lim_{|q|\to\infty} V(q) = +\infty.$

This hypothesis ensures that H_p is well defined and is selfadjoint on $\mathcal{D}(H_p) = \{\psi \in L^2(\mathbf{R}^d) | H_p\psi \in L^2(\mathbf{R}^d)\}$ ([8], Theorem X.28). We also know that H_f is selfadjoint on its domain $\mathcal{D}(H_f)$ ([7], Chapter VIII.10). One then easily proves that H_0 is selfadjoint on $\mathcal{D}(H_0) = (\mathcal{D}(H_p) \otimes \mathcal{F}) \cap (L^2(\mathbf{R}^d) \otimes \mathcal{D}(H_f))$. We now have the following result

Proposition 1 *Suppose that $n \geq 3$, and V satisfies condition (C). Then H is selfadjoint on $\mathcal{D}(H) = \mathcal{D}(H_0)$. Moreover, H is essentially selfadjoint on any core for H_0, and it is bounded from below.*

This result is a consequence of the Kato-Rellich theorem ([8], Theorem X.12). The only ingredient needed is that H_I is infinitesimally H_0-bounded, which follows from the following lemma.

Lemma 2 *Under the hypothesis of Proposition 1, for all $\Psi \in \mathcal{D}(H_0)$, we have:*

$$(i) \quad \| \int \mathrm{d}x\, \mathrm{d}k\, \frac{\hat{\rho}_2(k)}{\sqrt{\omega(k)}} \rho_1(x-Q) \otimes a(x,k)\Psi\|^2_{\mathcal{H}} \leq \Big[\int \mathrm{d}x\, \mathrm{d}k |\rho_1(x)|^2 \frac{|\hat{\rho}_2(k)|^2}{\omega(k)^2} \Big] \|(1 \otimes H_f^{\frac{1}{2}})\Psi\|^2_{\mathcal{H}}.$$

$$(ii) \quad \| \int \mathrm{d}x\, \mathrm{d}k\, \frac{\hat{\rho}_2(k)}{\sqrt{\omega(k)}} \rho_1(x-Q) \otimes a^*(x,k)\Psi\|^2_{\mathcal{H}} \leq \Big[\int \mathrm{d}x \mathrm{d}k |\rho_1(x)|^2 \frac{|\hat{\rho}_2(k)|^2}{\omega(k)^2} \Big] \|(1 \otimes H_f^{\frac{1}{2}})\Psi\|^2_{\mathcal{H}}$$

$$+ \Big[\int \mathrm{d}x\, \mathrm{d}k |\rho_1(x)|^2 \frac{|\hat{\rho}_2(k)|^2}{\omega(k)} \Big] \|\Psi\|^2_{\mathcal{H}}.$$

We now turn to the question of existence of a ground state. Using Proposition 1, we know that $E_0 := \inf \sigma(H) > -\infty$, where $\sigma(H)$ denotes the spectrum of H. E_0 is called the ground state energy. We will say that the Hamiltonian H admits a ground state if E_0 is an eigenvalue and then call ground state any corresponding eigenvector. The question of existence of a ground state is essential before studying questions such as scattering theory or return to equilibrium. Since E_0 is not isolated from the essential spectrum, it is not clear wether it is an eigenvalue or not.

It is well known that one of the main obstacles to the existence of a ground state, in those models where a particle interacts with a field, comes from the so-called *infrared catastrophe*, which is due to the behaviour of $\omega(k)$ for small k and in particular to the fact that $\omega(0) = \inf \omega(k) = 0$. This infimum is sometimes called the mass of the bosons. We will then need the following *infrared* condition on the coupling:

(IR) $\qquad \int_{\mathbb{R}^n} dk \frac{|\hat{\rho}_2(k)|^2}{\omega(k)^3} < +\infty.$

This condition is used to control the number of bosons which have low energy (soft bosons). Our main result is the following:

Theorem 3 *Suppose $n \geq 3$, V satisfies hypothesis (C), and $\hat{\rho}_2$ satisfies (IR). Then H has a ground state.*

On the other hand, if (IR) is not satisfied, we have the following reciprocal

Proposition 4 *Suppose $n \geq 3$, V satisfies hypothesis (C), $\hat{\rho}_2$ does not satisfy (IR) and $\hat{\rho}_1(0) \neq 0$, then H has no ground state.*

The proofs of these results can be found in [2].

We would like now to make some comment on these results. For that purpose, let us suppose that $\hat{\rho}_2(0) \neq 0$. Indeed, this is the only interesting case since the friction coefficient γ vanishes together with $\hat{\rho}_2(0) \neq 0$ (see (2)). One can therefore see that (IR) is satisfied if and only if $n \geq 4$. Thus, if the friction is non-linear, there is a ground state whereas if the friction is linear ($n = 3$) there is generically no ground state (if $\hat{\rho}_1(0) \neq 0$, which means that the total charge of the particle does not vanish). Hence, we have a class of models, depending on a parameter n, describing friction phenomena, linear or proportional to a power of the velocity of the particle, for which we are able to say wether they admit a ground state or not.

4. Comparison with the Nelson model

In this section, we would like to emphasize the difference between our model and the so called Nelson model (or more generally the Pauli-Fierz Hamiltonians following the terminology of [4]). The Hilbert space for the Nelson model is $\tilde{\mathcal{H}} = L^2(\mathbb{R}^d, \mathrm{d}q) \otimes \tilde{\mathcal{F}}$ where $\tilde{\mathcal{F}}$ is the bosonic Fock space over $L^2(\mathbb{R}^d, \mathrm{d}k)$, and the Hamiltonian is

$$H_{\mathrm{nels}} = (-\Delta + V) \otimes 1 + 1 \otimes \int_{\mathbb{R}^d} \mathrm{d}k \omega(k) a^*(k) a(k) + \int_{\mathbb{R}^d} \mathrm{d}k \left(\frac{\hat{\rho}(k) e^{-ikQ}}{\sqrt{2\omega(k)}} \otimes a^*(k) + \frac{\bar{\hat{\rho}}(k) e^{ikQ}}{\sqrt{2\omega(k)}} \otimes a(k) \right),$$

where $\omega(k) = |k|$.

Despite the formal similarity between our model and the classical Nelson model, we want to stress that they describe physically totally different phenomena. While our model leads to friction, the Nelson model exhibits radiation damping (the dissipation phenomenon which appears in electromagnetism). On the mathematical point of view, the main difference concerns the dispersion relation ω which describes the energy of the bosons. In the Nelson model, this energy goes to infinity when the momentum k of the bosons goes to infinity: a boson of large momentum has high energy. In particular, when we restrict ourselves to a low energy range, which is sufficient to study the ground state energy for instance, then we can control easily the momentum of the bosons. However, this is not the case in our model. Indeed, the dispersion relation does not depend on the momentum of the bosons in the "x-direction". We can a priori have bosons of arbitrarily low energy but arbitrarily large momentum. This is at the origin of some additional difficulties in the proof of existence of a ground state, we have to be more careful in order to control the momentum of the bosons. The proof otherwise uses the following well established strategy: we first prove a similar result for non zero mass (or massive) bosons, and then we let this mass tend to zero. The study of the massive case follows the lines of [1,6]: we first constrain the model to a finite box ($|x| < L$) and then control the error terms as L goes to infinity. Then, when we let the mass go to zero, we adapt the proof of [5].

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