Entropic Fluctuations in Quantum Statistical Mechanics: A Review

T. Benoist[1], L. Bruneau[2], V. Jakšić[3], A. Panati[4], C.-A. Pillet[4]

[1] Université de Toulouse, INUC, UT2J, INSA Toulouse, TSE, CNRS, IMT, Toulouse, France
tristan.benoist@math.univ-toulouse.fr

[2] Département de Mathématiques, CNRS UMR 8088
Cergy Paris Université, 95000 Cergy-Pontoise, France
laurent.bruneau@cyu.fr

[3]Dipartimento di Matematica, Politecnico di Milano
piazza Leonardo da Vinci 32, 20133 Milano, Italy
vojkan.jaksic@polimi.it

[4]Université de Toulon, CNRS, CPT, UMR 7332, 83957 La Garde, France
Aix-Marseille Univ, CNRS, CPT, UMR 7332, Case 907, 13288 Marseille, France
panati@univ-tln.fr, pillet@univ-tln.fr

Abstract. The celebrated Evans–Searles, respectively Gallavotti–Cohen, fluctuation theorem concerns certain universal statistical features of the entropy production rate of a classical system in a transient, respectively steady, state. In this paper, we review and compare several possible extensions of these fluctuation theorems to quantum systems. We present, in an introductory way, the approach proposed in a recent series of works [Lett. Math. Phys. 114, 32 (2024)], [Rev. Math. Phys 38, 2461006 (2026)], [Commun. Math. Phys. 406, 201 (2025)] and [J. Math. Phys. 67, 022101 (2026)], including a previously missing extension of the Gallavotti–Cohen (steady state) version of the FT to two-time measurement entropy production in the algebraic setting of quantum dynamical systems. We focus on the open quantum systems, where a small system interacts with several (infinitely extended) reservoirs, and discuss the following approaches: naive, two-time measurement, phase space contraction and ancilla state tomography. We review the connection with modular theory. The latter provides a way to extend the classical notions to the quantum domain. We discuss the quantum version of the principle of regular entropic fluctuations, introduced in the classical context in [Nonlinearity 24, 699 (2011)]. Finally, we relate the statistical properties of these various notions of entropy production to spectral resonances of quantum transfer operators and apply this approach to a paradigmatic model of non-equilibrium statistical mechanics: the spin-fermion model. The results presented here shed a new light on the nature of entropic fluctuations in quantum statistical mechanics.

Contents

1 Introduction **3**

2 Classical systems **7**

3 Quantum framework **11**

3.1 C^* quantum dynamical systems 11

3.2 Open quantum systems 12

3.3 Relative entropy and entropy observable in finite systems 13

3.4 Modular structure 14

3.5 Modular structure of finite systems 16

4 Finite systems and the transient fluctuation theorem **16**

4.1 Naive quantization 17

4.2 Two-time measurement (2TM) entropy production 18

4.3 Tasaki–Matsui relative modular approach 19

4.4 Quantum phase space contraction (QPSC) 20

4.5 Entropic Ancilla State Tomography (EAST) 21

5 Infinitely extended systems: 3 facets of entropic fluctuations **23**

5.1 2TM protocol in infinitely extended systems 23

5.2 Entropic cocycle and QPSC 27

5.3 Entropic Ancilla State Tomography (EAST) 28

5.4 Comparing the entropic functionals 29

6 The quantum principle of regular entropic fluctuations (PREF) **29**

6.1 Definition of the quantum PREF 29

6.2 PREF and 2TM 31

6.3 PREF and EAST 32

7 Quantum transfer operators approach to PREF **32**

7.1 Liouvillean representation of entropic functionals 32

7.2 Spin Fermion (simple version) 34

1 Introduction

Classical entropic fluctuation theorems (FT) have marked a turning point in non-equilibrium statistical mechanics. Since their discovery in the early 1990's [ECM93, ES94, Gal97], a new area of research has developed. The first numerical experiment [ECM93] were immediately followed by two separate sets of theoretical explanations corresponding to different conceptual frameworks, by Evans and Searles [ES94] on one side and Gallavotti and Cohen [Gal97] on the other, leading respectively to the transient and steady-state fluctuation relations. Particular attention has been given to the question of their universality. Three decades later, they are understood, for Time-Reversal Invariant (TRI) systems, as a statistical refinement of the second law of thermodynamics.

Note that a parallel literature concerning fluctuation relations for work (instead of entropy) developed in the late 1990's, following the seminal contributions of Jarzynski and Crook [Jar97, Cro98]. In this context, a first version of such relations was stated in the earlier works by Bochkov and Kuzovlev [BK77, BK79, BK81], and their contribution was then rediscovered. See the recent monograph [CJPS26] and references therein for more details.

The transient (Evans–Searles) and the steady (Gallavotti–Cohen) fluctuation theorems are two mathematically and physically distinct, although deeply connected, results. Both theorems involve a large deviation principle for the mean entropy production rate (in the sequel, whenever the meaning is clear, we will often abbreviate it as *entropy production*). Recall that, given two classical states μ, ν, *i.e.,* two probability measures on a phase space $\mathscr{X}$, their relative entropy $\mathrm{Ent}(\nu|\mu)$ is the expectation of a function of their Radon–Nikodym derivative $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, see Equation (2.1), and satisfies $\mathrm{Ent}(\nu|\mu) \leq 0$ with equality iff $\nu = \mu$. When applied to an initial state ω and its time evolution ω_t, the so-called entropy balance equation reads

$$\mathrm{Ent}(\omega_t|\omega) = -\omega(c^t)$$

for some suitable random variable c^t called entropic cocycle, and which corresponds to the phase space contraction over the time interval $[0,t]$ (see Equation (2.2) and below). The inequality $\omega(c^t) = -\mathrm{Ent}(\omega_t|\omega) \geq 0$ expresses the second law of thermodynamics in the form of non-negativity of the mean entropy produced over the interval $[0,t]$. This is a statement about the expectation value of c^t; it does not assert monotonicity in t or non-negativity of the instantaneous entropy production. Fluctuation theorems concern a more precise statistical description of the random process $(c^t)_{t\geq 0}$, also interpreted as the phase space contraction, and its law $Q_{\nu,t}$ with respect to a given measure ν.

For Time-Reversal Invariant (TRI) systems, the transient fluctuation relation corresponds to the choice $\nu = \omega$ and asserts that, for any finite time t,

$$\frac{\mathrm{d}\bar{Q}_{\omega,t}}{\mathrm{d}Q_{\omega,t}}(s) = \mathrm{e}^{-s}, \tag{1.1}$$

where $\bar{Q}_{\omega,t}(B) = Q_{\omega,t}(-B)$ is the reflected measure, a relation which directly reads as a refinement of the second law (see Section 2).

The terminology *"fluctuation theorem"* refers to the large time limit, in the Large Deviation Principle (LDP) regime, of the mean entropy production observable $\frac{c^t}{t}$. LDP roughly means (see Definition 2.1

for a precise definition) that in the large t limit one has

$$\mathrm{Prob}_\omega\left(\frac{c^t}{t}\in B\right)=Q_{\omega,t}(tB)\simeq \mathrm{e}^{-t\inf_{s\in B}I(s)},$$

for suitable intervals $B\subset\mathbb{R}$ and some function $I:\mathbb{R}\to[0,+\infty]$ called the rate function. The relation (1.1) then induces the corresponding symmetry

$$I(-s)=I(s)+s. \tag{1.2}$$

We emphasize that in the above formulation, the statistics of the random variable c^t is taken with respect to an initial, time-reversal invariant, state ω and it pertains to a transient process. Typically, the system is expected to reach a (non-equilibrium) stationary regime, *i.e.,* the state ω is expected to evolve in large time towards a non-equilibrium steady state ω_∞. In the Gallavotti–Cohen conceptual framework, the essence of the non-equilibrium regime is captured by the analysis of $Q_{\omega_\infty,t}$, *i.e.,* the statistics of the random process $(c^t)_{t\geq 0}$ with respect to the steady state ω_∞. The finite time symmetry, Relation (1.1), breaks down but Relation (1.2) is expected to hold true for a corresponding rate function I_∞ in the large time limit (provided an LDP holds).

To derive these LDP's, the Laplace transform $\mathfrak{F}_{\nu,t}(\alpha):-\int_{\mathbb{R}}\mathrm{e}^{-\alpha s}\mathrm{d}Q_{\nu,t}(s)$ of the measures $Q_{\nu,t}$ (think of ν being either ω or ω_∞), and in particular the large t limits

$$\boldsymbol{F}_\nu(\alpha)=\lim_{t\to\infty}\frac{1}{t}\log\mathfrak{F}_{\nu,t}(\alpha), \tag{1.3}$$

play a crucial role. Indeed, one common route to an LDP is via the Gärtner–Ellis theorem (see Theorem 2.2) which relies on $\boldsymbol{F}_\nu(\alpha)$. They will be the central objects of the theory and we will refer to the quantities $\mathfrak{F}_{\nu,t}(\alpha)$ with the generic name of *entropic functionals*. Note that for $\nu=\omega$ the symmetry (1.1) is equivalent to

$$\mathfrak{F}_{\omega,t}(\alpha)=\mathfrak{F}_{\omega,t}(1-\alpha),$$

while, in the Gärtner–Ellis setting where the rate function and the limiting entropic functional are Legendre duals, the rate function symmetry (1.2) is equivalent to $\boldsymbol{F}_\omega(\alpha)=\boldsymbol{F}_\omega(1-\alpha)$ on the corresponding domains.

In typical cases, the two states ω and ω_∞ are mutually singular, therefore the corresponding fluctuation theorems are mathematically very distinct statements. In spite of these differences, in all known examples where both the transient and steady fluctuation relation hold, it turns out that the respective LDP rate functions I and I_∞ are *identical*. Under the same duality hypotheses, the equality $I=I_\infty$ amounts to an exchange of limits related to the entropic functionals (see Equation (2.11)). This connection was explored in [JPRB11], where the authors advocate this identity should be regarded itself as a deep dynamical property of the system, and baptized *"Principle of Regular Entropic Fluctuations"* (PREF).

A lot of attention has then been devoted to the extension of fluctuation theorems to the quantum domain. This extension has proven to be more delicate than anticipated, both for conceptual and technical reasons. It came as a surprise that the natural route through standard quantization did not lead

to a theory satisfying FT. This has been first noted in the early work of Bochkov and Kuzovlev [BK77], and re-attempted in the 1990's where the approach was often referred to as "naive quantization." In the work related literature this fact is often expressed as "work is not an observable." This approach is briefly presented in Section 4.1.

A quantum version of the *transient FT* was first obtained in the early 2000's, thanks to the contributions by Kurchan and Hal Tasaki [Kur00], [Tas00] on one side and by Shuichi Tasaki and Matsui [TM03] on the other, these two sets of work following different lines of reasoning. From the physical point of view, Kurchan [Kur00] and Tasaki [Tas00] introduced an approach based on two-time measurement statistics. In this approach, the main object of interest is the distribution of the random variable expressing the entropy change *between two successive measurements*. This observation led to the first transient quantum fluctuation theorem in the context of finite quantum systems. We will refer to this approach as 2-Time Measurement (2TM) and we detail it in Section 4.2. From a mathematical point of view, starting with Araki's definition of relative entropy of states on a von Neumann algebra in terms of relative modular operator (see Section 3.4), Tasaki and Matsui [TM03] proposed a quantum analogue of $Q_{\omega,t}$ that satisfies FR, see Section 4.3. While appealing from a mathematical perspective, this approach, which reveals a deep link with the rich structure of modular theory, lacks an a priori physical interpretation. However, surprisingly the above two approaches turned out to lead to identical entropy-production distributions at each fixed time, shedding an unexpected new light both on quantum mechanics probabilistic rules and modular theory. This connection was explored in [JOPP10] where a pedagogical presentation is given (see also [DR09] for an early work on the subject).

One drawback of the proposal by Tasaki and Matsui is that the relative modular operator, on which the approach is based, is not an observable making it difficult to generalize to arbitrary states. One may however rewrite the Laplace transforms of the Tasaki and Matsui measures $Q^{\mathrm{TM}}_{\omega,t}$ in terms of the Araki–Connes cocycle observables (3.7). This opens another route to the extension of fluctuation theorems to the quantum domain. We will present this approach in Sections 4.4 and 5.2, and will refer to it as the *"quantum phase space contraction"* for reasons which will be transparent later.

Coming back to the physical perspective, note that the quantum measurement protocol now enters in the definition of the object of interest at a *fundamental level* . The distributions $Q_{\omega,t}$ correspond specifically to a two-time *direct* measurement. Such direct measurements have then caused debate due to a lack of experimental feasibility, raising interest in *indirect* measurement protocols, and it is therefore natural to study these other measurement protocols as well. In [BBJ$^+$25a], we have considered the Entropic Ancilla State Tomography (EAST) approach, where the system of interest is coupled to an additional one called ancilla on which the measurement is performed, and the corresponding entropic functional $\mathfrak{F}^{\mathrm{east}}_{\nu,t}$ is introduced (here ν refers to the state of the open system when it is coupled to the ancilla). As we shall see, when $\nu = \omega$, this approach also leads to the same distributions $Q_{\omega,t}$ (see Section 5.3).

Turning to the *steady* version of the FT, an extension to quantum two-time measurement entropy production was for a long time missing from the literature. The series of work [BBJ$^+$24, BBJ$^+$25b, BBJ$^+$25a, BBJ$^+$26] provides such an extension and explores the quantum version of the PREF. These works focus on open quantum systems, where a small system interacts with several infinitely extended reservoirs, and which play a privileged role in the study of non-equilibrium quantum statistical mechanics. The

purpose of this review is to present, in an introductory way, the approach proposed in these works.

First we underline that in order to obtain steady states with nonzero entropy production in the Hamiltonian setting considered here, one has to consider infinitely extended reservoirs, described either via a thermodynamic limit or directly using an algebraic approach, while the 2TM approach is by essence limited to reservoirs either finite dimensional or confined with discrete energy spectra. A procedure allowing to define entropy production statistics with respect to ω_∞, the quantum analogue of $Q_{\omega_\infty,t}$, is provided in the algebraic setting by [BBJ$^+$24]. The key formula (5.1) allows to construct, for infinitely extended reservoirs, finite time entropic functionals $\mathfrak{F}^{\mathrm{2tm}}_{\omega_T,t}(\alpha)$ associated to the time evolved states ω_T. They are the Laplace transform of Borel probability measures which are naturally identified as the entropy-production distributions of a 2TM protocol where the first measurement is taken at time T and the second at $T+t$. A large T limit finally allows to define the steady state entropic functionals $\mathfrak{F}^{\mathrm{2tm}}_{\omega_\infty,t}(\alpha)$ and associated probability distributions. The emerging quantum steady fluctuation theorem in 2TM however comes with a surprising degree of rigidity due to the invasive thermodynamic effect of the first measurement in the thermodynamic limit. For infinitely extended systems with ergodic reservoirs, the consequence is dramatic: the modular averaging that represents the first measurement resets the reservoir part of a normal initial state to the reference reservoir state, the unique invariant state in its normal folium (see Section 5.1). As a consequence, the PREF quantum analog trivializes in the 2TM approach.

The quantum phase space contraction and the indirect, ancilla assisted, measurement approaches are both also natural. They coincide with the 2TM description when statistics are considered with respect to the initial state ω, while this identity need not hold for other states. It turns out that, while the associated entropic functionals $\mathfrak{F}^{\mathrm{2tm}}_{\omega_\infty,t}(\alpha)$, $\mathfrak{F}^{\mathrm{east}}_{\omega_\infty,t}(\alpha)$ and $\mathfrak{F}^{\mathrm{qpsc}}_{\omega_\infty,t}(\alpha)$ generally differ for finite t, under additional strong, although natural, ergodic assumptions, their large t behavior as given by (1.3) actually coincide. Together with the rigidity of the 2TM approach this leads to a more general version of a quantum PREF, see Definition 6.1.

We emphasize that in the indirect measurement and quantum phase space contraction approaches, the primary objects are the functionals $\mathfrak{F}^{\mathrm{east}}_{\nu,t}$ and $\mathfrak{F}^{\mathrm{qpsc}}_{\nu,t}$. In other words, these approaches are defined at the level of "Laplace transform". However, when starting from an arbitrary state ν, the functionals $\mathfrak{F}^{\mathrm{east}}_{\nu,t}$ and $\mathfrak{F}^{\mathrm{qpsc}}_{\nu,t}$ are in general not the Laplace transform of any probability measure. Consequently there is no guarantee that these generalizations are associated to random processes.

It can also be easily seen that, when considering the indirect measurement approach, the quantum PREF actually relies on a suitable exchange of limits which is typically a deep dynamical problem that has to be proven model by model, see Section 6.3. This approach parallels in this sense the classical theory. We emphasize that the indirect measurement approach relies on a quantum state tomography (performed on the auxiliary system). One therefore needs to repeat the indirect measurement experiment with the *same* initial state, *i.e.,* the state of the open quantum system at the time where it is coupled to the ancilla. In this respect the steady state is a natural choice leading therefore to the study of the functional $\mathfrak{F}^{\mathrm{east}}_{\omega_\infty,t}(\alpha)$. The PREF in its strongest form, *i.e.,* for the steady state, is thus crucial.

This being said, a weaker version of the PREF, with finite waiting time, holds in greater generality in the following sense. If the open system, initially in the state ω, has already evolved for some time T before it is coupled to the ancilla, the large t behavior, again given by (1.3), of $\mathfrak{F}^{\mathrm{east}}_{\omega_T,t}(\alpha)$ does not

depend on T (see Equation (6.4)). This reflects a stability of the process all along the trajectory.

As mentioned above, in its indirect measurement version (but also in the phase space contraction one) the quantum PREF is a deep, ergodic type, dynamical problem. In [BBJ+25a], we considered a spectral approach to it and show how to characterize this quantum PREF in terms of spectral resonances of quantum transfer operators. We developed an axiomatic scheme to analyze these resonances, building on techniques introduced in [JP96b, JP96a, JP02]. The application of this scheme is carried out in [BBJ+26] for a paradigmatic model of open quantum system, namely the spin-fermion model. We provide a summary in Section 7.

This review is organized as follows. In Section 2 we briefly recall the entropic Fluctuation Theorem and Relation for classical systems. The quantum framework used in the rest of the paper is given in Section 3. In Section 4 we explain the various approaches to transient fluctuation relation in the context of finite quantum systems. Section 5 is devoted to the generalization of the two-time measurement, indirect measurement and quantum phase space contraction entropic functionals for open systems with infinitely extended reservoirs and associated to arbitrary states. The quantum PREF is discussed in Section 6 and the quantum transfer operator method in Section 7.

Acknowledgments. The work of CAP and VJ was partly funded by the CY Initiative grant Investissements d'Avenir, grant number ANR-16-IDEX-0008. VJ acknowledges the support of NSERC and the support of the MUR grant "Dipartimento di Eccellenza 2023-2027" of Dipartimento di Matematica, Politecnico di Milano. We also acknowledge the support of the ANR project DYNACQUS, grant number ANR-24-CE40-5714.

2 Classical systems

In this section we briefly recall the entropic fluctuation theorems for classical systems in order to put in perspective the issues and results which appear in the quantum context. This section is based on [JPRB11] where the reader can find a complete exposition and proofs.

The setup is a pair $(\mathscr{X},\phi)$, where $\mathscr{X}$ is a compact metric space and $\phi = \{\phi^t \mid t \in \mathbb{R}\}$ is a group of homeomorphisms of $\mathscr{X}$ such that the map

$$\mathbb{R} \times \mathscr{X} \ni (t,x) \mapsto \phi^t(x) \in \mathscr{X}$$

is continuous. We denote by $C(\mathscr{X})$ the vector space of all continuous complex-valued functions on $\mathscr{X}$ and equip it with the sup norm $\|f\|_\infty := \sup_{x\in\mathscr{X}} |f(x)|$. Observables are functions $f \in C(\mathscr{X})$, and they evolve in time as $f \mapsto f_t = f \circ \phi^t$. States are Borel probability measures on $\mathscr{X}$, and we write $\nu(f) = \int_{\mathscr{X}} f \mathrm{d}\nu$. They evolve in time as $\nu \mapsto \nu_t = \nu \circ \phi^{-t}$. A state ν is ϕ-invariant if $\nu_t = \nu$ for all t. The relative entropy of two states ν and μ is defined by

$$\mathrm{Ent}(\nu|\mu) = \begin{cases} -\int_{\mathscr{X}} \log\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) \mathrm{d}\nu & \text{if } \nu \ll \mu; \\ -\infty & \text{otherwise,} \end{cases} \tag{2.1}$$

and satisfies $\mathrm{Ent}(\nu|\mu) \leq 0$ with equality iff $\nu = \mu$.

A time-reversal of $(\mathscr{X},\phi)$ is an involutive homeomorphism $\iota:\mathscr{X}\to\mathscr{X}$ such that $\iota\circ\phi^t=\phi^{-t}\circ\iota$ for all $t\in\mathbb{R}$. Given such a time-reversal map ι, a state ν is called Time-Reversal Invariant (TRI) if $\nu\circ\iota=\nu$.

The starting point is then a triple $(\mathscr{X},\phi,\omega)$ where, to avoid triviality, the reference (initial) state ω is supposed not to be ϕ-invariant. The system is TRI if ω is TRI for some time-reversal of $(\mathscr{X},\phi)$. If for all $t\in\mathbb{R}$ the measures ω and ω_t are mutually absolutely continuous, we consider the following entropic cocycle

$$c^t:-\left(\log\frac{\mathrm{d}\omega_t}{\mathrm{d}\omega}\right)\circ\phi^t=-\log\frac{\mathrm{d}\omega_{-t}}{\mathrm{d}\omega}. \tag{2.2}$$

Its naturalness comes from the following entropy balance equation

$$\mathrm{Ent}(\omega_t|\omega)=-\omega(c^t).$$

It is also the phase space contraction between time 0 and time t: for any Borel set $B\subset X$ one has

$$\int_B \mathrm{e}^{-c^t}\,\mathrm{d}\omega=\int_{\phi^t(B)}\mathrm{d}\omega.$$

We will regard c^t as a random variable over the probability space $(\mathscr{X},\omega)$ and denote by $Q_{\omega,t}$ its law w.r.t. ω. The entropy balance equation then becomes

$$\int_{\mathbb{R}} s\,\mathrm{d}Q_{\omega,t}(s)=-\mathrm{Ent}(\omega_t|\omega). \tag{2.3}$$

As mentioned in the introduction, the non-positivity of relative entropy together with the entropy balance equation (2.3) gives the non-negativity of the mean entropy produced over $[0,t]$.

The family $(c^t)_{t\in\mathbb{R}}$ is an additive ϕ-cocycle, *i.e.,* $c^{t+s}=c^t+c^s\circ\phi^t$ for all $t,s\in\mathbb{R}$, and if we moreover assume that the map $\mathbb{R}\ni t\mapsto c^t\in C(\mathscr{X})$ is differentiable at $t=0$ the *entropy production* observable (or *phase space contraction rate*) of $(\mathscr{X},\phi,\omega)$ is defined by

$$\sigma=\frac{\mathrm{d}}{\mathrm{d}t}c^t|_{t=0}. \tag{2.4}$$

One then has $c^t=\int_0^t\sigma\circ\phi^s\,\mathrm{d}s$ so that $\mathrm{Ent}(\omega_t|\omega)=-\int_0^t\omega_s(\sigma)\mathrm{d}s$.

The cocycle $(c^t)_{t\in\mathbb{R}}$ enjoys the following additional properties:

1. Because $\int_{\mathscr{X}}\mathrm{e}^{-c^t}\,\mathrm{d}\omega=1$, either $c^t=0$ ω-almost surely or c^t takes both positive and negative values with positive ω-probability.

2. If $(\mathscr{X},\phi,\omega)$ is TRI we further have, for all $t\in\mathbb{R}$,

$$c^t\circ\iota=c^{-t}=-c^t\circ\phi^{-t},$$

   so that the image of c^t is actually symmetric around 0.

3. The inequality $\int_{\mathbb{R}} s\,\mathrm{d}Q_{\omega,t}(s)\geq 0$ shows that, with respect to ω, positive values of c^t are favored on average, even though c^t may take both positive and negative values (whether or not the system is TRI).

Entropic Fluctuation Theorem and Relation go beyond these simple observations. When the system is TRI one actually gets that for all $t \in \mathbb{R}$

$$\frac{\mathrm{d}\bar{Q}_{\omega,t}}{\mathrm{d}Q_{\omega,t}}(s) = \mathrm{e}^{-s}, \tag{2.5}$$

where $\bar{Q}_{\omega,t}(B) = Q_{\omega,t}(-B)$ is the reflected measure. All together, this can be summarized as

> Whenever its law is non-trivial, c^t takes both positive and negative values, in a symmetric range for TRI systems. Moreover, for such TRI systems and with respect to the initial ω, positive values of c^t are favored in an exponential way with a universal exponential rate.

The terminology *Entropic Fluctuation Theorem* actually refers to a (full or restricted) LDP for the family of measures $(P_{\omega,t})_{t>0}$, $P_{\omega,t}(B) = Q_{\omega,t}(tB)$, that is the law of the variables

$$\frac{c^t}{t} = \frac{1}{t}\int_0^t \sigma \circ \phi^s \, \mathrm{d}s.$$

For the sake of completeness we recall the basic definition and refer to [DZ00, Ell06] for more information about LDP.

Definition 2.1. *The family of probability measures $(P_t)_{t>0}$ satisfies a **full LDP** if there exists a lower-semicontinuous function $\mathbb{I} : \mathbb{R} \to [0,\infty]$, called the **rate function**, such that for any Borel set $B \subset \mathbb{R}$,*

$$-\inf_{s\in\mathrm{int}(B)} \mathbb{I}(s) \leq \liminf_{t\to\infty} \tfrac{1}{t}\log P_t(B) \leq \limsup_{t\to\infty} \tfrac{1}{t}\log P_t(B) \leq -\inf_{s\in\mathrm{cl}(B)} \mathbb{I}(s), \tag{2.6}$$

where $\mathrm{int}(B)/\mathrm{cl}(B)$ *denotes the interior/closure of* B.

*If (2.6) holds only for Borel sets $B \subseteq]a,b[$, where $]a,b[\neq \mathbb{R}$, we then say that a **restricted LDP** holds for $(P_t)_{t>0}$ on the interval $]a,b[$.*

If such an LDP holds for $(P_{\omega,t})_{t>0}$, the relation (2.5) and the induced symmetry

$$\mathbb{I}(-s) = \mathbb{I}(s) + s$$

of the LDP rate function for $(P_{\omega,t})_{t>0}$, on the domain where both s and $-s$ are covered by the LDP, constitute the so-called *classical Evans–Searles (or transient) fluctuation relation.*

The next step is then to go beyond the initial/reference state ω. For any state ν on $\mathscr{X}$, we denote by $Q_{\nu,t}$ the law of c^t w.r.t. ν, and $P_{\nu,t}(B) = Q_{\nu,t}(tB)$. Of particular interest is the case where ν is a NESS of $(\mathscr{X},\phi,\omega)$, *i.e.,* a weak-limit point of the net

$$\left\{ \frac{1}{T}\int_0^T \omega_t \mathrm{d}t \,\middle|\, T>0 \right\}$$

as $T \uparrow \infty$. The set of NESS is non-empty and any NESS is ϕ-invariant. Moreover, for any NESS ω_∞ one has

$$\omega_\infty(\sigma) \geq 0.$$

Suppose that the weak limit $\lim_{t\to\infty}\omega_t=\omega_\infty$ exists so that, in particular, ω_∞ is the unique NESS of the system $(\mathscr{X},\phi,\omega)$. If the full/restricted LDP holds for $(P_{\omega_\infty,t})_{t>0}$, we say that the *full/restricted classical Gallavotti–Cohen fluctuation theorem* holds for $(\mathscr{X},\phi,\omega)$. If in addition the respective rate function $\mathbb{I}_\infty$ satisfies

$$\mathbb{I}_\infty(-s)=\mathbb{I}_\infty(s)+s \tag{2.7}$$

on its domain of definition, we say that the *classical Gallavotti–Cohen fluctuation relation* holds. Unlike in the Evans–Searles case, (2.7) is not ensured by the LDP and time-reversal invariance. In any model where the symmetry (2.7) has been proven it turns out that $\mathbb{I}_\infty=\mathbb{I}$.

As mentioned in the introduction, one usual route to an LDP relies on the Laplace transform via the Gärtner–Ellis theorem.

Theorem 2.2. *Let $I=]\vartheta_-,\vartheta_+[\subset\mathbb{R}$ be an open interval containing 0, and suppose that the limit*

$$F(\alpha)=\lim_{t\to\infty}\frac{1}{t}\log\int_{\mathbb{R}}e^{-\alpha ts}\mathrm{d}P_t(s)$$

exists, is finite for $\alpha\in I$, and that the function $F:I\to\mathbb{R}$ is differentiable. If $I=\mathbb{R}$, then the full LDP holds with the rate function

$$\mathbb{I}(s)=\sup_{-\alpha\in I}(s\alpha-F(-\alpha)). \tag{2.8}$$

Otherwise, the restricted LDP with rate function (2.8) holds on the interval $]a,b[$ where

$$a=-\lim_{\alpha\uparrow\vartheta_+}F'(\alpha),\qquad b=-\lim_{\alpha\downarrow\vartheta_-}F'(\alpha).$$

For any state ν, $t\in\mathbb{R}$ and $\alpha\in\mathbb{C}$, let

$$\mathfrak{F}_{\nu,t}(\alpha):-\int_{\mathbb{R}}\mathrm{e}^{-\alpha ts}\mathrm{d}P_{\nu,t}(s)=\int_{\mathbb{R}}\mathrm{e}^{-\alpha s}\mathrm{d}Q_{\nu,t}(s)=\int_{\mathscr{X}}\mathrm{e}^{-\alpha c^t}\mathrm{d}\nu=\int_{\mathscr{X}}\left(\frac{\mathrm{d}\omega_{-t}}{\mathrm{d}\omega}\right)^{\alpha}\mathrm{d}\nu \tag{2.9}$$

denote the Laplace transform of $Q_{\nu,t}$. We further set

$$\boldsymbol{F}_\nu(\alpha):-\lim_{t\to\infty}\frac{1}{t}\log\mathfrak{F}_{\nu,t}(\alpha), \tag{2.10}$$

whenever the limit exists.

Note that the transient fluctuation relation (2.5) is equivalent to the symmetry

$$\mathfrak{F}_{\omega,t}(\alpha)=\mathfrak{F}_{\omega,t}(1-\alpha),\quad\forall\alpha\in\mathbb{R}.$$

When the rate functions and limiting entropic functionals are Legendre duals, as in the Gärtner–Ellis setting above, the basic properties of the Legendre transform show that $\mathbb{I}=\mathbb{I}_\infty$ is equivalent to $\boldsymbol{F}_\omega=\boldsymbol{F}_{\omega_\infty}$ on the corresponding domains. For each fixed T, the density $\mathrm{d}\omega_T/\mathrm{d}\omega$ is bounded above and bounded away from zero, so that $\boldsymbol{F}_{\omega_T}=\boldsymbol{F}_\omega$ whenever $\boldsymbol{F}_\omega$ exists. Moreover, the weak convergence $\omega_T\to\omega_\infty$ gives $\mathfrak{F}_{\omega_T,t}(\alpha)\to\mathfrak{F}_{\omega_\infty,t}(\alpha)$ for each fixed t. Thus, the equality $\boldsymbol{F}_\omega=\boldsymbol{F}_{\omega_\infty}$ is equivalent to the following exchange of limits:

$$\lim_{T\to\infty}\lim_{t\to\infty}\tfrac{1}{t}\log\mathfrak{F}_{\omega_T,t}(\alpha)=\lim_{t\to\infty}\lim_{T\to\infty}\tfrac{1}{t}\log\mathfrak{F}_{\omega_T,t}(\alpha). \tag{2.11}$$

This exchange of limits is the only known general mechanism that ensures the validity of the Gallavotti–Cohen fluctuation theorem and relation. The identities

$$\boldsymbol{F}_\omega = \boldsymbol{F}_{\omega_\infty}, \qquad \mathbb{I} = \mathbb{I}_\infty \tag{2.12}$$

have been raised to the classical *Principle of Regular Entropic Fluctuations* (PREF), see [JPRB11].

3 Quantum framework

3.1 C^* quantum dynamical systems

A C^*-*quantum dynamical system* is a triple $(\mathscr{O}, \tau, \omega)$ where:

- $\mathscr{O}$ is a C^*-algebra with a unit $\mathbb{1}$. Its elements $A \in \mathscr{O}$ describe the system observables.
- $\mathbb{R} \ni t \mapsto \tau^t$ is a C^*-dynamics, *i.e.,* a strongly continuous group of $*$-automorphisms of $\mathscr{O}$. It describes the Heisenberg time-evolution $A_t = \tau^t(A)$ of the system observables. The infinitesimal generator of a C^*-dynamics τ is a possibly unbounded $*$-derivation δ of $\mathscr{O}$. We use the convention $\tau^t = \mathrm{e}^{t\delta}$.
- ω is a state on $\mathscr{O}$, *i.e.,* a continuous linear functional such that $\omega(A^*A) \geq 0$ for all $A \in \mathscr{O}$, and $\omega(\mathbb{1}) = 1$. The number $\omega(A)$ is the quantum expectation value of the observable $A \in \mathscr{O}$, when the system is in the state ω. States evolve according to the Schrödinger picture $\omega_t = \omega \circ \tau^t$, so that $\omega_t(A) = \omega(A_t)$. ω is called τ-invariant whenever $\omega_t = \omega$ for all $t \in \mathbb{R}$.

Thermal equilibrium of the system at inverse temperature $\beta \in \mathbb{R}^*$ is described by a state ω satisfying the (τ, β)-KMS boundary condition: for any $A, B \in \mathscr{O}$ the function $F_{A,B}(t) = \omega(A\tau^t(B))$ has an analytic extension to the complex strip $\{z \mid 0 < \mathrm{sign}(\beta)\,\mathrm{Im}\, z < |\beta|\}$, which is bounded and continuous on its closure, and satisfies

$$F_{A,B}(t + \mathrm{i}\beta) = \omega(\tau^t(B)A)$$

for all $t \in \mathbb{R}$. Such states are said to be (τ, β)-KMS, and are τ-invariant. The quantum dynamical system $(\mathscr{O}, \tau, \omega)$ is said to be in thermal equilibrium at inverse temperature $\beta \in \mathbb{R}$ if its reference state ω is (τ, β)-KMS.

For the modular framework below, we assume that the GNS representation associated to ω is faithful; this is automatic for the CAR and spin algebras considered here. A state ω is called modular if it is $(\varsigma_\omega, -1)$-KMS for some C^*-dynamics ς_ω. When it exists ς_ω is unique and is called the modular group of ω. We denote its generator by δ_ω. If ω is (τ, β)-KMS then it is modular and $\varsigma_\omega^t = \tau^{-\beta t}$. This modular group will play a crucial role in the description of the two-time measurement approach to quantum Entropic Fluctuations, see Section 5.1.

The quantum dynamical system $(\mathscr{O}, \tau, \omega)$ is called ergodic if ω is τ-invariant and

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \omega\left(B^* \tau^t(A) B\right) \mathrm{d}t = \omega(A)\omega(B^*B)$$

holds for all $A, B \in \mathscr{O}$.

Finally, as we have seen in the classical case, time-reversal plays an important role. In the quantum domain, a time-reversal of $(\mathscr{O}, \tau)$ is an anti-linear involutive $*$-automorphism Θ of $\mathscr{O}$ satisfying

$$\Theta \circ \tau^t = \tau^{-t} \circ \Theta$$

for all $t \in \mathbb{R}$. A state ω is TRI if there exists a time-reversal Θ such that $\omega(\Theta(A)) = \omega(A^*)$ for all $A \in \mathscr{O}$.

Example 1: finite systems. For quantum systems with finite dimensional Hilbert space $\mathscr{K}$, $\mathscr{O}$ is the C^*-algebra of all linear maps on $\mathscr{K}$, states are identified with density matrices, the dynamics is given by $\tau^t(A) = \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH}$ where H is the Hamiltonian of the system, and thermal equilibrium states are given by the usual Gibbs states $\rho_\beta = \mathrm{e}^{-\beta H} / \mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right)$. Finite systems with $\dim \mathscr{K} > 1$ and a faithful reference state are never ergodic.

Example 2: ideal Fermi gas, with single particle Hilbert space $\mathfrak{h}$ and single particle Hamiltonian h. The algebra of observables is the CAR-algebra $\mathscr{O} = \mathrm{CAR}(\mathfrak{h})$, the C^*-algebra generated by creation/annihilation operators $a^*(f)/a(f)$, $f \in \mathfrak{h}$, satisfying the canonical anti-commutation relations

$$\{a(f), a^*(g)\} = \langle f, g\rangle \mathbb{1}, \qquad \{a(f), a(g)\} = 0.$$

The dynamics on $\mathscr{O}$ is the group of Bogoliubov $*$-automorphisms associated to h, *i.e.,* the C^*-dynamics defined by $\tau^t(a(f)) = a\left(\mathrm{e}^{\mathrm{i}th} f\right)$. The unique (τ, β)-KMS state on $\mathscr{O}$ is the gauge-invariant quasi-free state ω whose two-point function is given by $\omega(a^*(f)a(g)) = \langle g, (1 + \mathrm{e}^{\beta h})^{-1} f\rangle$. If h has purely absolutely continuous spectrum then $(\mathscr{O}, \tau, \omega)$ is ergodic (it is actually mixing).

3.2 Open quantum systems

In these notes, we will focus on a special class of quantum dynamical systems, the so-called open quantum systems, which play a privileged role in the study of non-equilibrium quantum statistical mechanics. They consist of a small system S interacting with several reservoirs. For simplicity of exposition, we will consider only the case of 2 reservoirs and refer the reader to the series [BBJ+24, BBJ+25a, BBJ+25b, BBJ+26] for the general situation.

As mentioned in the introduction, we will need to consider infinitely extended reservoirs. In the literature they are described either as a thermodynamic limit of finite dimensional systems, or directly using the algebraic setting described in the previous section. In this contribution, we shall use both descriptions. The finite dimensional approximations are needed to introduce the two-time measurement approach while the more general setting is needed to consider steady states.

In the algebraic general setting, each thermal reservoir R_j, $j = 1, 2$, is described by a C^*-quantum dynamical system $(\mathscr{O}_j, \tau_j, \omega_j)$, $\tau_j^t = \mathrm{e}^{t\delta_j}$. We assume them to be initially in thermal equilibrium at some inverse temperature $\beta_j > 0$, that is we assume that ω_j is a (τ_j, β_j)-KMS state. These reservoirs are coupled through a finite dimensional system S described by $(\mathscr{O}_\mathsf{S}, \tau_\mathsf{S}, \omega_\mathsf{S})$ where $\mathscr{O}_\mathsf{S} = \mathscr{B}(\mathscr{K}_\mathsf{S})$ and τ_S is generated by the Hamiltonian H_S. For convenience, we assume that ω_S is the tracial state. Analogous results hold for any faithful τ_S-invariant state.

In the absence of interaction, the combined system $\mathsf{S}+\mathsf{R}_1+\mathsf{R}_2$ is described by the quantum dynamical system $(\mathscr{O}, \tau_{\rm fr}, \omega)$, where[1]

$$\mathscr{O} = \mathscr{O}_\mathsf{S} \otimes \mathscr{O}_1 \otimes \mathscr{O}_2, \qquad \tau_{\rm fr} = \tau_\mathsf{S} \otimes \tau_1 \otimes \tau_2, \qquad \omega = \omega_\mathsf{S} \otimes \omega_1 \otimes \omega_2.$$

We write $\omega_\mathsf{R} = \omega_1 \otimes \omega_2$ for the product reservoir state. The state ω is modular with

$$\varsigma_\omega^\theta = \mathrm{Id}_\mathsf{S} \otimes \tau_1^{-\beta_1\theta} \otimes \tau_2^{-\beta_2\theta}. \tag{3.1}$$

The interaction of S with R_j is described by a self-adjoint element $V_j \in \mathscr{O}_\mathsf{S} \otimes \mathscr{O}_j$, and the full interaction by $V = V_1 + V_2$. The interacting dynamics τ is generated by

$$\delta = \delta_{\rm fr} + \mathrm{i}[V, \cdot], \qquad \delta_{\rm fr} = \mathrm{i}[H_\mathsf{S}, \cdot] + \delta_1 + \delta_2.$$

For later purposes, see Section 3.4, we shall also assume that

(Reg) $V_j \in \mathrm{Dom}(\delta_\mathrm{j})$, $j = 1, 2$.

When the reservoirs R_j are also finite dimensional systems, with non-interacting Hamiltonians H_{R_j}, this means that the evolution of the coupled system is generated by the Hamiltonian

$$H = H_{\rm fr} + V, \qquad H_{\rm fr} = H_\mathsf{S} + H_{\mathsf{R}_1} + H_{\mathsf{R}_2}, \tag{3.2}$$

and the initial state given by the density matrix

$$\omega = \frac{\mathrm{Id}_\mathsf{S}}{\dim \mathscr{K}_\mathsf{S}} \otimes \frac{\mathrm{e}^{-\beta_1 H_{\mathsf{R}_1}}}{\mathrm{tr}\left(\mathrm{e}^{-\beta_1 H_{\mathsf{R}_1}}\right)} \otimes \frac{\mathrm{e}^{-\beta_2 H_{\mathsf{R}_2}}}{\mathrm{tr}\left(\mathrm{e}^{-\beta_2 H_{\mathsf{R}_2}}\right)}. \tag{3.3}$$

When considering the Gallavotti–Cohen fluctuation theorem, we will also make the following assumption

(NESS) The system $(\mathscr{O}, \tau, \omega)$ has a unique NESS ω_∞, and for all $A \in \mathscr{O}$ one has

$$\lim_{t\to\infty} \omega \circ \tau^t(A) = \omega_\infty(A).$$

3.3 Relative entropy and entropy observable in finite systems

For finite systems, the relative entropy of a pair of states μ, ν [2] is given by

$$\mathrm{Ent}(\nu|\mu) = \begin{cases} \mathrm{tr}\left(\nu \log \mu - \nu \log \nu\right) & \text{if } \nu \ll \mu; \\ -\infty & \text{otherwise.} \end{cases} \tag{3.4}$$

[1] Whenever the meaning is clear within the context, we write A for $A \otimes \mathbb{1}$ and $\mathbb{1} \otimes A$, δ_j for $\delta_j \otimes \mathrm{Id}$, $\mathrm{Id} \otimes \delta_j$, etc.

[2] Recall that for finite systems states are identified with density matrices

where $\nu \ll \mu$ means $\mathrm{Ker}\,\mu \subset \mathrm{Ker}\,\nu$. The following entropy balance equation

$$\mathrm{Ent}(\omega_t|\omega) = -\int_0^t \omega_s(\sigma)\,\mathrm{d}s$$

relates the relative entropy of the evolved state ω_t with respect to ω to the entropy production observable

$$\sigma = -\beta_1\Phi_1 - \beta_2\Phi_2, \qquad \Phi_j = \mathrm{i}[H_{\mathrm{R}_j}, V_j]. \tag{3.5}$$

For general systems and an arbitrary pair of states, relative entropy has been introduced by Araki and requires modular structure, see Equation (3.6) in the next section.

3.4 Modular structure

In this section, we provide a brief account of the modular structure associated to a quantum dynamical system. This structure has played a fundamental role in the modern development of quantum statistical mechanics, including the entropic fluctuations theorems presented in this review. We shall only describe the main objects, mainly to fix notation. For a more complete introduction we refer the reader to the literature, see e.g. any of the references [BR81, BR87, DJP03, Haa96, OP93, Str81]. For the unfamiliar reader's convenience, the modular structure of finite systems will be described in the next section.

We denote by $(\mathscr{H}, \pi, \Omega)$ the GNS representation of $\mathscr{O}$ associated to the state ω, by $\mathfrak{M} = \pi(\mathscr{O})''$ its enveloping von Neumann algebra.[3] The state ω has a unique normal extension to $\mathfrak{M}$ which we denote by the same symbol. A density matrix ρ on $\mathscr{H}$ defines a state on $\mathfrak{M}$ by the familiar quantum mechanical rule $\mathfrak{M} \ni A \mapsto \mathrm{tr}(\rho A)$. Such states on $\mathfrak{M}$ are called *normal,* and their restriction to $\pi(\mathscr{O})$ induce states on $\mathscr{O}$ which are called ω-*normal.* The *folium* of ω is the set $\mathscr{N}$ of all ω-normal states on $\mathscr{O}$ (it is the quantum analogue of the set of measures absolutely continuous with respect to a given measure ω). We assume in this modular framework that ω is modular and that τ extends to a pointwise σ-weakly continuous group of $*$-automorphisms of $\mathfrak{M}$. These regularity properties hold for the bounded open-system perturbations considered here.

Since the state ω is modular (see Section 3.1), its normal extension to $\mathfrak{M}$ is faithful, *i.e.,* the map $\mathfrak{M} \ni A \mapsto A\Omega \in \mathscr{H}$ is injective. In this case, it follows from Tomita–Takesaki's theory that the GNS Hilbert space $\mathscr{H}$ comes with a modular structure:

- The anti-linear operator defined on the dense subspace $\pi(\mathscr{O})\Omega \subset \mathscr{H}$ by $\pi(A)\Omega \mapsto \pi(A)^*\Omega$ has a closure S with the unique polar decomposition $S = J\Delta_\omega^{1/2}$. There, Δ_ω is a densely defined positive linear operator and J an anti-unitary involution on $\mathscr{H}$. Δ_ω is called *modular operator* and J *modular conjugation* of the pair $(\mathfrak{M}, \omega)$. The closure $\mathscr{H}_+$ of the subset $\{AJA\Omega \mid A \in \mathfrak{M}\}$ is its *natural cone*. The modular operator implements the modular group on $\pi(\mathscr{O})$, *i.e.,*

$$\pi\big(\varsigma_\omega^\theta(A)\big) = \Delta_\omega^{\mathrm{i}\theta}\pi(A)\Delta_\omega^{-\mathrm{i}\theta},$$

  and thus allows to extend this group to $\mathfrak{M}$.

[3] $\mathfrak{A}'$ denotes the commutant of a subset $\mathfrak{A} \subset \mathscr{B}(\mathscr{H})$, *i.e.,* the set of elements of $\mathscr{B}(\mathscr{H})$ which commute with all elements of $\mathfrak{A}$.

- Any normal state ν on $\mathfrak{M}$ has a unique vector representative $\Omega_\nu \in \mathscr{H}_+$ such that, for all $A \in \mathfrak{M}$, $\nu(A) = \langle \Omega_\nu, A\Omega_\nu \rangle$. Moreover, a unit vector $\Psi \in \mathscr{H}_+$ is cyclic for $\mathfrak{M}$, *i.e.,* $\mathfrak{M}\Psi$ is dense in $\mathscr{H}$, iff the state it induces on $\mathfrak{M}$ is faithful, *i.e.,* the map $\mathfrak{M} \ni A \mapsto A\Psi \in \mathscr{H}$ is injective.

- If μ and ν are faithful normal states on $\mathfrak{M}$, the densely defined anti-linear map $A\Omega_\nu \mapsto A^*\Omega_\mu$ has a closure $S_{\mu|\nu}$ whose polar decomposition reads $S_{\mu|\nu} = J\Delta_{\mu|\nu}^{1/2}$, J being the modular conjugation of $(\mathfrak{M}, \omega)$ and $\Delta_{\mu|\nu}$ a densely defined positive linear operator: the *relative modular operator* of the pair (μ, ν).

- Let (μ, ν) be a pair of faithful normal states on $\mathfrak{M}$. Their relative entropy is defined by

$$\mathrm{Ent}(\nu|\mu) = \langle \Omega_\nu, \log \Delta_{\mu|\nu} \Omega_\nu \rangle. \tag{3.6}$$

  This is the original definition by Araki [Ara76, Ara77] with the sign and ordering convention of [JP01].

- The Araki–Connes cocycle of this pair is the strongly continuous one-parameter family of unitaries in $\mathfrak{M}$ given by[4]

$$[D\mu : D\nu]_\alpha = \Delta_{\mu|\nu}^\alpha \Delta_\nu^{-\alpha}, \qquad (\alpha \in \mathrm{i}\mathbb{R}). \tag{3.7}$$

  We mention that, contrary to the unitaries $\Delta_{\mu|\nu}^\alpha$ and $\Delta_\nu^{-\alpha}$, the Araki–Connes cocycle belongs to $\mathfrak{M}$, *i.e.,* is an observable. This cocycle, and in particular the quantities $[D\omega_{-t} : D\omega]_\alpha$, will be central to the theory, see Section 5. Assumption **(Reg)** ensures that the latter actually belong to $\pi(\mathscr{O})$, see e.g. [BBJ+25b, Lemma 2.4]. Denoting $\pi^{-1}([D\omega_{-t} : D\omega]_\alpha) \in \mathscr{O}$ by $[D\omega_{-t} : D\omega]_\alpha$ this allows to consider quantities like $\nu([D\omega_{-t} : D\omega]_\alpha)$ for arbitrary states on $\mathscr{O}$, not necessary ω-normal, in particular the NESS ω_∞.

- The standard Liouvillean $\mathscr{L}$ is the unique self-adjoint operator on the GNS space $\mathscr{H}$ satisfying

$$\mathrm{e}^{-\mathrm{i}t\mathscr{L}} \mathscr{H}_+ = \mathscr{H}_+, \qquad \tau^t(A) = \mathrm{e}^{\mathrm{i}t\mathscr{L}} A \mathrm{e}^{-\mathrm{i}t\mathscr{L}},$$

  for all $t \in \mathbb{R}$ and $A \in \mathfrak{M}$. In particular, $\Omega_t :- \mathrm{e}^{-\mathrm{i}t\mathscr{L}}\Omega$ is the vector representative of $\omega_t = \omega \circ \tau^t$ in the natural cone. When ω is (τ, β)-KMS (this is the case for the reservoir Liouvilleans $\mathscr{L}_{\mathsf{R}_j}$), then $\mathscr{L} = -\frac{1}{\beta} \log \Delta_\omega$. Let us just mention that, see *e.g.* [JOPS12], for any two faithful normal states μ and ν one has

$$\mathrm{e}^{-\mathrm{i}t\mathscr{L}} \Delta_{\nu|\mu} \mathrm{e}^{\mathrm{i}t\mathscr{L}} = \Delta_{\nu\circ\tau^t|\mu\circ\tau^t}. \tag{3.8}$$

  Finally, we note that the ergodic properties of a quantum dynamical system $(\mathscr{O}, \tau, \omega)$ are related to the spectral properties of $\mathscr{L}$. For a τ-invariant reference state ω, $(\mathscr{O}, \tau, \omega)$ is ergodic if and only if 0 is a simple eigenvalue of $\mathscr{L}$.

  For the open systems considered in Section 3.2 the standard Liouvillean of the interacting system has the form

$$\mathscr{L} = \mathscr{L}_{\mathrm{fr}} + V - JVJ, \quad \mathscr{L}_{\mathrm{fr}} = \mathscr{L}_{\mathsf{S}} + \mathscr{L}_{\mathsf{R}_1} + \mathscr{L}_{\mathsf{R}_2}.$$

[4] Here, we depart from the traditional convention, denoting by $[D\mu : D\nu]_{\mathrm{i}t}$ what is usually written $[D\mu : D\nu]_t$.

3.5 Modular structure of finite systems

For the unfamiliar reader's convenience, let us give the explicit form of a GNS representation and of the modular data for quantum systems on a finite dimensional Hilbert space $\mathscr{K}$ and initial state given by the positive definite density matrix ω. In this case $\mathscr{H} = \mathscr{B}(\mathscr{K})$ endowed with the scalar product $\langle X, Y\rangle = \mathrm{tr}(X^* Y)$, $\pi(A)$ is the left multiplication by A, $\pi(A) : X \mapsto AX$ and $\Omega = \omega^{1/2}$. The modular operator and conjugation are

$$\Delta_\omega : X \mapsto \omega X \omega^{-1}, \qquad J : X \mapsto X^*,$$

and the modular group is given by

$$\varsigma_\omega^t(A) = \omega^{it} A \omega^{-it}.$$

The relative modular operator of a pair (μ, ν) is

$$\Delta_{\mu|\nu} : X \mapsto \mu X \nu^{-1},$$

and (3.6) indeed coincides with (3.4). Finally the Araki–Connes cocycle is

$$[D\mu : D\nu]_\alpha : X \mapsto \mu^\alpha \nu^{-\alpha} X$$

which indeed belongs to $\pi(\mathscr{O})$. Of particular importance in the next sections will be the observable $[D\omega_{-t} : D\omega]_\alpha = \omega_{-t}^\alpha \omega^{-\alpha}$. Finally the standard Liouvillean associated to the dynamics generated by the Hamiltonian H is $\mathscr{L} = [H, \cdot]$.

Remark 1. As already mentioned the Araki–Connes cocycles $[D\mu : D\nu]_\alpha$ will play a fundamental role. For finite systems they are well defined and belong to $\pi(\mathscr{O})$ for all $\alpha \in \mathbb{C}$. In the general case it is initially defined only for $\alpha \in \mathrm{i}\mathbb{R}$, and additional hypothesis will be needed to guarantee that they belong to $\pi(\mathscr{O})$. The latter property is needed to consider their expectation value with respect to the (non ω-normal) steady state ω_∞.

Remark 2. Classical systems also fit into the algebraic framework, the algebra of observables being $C(\mathscr{X})$. In that case the modular operator associated to a state ω is trivial (it is the identity), the relative modular operator of a pair (μ, ν) is related to their Radon–Nikodym derivative, namely $\Delta_{\mu|\nu} : g \mapsto \frac{\mathrm{d}\mu}{\mathrm{d}\nu} g$ while the Connes cocycle is given by

$$[D\mu : D\nu]_\alpha = \left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\right)^\alpha. \tag{3.9}$$

4 Finite systems and the transient fluctuation theorem

Although infinitely extended systems are required to consider situations which are thermodynamically non-trivial, in this section we restrict ourselves to finite dimensional systems in order to make the definition of the various entropic functionals under consideration more transparent (this is particularly important for the two-time measurement approach). Throughout this section, we therefore assume that the reservoirs, and consequently the entire system, are finite dimensional. We denote by H_{R_j} the non-interacting Hamiltonian of R_j.

4.1 Naive quantization

The entropic cocycle

$$c^t = -\log\omega_{-t} + \log\omega, \tag{4.1}$$

is the natural quantization of its classical counterpart, see (2.2), and the entropy production observable σ defined in (3.5) satisfies (compare with (2.4))

$$\sigma = \left.\frac{\mathrm{d}c_t}{\mathrm{d}t}\right|_{t=0}. \tag{4.2}$$

It is thus natural to consider, as the quantum analogue of the measure $Q_{\nu,t}$ of Section 2, the spectral measure $Q^{\mathrm{naive}}_{\nu,t}$ of c^t with respect to the state ν whose Laplace transform is thus given by

$$\mathfrak{F}^{\mathrm{naive}}_{\nu,t}(\alpha) = \nu\left(\mathrm{e}^{-\alpha c^t}\right) = \nu\left(\mathrm{e}^{\alpha\log\omega_{-t}-\alpha\log\omega}\right). \tag{4.3}$$

The basic properties of c^t hold at the quantum level:

1. It follows from $\mathrm{tr}(c^t) = 0$ that, unless $c^t = 0$, its spectrum contains both positive and negative values,
2. One still has $0 \geq \mathrm{Ent}(\omega_t|\omega) = -\omega(c^t)$ so that, with respect to ω, positive values of c^t are favored on average,
3. If the system is TRI with time-reversal Θ then $\Theta(c^t) = -\tau^{-t}(c^t)$ so that the spectrum of c^t is symmetric around 0.

However, the transient fluctuation relation fails. If $\bar{Q}^{\mathrm{naive}}_{\omega,t}$ is the reflected measure associated to $Q^{\mathrm{naive}}_{\omega,t}$, one gets, using TRI, that for any function f

$$\int_{\mathbb{R}} f(s)\,\mathrm{d}\bar{Q}^{\mathrm{naive}}_{\omega,t} = \omega\left(f(c^t)\mathrm{e}^{\log\omega_{-t}}\mathrm{e}^{-\log\omega}\right)$$

while

$$\int_{\mathbb{R}} f(s)\mathrm{e}^{-s}\,\mathrm{d}Q^{\mathrm{naive}}_{\omega,t} = \omega\left(f(c^t)\mathrm{e}^{\log\omega_{-t}-\log\omega}\right).$$

At the level of the Laplace transform this is reflected by the fact that

$$\mathfrak{F}^{\mathrm{naive}}_{\omega,t}(\alpha) = \omega\left(\mathrm{e}^{\alpha\log\omega_{-t}-\alpha\log\omega}\right) \tag{4.4}$$

does not satisfy the symmetry $\mathfrak{F}^{\mathrm{naive}}_{\omega,t}(\alpha) = \mathfrak{F}^{\mathrm{naive}}_{\omega,t}(1-\alpha)$.

From a mathematical perspective, the non-commutativity of ω_{-t} and ω is at the origin of the failure, in general, of the fluctuation relation:

$$\frac{\mathrm{d}\bar{Q}^{\mathrm{naive}}_{\omega,t}}{\mathrm{d}Q^{\mathrm{naive}}_{\omega,t}}(s) \neq \mathrm{e}^{-s}.$$

Physically, the naive approach describes a single measurement of c^t, whereas the two-time measurement approach describes the entropy change between two measurements.

This failure motivated the early searches [Kur00, Tas00, TM03] for alternative candidates for quantum Evans–Searles Fluctuation Theorem and Relation. In [Kur00, Tas00] the physical aspects are considered, and their proposal is based on a line of reasoning that takes into account the measurement issue. The proposal [TM03] is mathematically oriented and, in a sense, resolves the non-commutativity issue. We shall discuss the two approaches respectively in Sections 4.2 and 4.3. Surprisingly these two proposals turned out to be equivalent, shedding an unexpected light on both quantum mechanical probabilistic rules and modular theory. A pedagogical discussion of this topic can be found in the lecture notes [JOPP10].

4.2 Two-time measurement (2TM) entropy production

This notion of entropy production was introduced in [Kur00, Tas00]. It consists in two consecutive measurements of the entropy observable $S = -\log\omega$ associated to the reference state ω given by (3.3). For the open systems considered here,

$$S = -\log\omega = \beta_1 H_{\mathrm{R}_1} + \beta_2 H_{\mathrm{R}_2} + \mathrm{const.} \tag{4.5}$$

For the reference state ω, the same entropy-change statistics are obtained by measuring the energies of both reservoirs at the beginning and end of the evolution, with

$$\Delta S = \beta_1 \Delta E_1 + \beta_2 \Delta E_2,$$

where ΔE_j is the measured change in energy of the j^{th} reservoir.

Note that the entropic cocycle can then be written as $c^t = \tau^t(S) - S$, the entropy production observable as $\sigma = \frac{\mathrm{d}}{\mathrm{d}t}\tau^t(S)\restriction_{t=0}$ and that

$$\mathrm{Ent}(\omega_t|\omega) = -\omega\left(\tau^t(S) - S\right).$$

The two-time measurement protocol goes as follows. The system is in a state ν at time t_i where a first measurement of S is performed. The outcome $s_i \in \mathrm{sp}(S)$ is observed with probability $p_\nu(s_i) = \nu(P_{s_i})$ where P_{s_i} is the eigenprojection corresponding to the eigenvalue $s_i \in \mathrm{sp}(S)$. After the measurement the system is in the reduced state

$$\frac{1}{p_\nu(s_i)} P_{s_i} \nu P_{s_i},$$

which evolves under the system dynamics over the time interval of length $t = t_f - t_i$. The second measurement, performed at time t_f, yields the outcome $s_f \in \mathrm{sp}(S)$ with probability

$$p_{\nu,t}(s_f|s_i) = \frac{1}{p_\nu(s_i)} \mathrm{tr}\left(\mathrm{e}^{-\mathrm{i}tH} P_{s_i} \nu P_{s_i} \mathrm{e}^{\mathrm{i}tH} P_{s_f}\right).$$

Finally, the probability of observing the pair (s_f, s_i) in this two-time measurement protocol is

$$p_{\nu,t}(s_f, s_i) = p_{\nu,t}(s_f|s_i) p_\nu(s_i) = \mathrm{tr}\left(\mathrm{e}^{-\mathrm{i}tH} P_{s_i} \nu P_{s_i} \mathrm{e}^{\mathrm{i}tH} P_{s_f}\right). \tag{4.6}$$

The entropy production random variable $\mathscr{E} : \mathrm{sp}(S) \times \mathrm{sp}(S) \to \mathbb{R}$ is defined by

$$\mathscr{E}(s_f, s_i) = s_f - s_i,$$

and its probability distribution with respect to $p_{\nu,t}$ is denoted by $Q^{\mathrm{2tm}}_{\nu,t}$,

$$Q^{\mathrm{2tm}}_{\nu,t}(s) = \sum_{s_f - s_i = s} p_{\nu,t}(s_f, s_i). \tag{4.7}$$

The family $(Q^{\mathrm{2tm}}_{\nu,t})_{t>0}$ thus describes the statistics of the two-time measurement entropy production. It follows from (4.6)-(4.7) and the functional calculus that their Laplace transforms are given by

$$\mathfrak{F}^{\mathrm{2tm}}_{\nu,t}(\alpha) \;=\; \int_{\mathbb{R}} \mathrm{e}^{-\alpha s} \mathrm{d}Q^{\mathrm{2tm}}_{\nu,t}(s) \;=\; \bar{\nu}(\omega^{\alpha}_{-t}\omega^{-\alpha}), \qquad (\alpha \in \mathbb{C}), \tag{4.8}$$

where $\bar{\nu} = \sum_{s \in \mathrm{sp}(S)} P_s \nu P_s$ is the diagonal part of ν with respect to S.[5] The fact that one replaces the initial state ν by its diagonal part $\bar{\nu}$ reflects the decoherence effect induced by the first measurement.

When $\nu = \omega$, we simply have

$$\mathfrak{F}^{\mathrm{2tm}}_{\omega,t}(\alpha) = \omega(\omega^{\alpha}_{-t}\omega^{-\alpha}) = \mathrm{tr}(\omega^{\alpha}_{-t}\omega^{1-\alpha}), \tag{4.9}$$

compare with (4.3). When the system is TRI one easily infers that $\mathfrak{F}^{\mathrm{2tm}}_{\omega,t}(\alpha) = \mathfrak{F}^{\mathrm{2tm}}_{\omega,t}(1-\alpha)$ for any $\alpha \in \mathbb{R}$ restoring the transient fluctuation relation

$$\frac{\mathrm{d}\bar{Q}^{\mathrm{2tm}}_{\omega,t}}{\mathrm{d}Q^{\mathrm{2tm}}_{\omega,t}}(s) = \mathrm{e}^{-s}. \tag{4.10}$$

Note also that the two distributions $Q^{\mathrm{2tm}}_{\omega,t}$ and $Q^{\mathrm{naive}}_{\omega,t}$ actually have the same first and second moment. In particular one still has

$$\mathrm{Ent}(\omega_t|\omega) = -\int_{\mathbb{R}} s \, \mathrm{d}Q^{\mathrm{2tm}}_{\omega,t}(s).$$

4.3 Tasaki–Matsui relative modular approach

Motivated by Araki's definition of relative entropy

$$\mathrm{Ent}(\omega_t|\omega) = \langle \Omega_t | \log \Delta_{\omega|\omega_t} \Omega_t \rangle = \langle \Omega | \log \Delta_{\omega_{-t}|\omega} \Omega \rangle,$$

(the second identity follows from (3.8)), in [TM03] the authors considered the operator $-\log\Delta_{\omega_{-t}|\omega}$ as the entropy production observable between time 0 and time t, although the latter does not belong to the algebra of observables. If $Q^{\mathrm{TM}}_{\omega,t}$ denotes the associated spectral measures with respect to the vector Ω, its Laplace transform is therefore

$$\mathfrak{F}^{\mathrm{TM}}_{\omega,t}(\alpha) = \langle \Omega, \Delta^{\alpha}_{\omega_{-t}|\omega} \Omega \rangle.$$

[5] Since $S = -\log\omega$, $\bar{\nu}$ is equivalently the diagonal part of ν with respect to ω.

For finite systems the relative modular operator acts as, see the end of Section 3,

$$\Delta_{\omega_{-t}|\omega} : X \mapsto \omega_{-t} X \omega^{-1}.$$

Hence (recall $\Omega = \omega^{1/2}$)

$$\mathfrak{F}^{\mathrm{TM}}_{\omega,t}(\alpha) = \mathrm{tr}(\omega^{\alpha}_{-t}\omega^{1-\alpha}) = \omega(\omega^{\alpha}_{-t}\omega^{-\alpha}), \tag{4.11}$$

which, for TRI systems, satisfies the symmetry $\mathfrak{F}^{\mathrm{TM}}_{\omega,t}(\alpha) = \mathfrak{F}^{\mathrm{TM}}_{\omega,t}(1-\alpha)$. The family of spectral measures $(Q^{\mathrm{TM}}_{\omega,t})_t$ therefore satisfies the transient fluctuation relation.

Note also that

$$-\log\Delta_{\omega_{-t}|\omega} : X \mapsto -\log(\omega_{-t})X + X\log\omega,$$

so the entropy production "observable" $-\log\Delta_{\omega_{-t}|\omega}$ is actually the sum of two *commuting* maps, the left multiplication by $-\log\omega_{-t}$ and the right multiplication by $\log\omega$. Although it was not mentioned in [TM03], the choice of $-\log\Delta_{\omega_{-t}|\omega}$ can thus be seen as a mathematical trick to restore the "lack of commutativity" mentioned in Section 4.1. Indeed, see Section 3.5, $-\log\Delta_{\omega_{-t}|\omega}$ differs from the entropic cocycle observable

$$\pi(c^t) : X \mapsto (-\log(\omega_{-t}) + \log\omega)X$$

by

$$\log\Delta_{\omega} : X \mapsto \log(\omega)\, X - X\log(\omega). \tag{4.12}$$

Namely,

$$-\log\Delta_{\omega_{-t}|\omega} = \pi(c^t) - \log\Delta_{\omega}. \tag{4.13}$$

Since $\langle\Omega, \log\Delta_{\omega}\Omega\rangle = 0$, we get that

$$-\omega(c^t) = -\langle\Omega, \pi(c^t)\Omega\rangle = \langle\Omega|\log\Delta_{\omega_{-t}|\omega}\Omega\rangle$$

are two equivalent ways to express the relative entropy $\mathrm{Ent}(\omega_t|\omega)$. The difference between the two quantities $\pi(c^t)$ and $-\log\Delta_{\omega_{-t}|\omega}$ however reappears at the level of the Laplace transforms and is at the origin of the difference between (4.4) and (4.11). The non-triviality of the modular operator Δ_{ω} is the crucial difference between the classical (commutative) and the quantum (non-commutative) case.

One potential objection to this approach could be the lack of physical interpretation. In particular, as previously mentioned, $\log\Delta_{\omega_{-t}|\omega}$ does not even belong to the algebra of observables. However, its unexpected connection with the two-time measurement process resolves this issue. Indeed, as the reader may have noticed, the quantities $\mathfrak{F}^{\mathrm{TM}}_{\omega,t}(\alpha)$ and $\mathfrak{F}^{\mathrm{2tm}}_{\omega,t}(\alpha)$ coincide. Therefore, the spectral measure $Q^{\mathrm{TM}}_{\omega,t}$ is nothing but the probability distribution $Q^{\mathrm{2tm}}_{\omega,t}$ obtained via the two-time measurement protocol giving an observational status to the spectral measure of the relative modular operator. This deep link between the statistics of the two-time measurement entropy production and modular theory has a somewhat unusual history and was discussed in detail in [JOPP10].

4.4 Quantum phase space contraction (QPSC)

The spectral projections of $\log\Delta_{\omega_{-t}|\omega}$ do not, in general, belong to the algebra of observables. The associated spectral measures can therefore be defined only for states which are ω normal, leaving

the steady state ω_∞ typically out of reach. One can however further rewrite the functional $\mathfrak{F}^{\mathrm{TM}}_{\omega,t}$ in terms of the Araki–Connes cocycle, the latter having the advantage to be an observable. Namely, since $\Delta_\omega\Omega = \Omega$ one may write

$$\mathfrak{F}^{\mathrm{TM}}_{\omega,t}(\alpha) = \langle\Omega, \Delta^{\alpha}_{\omega_{-t}|\omega}\Delta_\omega^{-\alpha}\Omega\rangle = \langle\Omega, [D\omega_{-t} : D\omega]_\alpha\Omega\rangle = \omega([D\omega_{-t} : D\omega]_\alpha).$$

Now this allows to define an analogous family for any state ν, namely

$$\mathfrak{F}^{\mathrm{qpsc}}_{\nu,t}(\alpha) :- \nu([D\omega_{-t} : D\omega]_\alpha) = \nu(\omega^{\alpha}_{-t}\omega^{-\alpha}), \tag{4.14}$$

which we will call the *Quantum Phase Space Contraction* (QPSC) functional.

Remark 1. In the classical setting, $\mathfrak{F}_{\nu,t}(\alpha)$ is the Laplace transform of the classical phase space contraction c^t. Combining (2.9) with (3.9) one sees that it can be expressed as

$$\mathfrak{F}_{\nu,t}(\alpha) = \nu([D\omega_{-t} : D\omega]_\alpha).$$

This makes the quantity (4.14) a natural quantum analogue and explains the terminology QPSC.

Remark 2. When $\nu = \omega$ the QPSC functional $\mathfrak{F}^{\mathrm{qpsc}}_{\omega,t}$ coincides with the Tasaki–Matsui one $\mathfrak{F}^{\mathrm{TM}}_{\omega,t}$, hence with the two-time measurement functional $\mathfrak{F}^{\mathrm{2tm}}_{\omega,t}$. However, for $\nu \neq \omega$ the equality with the two-time measurement functional $\mathfrak{F}^{\mathrm{2tm}}_{\nu,t}$ is usually broken unless $\nu = \bar{\nu}$ (compare with (4.8)), *i.e.,* when ν is completely decoherent with respect to ω.

Finally, let us emphasize that, contrary to $\mathfrak{F}^{\mathrm{2tm}}_{\nu,t}$, $\mathfrak{F}^{\mathrm{qpsc}}_{\nu,t}$ is not necessarily the Laplace transform of a Borel probability measure when $\nu \neq \omega$ (or more generally when $\nu \neq \bar{\nu}$). Thus, this family does not in general define entropy-production probability distributions.

4.5 Entropic Ancilla State Tomography (EAST)

As mentioned in the introduction, the two-time measurement protocol described in Section 4.2 has caused some debate due to a lack of experimental feasibility, in particular for large quantum systems. This motivated the search for more operational approaches, raising interest in *indirect* measurement protocols. In the closely related literature concerning work, these points are subject to an ongoing discussion aiming to provide an unambiguous definition of work, see e.g. [DCSCR18] and references therein, and a successful protocol has been introduced. The idea is to couple the system to an additional ancilla in order to implement physically the Fourier transform. A tomography on the ancilla's part then provides access to the Fourier transform of the desired distribution. To understand this protocol let us first consider the case where one would like to have access to the distribution $P_{A,\nu}$ of a given measured observable A with respect to a state ν. Its Fourier transform is then given by

$$\mathfrak{F}_{A,\nu}(\alpha) = \int_{\mathbb{R}} \mathrm{e}^{-s\alpha}\mathrm{d}P_{A,\nu}(s) = \nu\left(\mathrm{e}^{-\alpha A}\right), \qquad \alpha = \mathrm{i}u \in \mathrm{i}\mathbb{R}.$$

The ancilla coupling procedure goes as follows:

1. Add a qubit and initialize it to the state $\rho_a = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

2. Let the system and qubit interact for a time $u = -\mathrm{i}\alpha$ with interaction Hamiltonian $\frac{1}{2}A\otimes\sigma_z$. The qubit is then in the state

$$\rho_a(\alpha) = \mathrm{tr}_{\mathscr{H}}\left(\mathrm{e}^{-\frac{\alpha}{2}A\otimes\sigma_z}\nu\otimes\rho_a\mathrm{e}^{\frac{\alpha}{2}A\otimes\sigma_z}\right) = \frac{1}{2}\begin{bmatrix} 1 & \mathfrak{F}_{A,\nu}(\alpha) \\ \overline{\mathfrak{F}_{A,\nu}(\alpha)} & 1 \end{bmatrix}.$$

3. A quantum state tomography of the qubit (using the σ_x and σ_y observables) gives then access to the Fourier transform $\mathfrak{F}_{A,\nu}(\alpha)$.

In the two-time measurement protocol, one does not measure the observable $S = -\log\omega$ once but twice, the system having evolved in the meantime. We then consider the distribution of the differences between the two measured values (the first value has thus to be taken into account with a minus sign). The corresponding ancilla coupling procedure is similar to the previously described one, replacing the evolution $\mathrm{e}^{-\frac{\alpha}{2}A\otimes\sigma_z}$ by

$$\mathrm{e}^{-\frac{\alpha}{2}S\otimes\sigma_z}\mathrm{e}^{-\mathrm{i}tH\otimes\mathbb{1}}\mathrm{e}^{+\frac{\alpha}{2}S\otimes\sigma_z}, \qquad S = -\log\omega.$$

The experimental feasibility of such an evolution is again a priori not obvious. However, for open systems, using (3.2) and (4.5) one simply has

$$\mathrm{e}^{-\frac{\alpha}{2}S\otimes\sigma_z}\mathrm{e}^{-\mathrm{i}tH\otimes\mathbb{1}}\mathrm{e}^{\frac{\alpha}{2}S\otimes\sigma_z} = \mathrm{e}^{-\mathrm{i}t\widehat{H}_\alpha}, \qquad \widehat{H}_\alpha = H\otimes\mathbb{1} + \widehat{W}_\alpha, \tag{4.15}$$

where

$$\widehat{W}_\alpha = \tfrac{1}{2}W_\alpha\otimes(\mathbb{1}+\sigma_z) + \tfrac{1}{2}W_{-\alpha}\otimes(\mathbb{1}-\sigma_z), \qquad W_\alpha = \varsigma_\omega^{-\mathrm{i}\alpha/2}(V) - V. \tag{4.16}$$

In a 2×2 block matrix form this corresponds to

$$\widehat{H}_\alpha = \begin{bmatrix} H+W_\alpha & 0 \\ 0 & H+W_{-\alpha}\end{bmatrix} = \begin{bmatrix} H_{\mathrm{fr}}+\varsigma_\omega^{-\mathrm{i}\alpha/2}(V) & 0 \\ 0 & H_{\mathrm{fr}}+\varsigma_\omega^{\mathrm{i}\alpha/2}(V)\end{bmatrix}.$$

Only the interaction term V is modified, to $\varsigma_\omega^{\pm\mathrm{i}\alpha/2}(V)$ depending on the qubit state.

A simple calculation shows that the qubit is then in the state

$$\rho_{a,t}(\alpha) = \frac{1}{2}\begin{bmatrix} 1 & \mathfrak{F}^{\mathrm{east}}_{\nu,t}(\alpha) \\ \overline{\mathfrak{F}^{\mathrm{east}}_{\nu,t}(\alpha)} & 1\end{bmatrix}, \tag{4.17}$$

with

$$\mathfrak{F}^{\mathrm{east}}_{\nu,t}(\alpha) = \nu\left(\omega^{-\alpha/2}\omega^{\alpha}_{-t}\omega^{-\alpha/2}\right), \tag{4.18}$$

the latter being again accessible using a quantum state tomography of the qubit. When the initial state of the system is ω we simply have

$$\mathfrak{F}^{\mathrm{east}}_{\omega,t}(\alpha) = \mathrm{tr}\left(\omega^{\alpha}_{-t}\omega^{1-\alpha}\right),$$

which coincides with the Fourier transform of the two-time measurement probability distribution $Q^{\mathrm{2tm}}_{\omega,t}$, see Equation (4.9), giving the announced more operational approach than the two-time measurement.

Note however that when $\nu\neq\omega$ the ancilla functional $\mathfrak{F}^{\mathrm{east}}_{\nu,t}$ generally differs from both the 2TM and the QPSC functionals (compare (4.18) with (4.8) and (4.14)). Moreover we stress that, as it is the case for the QPSC functional, the ancilla functional $\mathfrak{F}^{\mathrm{east}}_{\nu,t}$ is not necessarily the Laplace transform of a Borel probability measure when $\nu\neq\omega$.

5 Infinitely extended systems: 3 facets of entropic fluctuations

Besides their own interest, infinitely extended systems are needed to consider entropic fluctuations with respect to steady states. In this section we explain how the 2TM, QPSC and ancilla approaches presented in the previous section for finite systems can be generalized to infinitely extended systems. The QPSC and ancilla approaches are rather straightforward to extend, although not completely obvious. The 2TM approach is more problematic. The two-time measurement entropy production protocol is always introduced in the context of finite quantum systems (or, slightly more generally, confined quantum systems with possibly infinite but discrete energy spectra), and its definition for infinitely extended systems obtained via a thermodynamic limit (TDL). For a long time such a limiting procedure was unknown when the state ν at the instant of the first measurement is arbitrary, in particular for the steady state $\nu = \omega_\infty$. To the best of our knowledge, the case of a general state ν was considered for the first time in [BBJ+24].

For the sake of simplicity, and motivated by the Gallavotti–Cohen fluctuation theorem, in the rest of this review we will only consider the case where the state ν is either the time-evolved state ω_T, for some $T \geq 0$, or the NESS ω_∞. The conceptual and technical difference between these two states is that, while ω_T is ω-normal for all $T \geq 0$, the NESS ω_∞ and ω are typically mutually singular. Most of the results presented below hold for arbitrary state ν, with the distinction T finite or infinite being replaced by ν is ω-normal or arbitrary.

In the sequel the functions $\mathfrak{F}^{\#}_{\omega_T,t}$, $\# \in \{2\mathrm{tm}, \mathrm{qpsc}, \mathrm{east}\}$ and $T \in \mathbb{R}_+ \cup \{\infty\}$, will be called the *entropic functionals*. As for classical systems, they are the central objects to study LDP through the use of the Gärtner–Ellis theorem.

5.1 2TM protocol in infinitely extended systems

5.1.1 Definition and stability

For finite systems the entropic functional $\mathfrak{F}^{2\mathrm{tm}}_{\omega_T,t}$ is given by (4.8). In terms of modular data[6] it becomes

$$\mathfrak{F}^{2\mathrm{tm}}_{\omega_T,t}(\alpha) = \overline{\omega}_T\left([D\omega_{-t} : D\omega]_\alpha\right),$$

where we recall that $\overline{\omega}_T$ is the completely decoherent part of ω_T with respect to ω.

When $T = 0$ one simply has $\mathfrak{F}^{2\mathrm{tm}}_{\omega,t}(\alpha) = \omega\left([D\omega_{-t} : D\omega]_\alpha\right)$ and the generalization to extended systems is straightforward. When $T \neq 0$ the meaning of $\overline{\omega}_T$ is not clear anymore and the key observation is that one can further write, see [BBJ+24],

$$\mathfrak{F}^{2\mathrm{tm}}_{\omega_T,t}(\alpha) = \lim_{R\to\infty} \frac{1}{R}\int_0^R \omega_T \circ \varsigma^\theta_\omega\left([D\omega_{-t} : D\omega]_\alpha\right) \mathrm{d}\theta. \tag{5.1}$$

The ergodic average over the modular group $(\varsigma^\theta_\omega)_\theta$, see (3.1), reflects the effect of a first measurement and replaces the projection to the diagonal part with respect to ω.

[6] Modular theory is the natural language to describe infinitely extended systems where quantities like $-\log\omega$ do not have a meaning.

Remark. Since the initial state ω is invariant under the action of the modular group ς_ω, for $T = 0$ one gets as expected that

$$\mathfrak{F}^{\mathrm{2tm}}_{\omega,t}(\alpha) = \omega\left([D\omega_{-t} : D\omega]_\alpha\right), \tag{5.2}$$

which can also be rewritten as

$$\mathfrak{F}^{\mathrm{2tm}}_{\omega,t}(\alpha) = \omega\left([D\omega_{-t} : D\omega]^*_{\frac{\alpha}{2}}[D\omega_{-t} : D\omega]_{\frac{\alpha}{2}}\right). \tag{5.3}$$

The latter expression will be useful in comparison with the ancilla assisted protocol, see Section 5.3.

In [BBJ+24, Theorem 1.3] it is proven that, in the modular setting recalled in Section 3, under the regularity hypotheses stated there, for any $T \in \mathbb{R}$, $t \in \mathbb{R}$ and $\alpha \in \mathrm{i}\mathbb{R}$ the limit in the right-hand side of (5.1) exists. Moreover there is a unique Borel probability measure $Q^{\mathrm{2tm}}_{\omega_T,t}$ on $\mathbb{R}$ such that

$$\mathfrak{F}^{\mathrm{2tm}}_{\omega_T,t}(\alpha) = \int_{\mathbb{R}} \mathrm{e}^{-\alpha s}\mathrm{d}Q^{\mathrm{2tm}}_{\omega_T,t}(s).$$

The measure $Q^{\mathrm{2tm}}_{\omega_T,t}$ gives the statistics of an idealized two-time measurement of the entropy production in the system $(\mathscr{O}, \tau, \omega)$ in a protocol where the first measurement is done at time T and the second at time $T + t$.

This definition of such an idealized two-time measurement entropy production actually comes with a somewhat surprising rigidity result: under mild ergodic assumptions on the reservoirs, it is essentially independent of T. More precisely, the following is proven in [BBJ+24, Theorem 1.6].

Theorem 5.1. *Suppose that each reservoir subsystem is ergodic. For all T denote by $\omega_{T,\mathsf{S}}$ the restriction of the evolved state ω_T to $\mathscr{O}_\mathsf{S}$, and by $\Omega_{T,\mathsf{S}}$ its representative vector. We write $\Omega_\mathsf{R} = \Omega_1 \otimes \Omega_2$ for the representative vector of ω_R. Then for all $\alpha \in \mathrm{i}\mathbb{R}$,*

$$\mathfrak{F}^{\mathrm{2tm}}_{\omega_T,t}(\alpha) = \omega_{T,\mathsf{S}} \otimes \omega_\mathsf{R}\left([D\omega_{-t} : D\omega]_\alpha\right) = \langle \Omega_{T,\mathsf{S}} \otimes \Omega_\mathsf{R}, \Delta^\alpha_{\omega_{-t}|\omega}\, \Omega_{T,\mathsf{S}} \otimes \Omega_\mathsf{R}\rangle. \tag{5.4}$$

In particular, $Q^{\mathrm{2tm}}_{\omega_T,t}$ is the spectral measure of $-\log\Delta_{\omega_{-t}|\omega}$ for the vector $\Omega_{T,\mathsf{S}} \otimes \Omega_\mathsf{R}$. Moreover, for any $t \in \mathbb{R}$,

$$\min \mathrm{sp}(\omega_{T,\mathsf{S}}) \times \dim \mathscr{K}_\mathsf{S} \leq \frac{\mathrm{d}Q^{\mathrm{2tm}}_{\omega_T,t}}{\mathrm{d}Q^{\mathrm{2tm}}_{\omega,t}} \leq \dim \mathscr{K}_\mathsf{S}, \tag{5.5}$$

where $\mathrm{sp}(\omega_{T,\mathsf{S}})$ denotes the spectrum of the density matrix associated to the state $\omega_{T,\mathsf{S}}$.

Corollary 5.2. *Under the same hypotheses, if in addition Assumption* **(NESS)** *holds then Equation (5.4) allows to define $\mathfrak{F}^{\mathrm{2tm}}_{\omega_\infty,t} = \lim_{T\to\infty} \mathfrak{F}^{\mathrm{2tm}}_{\omega_T,t}$. The associated measure $Q^{\mathrm{2tm}}_{\omega_\infty,t}$ is obtained as the weak limit of $Q^{\mathrm{2tm}}_{\omega_T,t}$ as $T \to \infty$ and (5.5) remains in force.*

Remark 1. If one allows a direct interaction $V \in \mathscr{O}_1 \otimes \mathscr{O}_2$ between the reservoirs and takes $\dim \mathscr{K}_\mathsf{S} = 1$, one actually has the even stronger rigidity property that $Q^{\mathrm{2tm}}_{\omega_T,t} = Q^{\mathrm{2tm}}_{\omega,t}$ for all T and t.

Remark 2. The needed ergodicity assumption is actually the one of the modular groups of the reservoir states, *i.e.,* the $(\mathscr{O}_j, \omega_j, \varsigma_{\omega_j})$'s need to be ergodic [BBJ+24]. This is equivalent to the above mentioned assumption because the ω_j's are (τ_j, β_j)-KMS.

Remark 3. The first inequality in (5.5) is non-trivial only when $\omega_{T,\mathsf{S}}$ is faithful. This is automatic when T is finite.

The two-time measurement entropy production statistics exhibits a striking rigidity, as shown by Inequality (5.5), which can be interpreted as the memory erasing effect due to the decoherence induced by the first measurement on the reservoirs. For infinitely extended systems this effect is dramatic as the modular averaging associated with the first measurement resets the reservoir part of a normal initial state to the reference reservoir state, which is the unique invariant state in its normal folium, while the state of the small system S has a marginal effect, being finite dimensional. In Section 6 we shall explore its consequences concerning the related LDP's.

Remark 4. As mentioned, this rigidity result is due to the ergodicity of the modular groups (of the reservoir initial states). It has no classical analog[7], see [BBJ+25a, Section 2.8] for a discussion of this point.

We conclude this section with the following proposition when $T = 0$. It shows that in the two-time measurement approach the entropic functional can be extended to real values of α, leading to a transient fluctuation relation for infinitely extended systems.

Proposition 5.3. *The map* $\mathrm{i}\mathbb{R} \ni \alpha \mapsto \mathfrak{F}^{2\mathrm{tm}}_{\omega,t}(\alpha)$ *has an analytic extension to the vertical strip* $0 < \mathrm{Re}\,\alpha < 1$ *which is bounded and continuous on its closure.*

If moreover the system is time-reversal invariant then, for any $0 \le \alpha \le 1$,

$$\mathfrak{F}^{2\mathrm{tm}}_{\omega,t}(\alpha) = \mathfrak{F}^{2\mathrm{tm}}_{\omega,t}(1-\alpha).$$

As a consequence, the measures $Q^{2\mathrm{tm}}_{\omega,t}$ *satisfy the transient fluctuation relation* (4.10).

5.1.2 Thermodynamic justification of 2TM

In order to define the two-time measurement protocol for infinitely extended systems we have used the following customary route: a physical notion, introduced in the context of finite quantum systems, is expressed in a modular form (here it is (5.1)) and through this form is directly extended, by definition, to a general quantum dynamical system. In concrete physical models this definition is justified by the TDL limit. The latter has been extensively studied in early days of quantum statistical mechanics and the wealth of obtained results make its implementation in modern literature most often a routine exercise which is generally skipped. However the mixture of quantum measurements and thermodynamics, which is central to the definition of two-time quantum measurement entropy production, has a number of unexpected features among which is the above-mentioned drastic decoherence effect due to the first measurement. For this reason the TDL justification of this idealized two-time measurement needs to be put on solid physical grounds.

In more mathematical terms the first measurement decoherence corresponds to a projection onto $\mathrm{Ker}\log\Delta_\omega$, see Equation (5.7) below. Indeed, assume that for some $B_T \in \mathfrak{M}'$ with $\|B_T\Omega\| = 1$, one has $\Omega_T = B_T\Omega$. This holds true, see [BBJ+25b, Proposition 2.1], if the map

$$\mathrm{i}\mathbb{R} \ni z \mapsto [D\omega_T : D\omega]_z$$

[7] The modular groups are then trivial and hence not ergodic, except for the one-dimensional algebra.

has an analytic continuation to the vertical strip $0 < \mathrm{Re}\, z < \frac{1}{2}$ that is continuous and bounded on its closure, taking

$$B_T = J[D\omega_T : D\omega]_{\frac{1}{2}} J \in \mathfrak{M}'. \tag{5.6}$$

One can then write (5.1) as, using that $\varsigma^\theta([D\omega_{-t} : D\omega]_\alpha) \in \mathfrak{M}$ commutes with B_T,

$$\begin{aligned} \mathfrak{F}^{\mathrm{2tm}}_{\omega_T,t}(\alpha) &= \lim_{R\to\infty} \frac{1}{R} \int_0^R \langle B_T^* B_T \Omega, \Delta_\omega^{\mathrm{i}\theta} [D\omega_{-t} : D\omega]_\alpha \Delta_\omega^{-\mathrm{i}\theta} \Omega \rangle \, \mathrm{d}\theta \\ &= \lim_{R\to\infty} \frac{1}{R} \int_0^R \langle B_T^* B_T \Omega, \Delta_\omega^{\mathrm{i}\theta} [D\omega_{-t} : D\omega]_\alpha \Omega \rangle \, \mathrm{d}\theta \\ &= \langle B_T^* B_T \Omega, \mathbb{1}_{\{0\}}(\log \Delta_\omega) [D\omega_{-t} : D\omega]_\alpha \Omega \rangle, \end{aligned} \tag{5.7}$$

where $\mathbb{1}_{\{0\}}(\log \Delta_\omega)$ is the mentioned projection on $\mathrm{Ker} \log \Delta_\omega$.

For infinitely extended reservoirs, and due to the ergodicity assumptions (see the end of Section 3.4), this projection is given by

$$\mathbb{1}_{\{0\}}(\log \Delta_\omega) = \mathrm{Id}_{\mathsf{S}} \otimes |\Omega_{R_1}\rangle\langle\Omega_{R_1}| \otimes |\Omega_{R_2}\rangle\langle\Omega_{R_2}|$$

and its rank equals $(\dim \mathscr{K}_{\mathsf{S}})^2$. On the contrary, for finite dimensional reservoirs and using (3.3) and (4.12), one has

$$\log \Delta_\omega = -\beta_{\mathsf{R}_1}[H_{\mathsf{R}_1}, \cdot] - \beta_{\mathsf{R}_2}[H_{\mathsf{R}_2}, \cdot],$$

so the rank of this projection is at least $(\dim \mathscr{K}_{\mathsf{S}})^2 \times \dim \mathscr{K}_{\mathsf{R}_1} \times \dim \mathscr{K}_{\mathsf{R}_2}$. Along a sequence of TDL approximations the size of the kernel $\mathrm{Ker} \log \Delta_\omega$ thus grows to infinity together with the dimension of the reservoirs Hilbert spaces to suddenly drastically shrink in the limiting state. This dimension reduction is at the core of the need for the TDL justification of the definition (5.1) of the 2TM.

Remark. The stability result given in Theorem 5.1 relies on the representation of $\mathfrak{F}^{\mathrm{2tm}}_{\omega_T,t}(\alpha)$ given in (5.7).

If, for each reservoir R_j, $(\mathscr{O}_{j,\Lambda}, \tau_{j,\Lambda}, \omega_{j,\Lambda})_\Lambda$ is a net of finite dimensional approximations of the infinitely extended C^* quantum dynamical system $(\mathscr{O}_j, \tau_j, \omega_j)$, one thus needs to prove that[8]

$$\lim_\Lambda \langle B_{T,\Lambda}^* B_{T,\Lambda} \Omega_\Lambda, \mathbb{1}_{\{0\}}(\log \Delta_{\omega,\Lambda}) [D\omega_{-t,\Lambda} : D\omega_\Lambda]_\alpha \Omega_\Lambda \rangle = \langle B_T^* B_T \Omega, \mathbb{1}_{\{0\}}(\log \Delta_\omega) [D\omega_{-t} : D\omega]_\alpha \Omega \rangle.$$

In [BBJ+25b] a general scheme is proposed, which can then be used on concrete models. We briefly comment on two specific aspects of this TDL problem and refer the reader to [BBJ+25b] for more details.

One of the issues is the control of the projections $\mathbb{1}_{\{0\}}(\log \Delta_{\omega,\Lambda})$. One proves that they converge weakly to $\mathbb{1}_{\{0\}}(\log \Delta_\omega)$ by combining Araki's results on the continuity of modular structure [AI74, Section 2] with the dynamical ergodicity assumption on the infinitely extended reservoirs. In the models considered, the latter ensures the inclusion $\mathrm{Ker} \log \Delta_\omega \subset \mathrm{Ker} \log \Delta_{\omega,\Lambda}$ needed in the projection convergence argument. We emphasize that such an ergodicity assumption is unusual in TDL type results.

[8] A quantity with an index Λ means that it is the quantity obtained in the corresponding finite dimensional approximation.

The second main issue concerns the control of the Araki–Connes cocycles $[D\omega_{-t,\Lambda} : D\omega_\Lambda]_\alpha$, not only for $\alpha \in \mathrm{i}\mathbb{R}$ but also, in view of (5.6), of their analytic extensions to a vertical strip up to $\mathrm{Re}(\alpha) = \frac{1}{2}$.

The general TDL scheme mentioned above is applied in [BBJ+25b] to extended black-box fermionic models and quantum spin systems, under the hypotheses specified there. For quantum spin systems these include interaction decay, high temperature and reservoir ergodicity. The finite-volume reservoir states are restrictions of the infinite-volume KMS states, with effective Hamiltonians $\widehat{H}_{j,\Lambda} = -\beta_j^{-1} \log(\omega_j \lceil \mathscr{O}_{j,\Lambda})$. The general-state TDL for the conventional finite-volume Gibbs approximation remains open. Even with the restriction approximation, one does not prove that

$$\lim_\Lambda \mathfrak{F}^{\mathrm{2tm}}_{\omega_{T,\Lambda},t}(\alpha) = \mathfrak{F}^{\mathrm{2tm}}_{\omega_T,t}(\alpha), \qquad T, t \in \mathbb{R},\ \alpha \in \mathrm{i}\mathbb{R},$$

and consequently $\lim_\Lambda Q^{\mathrm{2tm}}_{\omega_{T,\Lambda},t} = Q^{\mathrm{2tm}}_{\omega_T,t}$ weakly, for the net of states $\omega_{T,\Lambda} = \omega_\Lambda \circ \tau_\Lambda^T$, *i.e.,* the evolution of the restriction ω_Λ of ω to the finite volume Λ under the finite volume evolution τ_Λ. Instead, one gets

$$\lim_\Lambda \mathfrak{F}^{\mathrm{2tm}}_{\omega_T \lceil \Lambda,t}(\alpha) = \mathfrak{F}^{\mathrm{2tm}}_{\omega_T,t}(\alpha), \qquad T, t \in \mathbb{R},\ \alpha \in \mathrm{i}\mathbb{R},$$

where $\omega_T \lceil \Lambda$ is the restriction to the finite volume Λ of the infinite volume evolved state ω_T. Except in trivial cases $\omega_T \lceil \Lambda \neq \omega_{T,\Lambda}$.

5.2 Entropic cocycle and QPSC

The second approach to fluctuation relation, the QPSC approach presented in Section 4.4, is more straightforward to extend to infinitely extended systems than the 2TM approach. Indeed, compare with (4.14), we consider the map

$$\mathrm{i}\mathbb{R} \ni \alpha \mapsto [D\omega_{-t} : D\omega]_\alpha \in \mathscr{O}$$

as a characterization of the *quantum phase space contraction* of $(\mathscr{O}, \tau, \omega)$ at time t, and we set

$$\mathfrak{F}^{\mathrm{qpsc}}_{\omega_T,t}(\alpha) :- \omega_T([D\omega_{-t} : D\omega]_\alpha), \qquad T \in \mathbb{R} \cup \{\infty\}. \tag{5.8}$$

Note that when $T = 0$ this functional coincides with the one obtained from the 2TM protocol, see Equation (5.2),

$$\mathfrak{F}^{\mathrm{qpsc}}_{\omega,t} = \mathfrak{F}^{\mathrm{2tm}}_{\omega,t}.$$

This identity need not hold when $T \neq 0$.

To motivate further the above definition and the relevance of (5.8), we shall relate it to the entropic cocycle c^t. As already pointed out, the definition (4.1) of c^t is meaningless in infinitely extended systems. Instead it can be defined through the use of the Araki–Connes family $([D\omega_t : D\omega]_\alpha)_{t\in\mathbb{R},\alpha\in\mathrm{i}\mathbb{R}}$, which can be regarded as a non-commutative version of the Radon–Nikodym derivative $\frac{\mathrm{d}\omega_t}{\mathrm{d}\omega}$.

The regularity Assumption **(Reg)** guarantees that the map $\mathbb{R} \ni \theta \mapsto [D\omega_t : D\omega]_{\mathrm{i}\theta} \in \mathscr{O}$ is differentiable at $\theta = 0$ for all $t \in \mathbb{R}$, see [BBJ+25b, Lemma 2.4]. The entropic cocycle c^t is then defined by

$$c^t :- \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}[D\omega_{-t} : D\omega]_{\mathrm{i}\theta}\Big|_{\theta=0}. \tag{5.9}$$

Note that c^t is a self-adjoint element of $\mathscr{O}$ and indeed coincides with (4.1) for finite systems, compare also with (2.2) in the classical case. The map $\mathbb{R} \ni t \mapsto c^t \in \mathscr{O}$ is then differentiable at $t = 0$ and the entropy production observable σ defined by, compare with (2.4) in the classical setting and with (3.5) and (4.2) for finite quantum systems,

$$\sigma := -\left.\frac{\mathrm{d}c_t}{\mathrm{d}t}\right|_{t=0} = -\beta_1\delta_1(V_1) - \beta_2\delta_2(V_2).$$

The properties of $(c^t)_t$ and σ then completely parallel those observed in classical systems, see Section 4.1. Namely, we have:

1. the family $(c^t)_{t\in\mathbb{R}}$ is an additive τ-cocycle, *i.e.*, $c^{t+s} = c^t + \tau^t(c^s)$ for all $t, s \in \mathbb{R}$,
2. $-\log\Delta_{\omega_{-t}|\omega} = c^t - \log\Delta_\omega$ (compare with (4.13)) so that $\mathrm{Ent}(\omega_t|\omega) = -\omega(c^t)$.
3. $c^t = \int_0^t \sigma_s \mathrm{d}s$ so that $\mathrm{Ent}(\omega_t|\omega) = -\int_0^t \omega_s(\sigma)\mathrm{d}s$.
4. If $(\mathscr{O}, \tau, \omega)$ is TRI with time-reversal Θ, then $\Theta(c^t) = c^{-t} = -\tau^{-t}(c^t)$.

5.3 Entropic Ancilla State Tomography (EAST)

The EAST functional presented in Section 4.5 can similarly be generalized to infinitely extended systems where it becomes even more relevant. Indeed, for such extended systems, the two-time measurement approach raises both the question of experimental feasibility and that of the invasive decoherence due to the first measurement, with its induced rigidity on the probability distributions $Q^{\mathrm{2tm}}_{\omega_T,t}$.

The procedure described for finite systems can be easily generalized to infinitely extended systems, the ancilla remaining a qubit. The algebra of observables of the system coupled to a qubit is $\widehat{\mathscr{O}} := \mathscr{O} \otimes \mathrm{Mat}_2(\mathbb{C})$, where $\mathrm{Mat}_2(\mathbb{C})$ is the algebra of complex 2×2 matrices. The interacting system+ancilla dynamics on $\widehat{\mathscr{O}}$ is given by, compare with (4.15),

$$\widehat{\tau}^t_\alpha = \mathrm{e}^{t\widehat{\delta}_\alpha}, \qquad \widehat{\delta}_\alpha = \delta \otimes \mathrm{Id} + \mathrm{i}[\widehat{W}_\alpha, \cdot],$$

where $\widehat{W}_\alpha$ is the perturbation defined by (4.16) (note that it is self-adjoint). The initial state of the ancilla is ρ_a as above. For any state ν on $\mathscr{O}$ we set $\widehat{\nu} = \nu \otimes \rho_a$, $\widehat{\nu}_t(\alpha) = \widehat{\nu} \circ \widehat{\tau}^t_\alpha$, and $\rho_{a,t}(\alpha) = \widehat{\nu}_t|_{\mathrm{Mat}_2(\mathbb{C})}$. We then have:

Proposition 5.4. *Suppose that $V \in \mathrm{Dom}(\delta_\omega)$. Then, for all $\alpha \in \mathrm{i}\mathbb{R}$ and any state ν on $\mathscr{O}$, the ancilla's state $\rho_{a,t}(\alpha)$ at time $t \in \mathbb{R}$ is given by the right-hand side of (4.17) with*

$$\mathfrak{F}^{\mathrm{east}}_{\nu,t}(\alpha) = \nu\left([D\omega_{-t} : D\omega]^*_{\frac{\alpha}{2}}\,[D\omega_{-t} : D\omega]_{\frac{\alpha}{2}}\right). \tag{5.10}$$

Here again, when $\nu = \omega$ we get $\mathfrak{F}^{\mathrm{east}}_{\omega,t}(\alpha) = \mathfrak{F}^{\mathrm{2tm}}_{\omega,t}(\alpha)$ (see Equation (5.3)), and the ancilla coupling protocol indeed gives the announced more operational definition of entropy production. We will consider this ancilla protocol starting with $\nu = \omega_T$, $T \in \mathbb{R}_+ \cup \{\infty\}$. For $T \neq 0$ it generally differs from both the 2TM functional (5.1) and the QPSC one (5.8), and the functional $\mathfrak{F}^{\mathrm{east}}_{\omega_T,t}$ need not be the Laplace transform of a probability measure. As we shall see in Section 6.3 it will also be less invasive than the two-time measurement.

5.4 Comparing the entropic functionals

In the previous sections we have introduced several approaches to study entropic fluctuations (naive, 2TM, QPSC and EAST), and we shall give a first comparison between them.

For infinitely extended systems the naive approach of Section 4.1, with c^t defined by (5.9), would lead to the following functionals (compare with (4.3))

$$\mathfrak{F}^{\text{naive}}_{\omega_T,t}(\alpha) := -\,\omega_T\left(\mathrm{e}^{-\alpha c^t}\right),$$

while the 2TM, QPSC and EAST functionals are respectively given by (5.1), (5.8) and (5.10).

When T is finite ω_T is ω-normal and the different functionals can be further rewritten in the GNS representation. Namely,

$$\begin{aligned}
\mathfrak{F}^{\text{naive}}_{\omega_T,t}(\alpha) &= \left\langle \Omega_T, \mathrm{e}^{-\alpha c^t}\Omega_T\right\rangle,\\
\mathfrak{F}^{\text{2tm}}_{\omega_T,t}(\alpha) &= \lim_{R\to\infty}\frac{1}{R}\int_0^R \left\langle \Omega_T, \Delta_\omega^{i\theta}\mathrm{e}^{-\alpha c^t+\alpha\log\Delta_\omega}\mathrm{e}^{-\alpha\log\Delta_\omega}\Delta_\omega^{-i\theta}\Omega_T\right\rangle \mathrm{d}\theta,\\
\mathfrak{F}^{\text{qpsc}}_{\omega_T,t}(\alpha) &= \left\langle \Omega_T, \mathrm{e}^{-\alpha c^t+\alpha\log\Delta_\omega}\mathrm{e}^{-\alpha\log\Delta_\omega}\Omega_T\right\rangle,\\
\mathfrak{F}^{\text{east}}_{\omega_T,t}(\alpha) &= \left\langle \Omega_T, \mathrm{e}^{-\frac{\alpha}{2}\log\Delta_\omega}\mathrm{e}^{-\alpha c^t+\alpha\log\Delta_\omega}\mathrm{e}^{-\frac{\alpha}{2}\log\Delta_\omega}\Omega_T\right\rangle.
\end{aligned}$$

In the classical setting all these quantities coincide due to the triviality of Δ_ω, and are given by (2.9). In other words, the various approaches are indistinguishable in the classical/commutative framework, and the naive, 2TM, QPSC and EAST functionals discussed above are natural non-commutative extensions of the classical theory. The diversity of non-commutative/quantum theory of entropic fluctuations then stems from the non triviality and the rich KMS structure associated to ω via Δ_ω. In fact one can construct an entire host of mathematically natural non-commutative extensions of $\mathfrak{F}_{\omega_T,t}$; see [JOPP10, Section 3.3]. Some of these non-commutative extensions have direct quantum mechanical interpretations, like the 2TM and EAST ones, and some do not, or such interpretations are not known at the moment. For example, if $\nu\neq\omega$, then to the best of our knowledge, the functional $\mathfrak{F}^{\text{qpsc}}_{\nu,t}$ does not have any quantum mechanical or stochastic interpretation.

6 The quantum principle of regular entropic fluctuations (PREF)

In this section we elaborate on the classical PREF described at the end of Section 2.

6.1 Definition of the quantum PREF

As we have already stressed, while the 2TM functionals are Fourier transforms of probability measures, the EAST and QPSC functionals need not have this property (at $T=0$ the three functionals coincide). Hence, an associated LDP is not available in general. Concerning the EAST entropic functionals, when $T=0$ they provide natural operational access to the 2TM functional. The question of

their connection for later time, eventually in the NESS, is therefore natural. Moreover, as mentioned in the introduction and in view of the tomography part of the EAST protocol, in this indirect measurement approach one needs to repeat the experiment with the *same* "initial" state, *i.e.,* the state of the open quantum system at the time when it is coupled to the ancilla. In this respect the NESS is a natural choice putting emphasis on $\mathfrak{F}^{\mathrm{east}}_{\omega_\infty,t}$. Concerning the QPSC approach, its close parallel with the classical setting makes it natural to incorporate it in the quantum version of PREF which we define below, compare with (2.10) and (2.12) in Section 2.

The quantum version of PREF given in Definition 6.1 below, requires extending the various entropic functionals to α's in a real interval (they are up to now defined only for $\alpha \in \mathrm{i}\mathbb{R}$). This is possible under the following assumption (ϑ will be specified later on).

> **(AnV(ϑ))** The connecting perturbation V is such that the map $\mathrm{i}\mathbb{R} \ni \theta \mapsto \varsigma_\omega^{-\mathrm{i}\theta}(V) \in \mathscr{C}$ has an analytic extension to the vertical strip $\mathfrak{S}(\vartheta) = \{z \in \mathbb{C} \mid |\mathrm{Re}\, z| < \vartheta\}$.

If Assumption **(AnV(ϑ))** holds then, for any $t \in \mathbb{R}$, the function

$$\mathrm{i}\mathbb{R} \ni \alpha \mapsto [D\omega_t : D\omega]_\alpha \in \mathscr{C}$$

has an analytic extension to $\mathfrak{S}(\vartheta)$, and hence so do the various entropic functionals.[9] At this point, the sole analytic extension of $[D\omega_t : D\omega]_\alpha$ would be sufficient, see [BBJ$^+$25a]. Since we will need the stronger Assumption **(AnV(ϑ))** later on, for the sake of simplicity in the exposition we assume it holds throughout this section.

We also assume **(NESS)** and ergodicity of the reservoir systems, so that the NESS 2TM functional is defined as above. We use the formulation of PREF in [BBJ$^+$26], which adds the QPSC equality to the strong PREF of [BBJ$^+$25a]. In the latter work, weak PREF means that the transient and steady-state 2TM statistics satisfy large deviation principles with the same rate function.

Definition 6.1. *Let $I =]\vartheta_-, \vartheta_+[$ be an open interval containing 0 and contained in $]-\vartheta, \vartheta[$. We say that $(\mathscr{C}, \tau, \omega)$ satisfies the PREF on the interval I if, for all $\alpha \in I$, the limits*

$$\begin{aligned}
\boldsymbol{F}^{\mathrm{2tm}}_{\omega}(\alpha) &= \lim_{t\to\infty} \frac{1}{t} \log \mathfrak{F}^{\mathrm{2tm}}_{\omega,t}(\alpha), \\
\boldsymbol{F}^{\mathrm{2tm}}_{\omega_\infty}(\alpha) &= \lim_{t\to\infty} \frac{1}{t} \log \mathfrak{F}^{\mathrm{2tm}}_{\omega_\infty,t}(\alpha), \\
\boldsymbol{F}^{\mathrm{east}}_{\omega_\infty}(\alpha) &= \lim_{t\to\infty} \frac{1}{t} \log \mathfrak{F}^{\mathrm{east}}_{\omega_\infty,t}(\alpha), \\
\boldsymbol{F}^{\mathrm{qpsc}}_{\omega_\infty}(\alpha) &= \lim_{t\to\infty} \frac{1}{t} \log \mathfrak{F}^{\mathrm{qpsc}}_{\omega_\infty,t}(\alpha)
\end{aligned} \tag{6.1}$$

exist, and define differentiable functions on I, satisfying

$$\boldsymbol{F}^{\mathrm{2tm}}_{\omega} = \boldsymbol{F}^{\mathrm{2tm}}_{\omega_\infty} = \boldsymbol{F}^{\mathrm{east}}_{\omega_\infty} = \boldsymbol{F}^{\mathrm{qpsc}}_{\omega_\infty}. \tag{6.2}$$

For the QPSC functional, the logarithm is understood as the principal value. We denote by $\boldsymbol{F}$ the common function in (6.2).

[9] The EAST functional actually extends to the strip $\mathfrak{S}(2\vartheta)$.

6.2 PREF and 2TM

Using the Gärtner–Ellis Theorem, if the limit

$$\boldsymbol{F}^{\mathrm{2tm}}_{\omega}(\alpha) := \lim_{t\to\infty} \tfrac{1}{t} \log \mathfrak{F}^{\mathrm{2tm}}_{\omega,t}(\alpha)$$

exists for all $\alpha \in]\vartheta_-, \vartheta_+[$, with $\max\{|\vartheta_-|, \vartheta_+\} < \vartheta$, and defines a differentiable function on this interval, then the family $(P^{\mathrm{2tm}}_{\omega,t})_t$ satisfies the restricted LDP on the interval of slopes specified by the Gärtner–Ellis Theorem in Section 2. Under the assumptions of Theorem 5.1, when $\omega_{T,\mathrm{S}}$ is faithful it is an immediate consequence of Equation (5.5) that all the limits

$$\boldsymbol{F}^{\mathrm{2tm}}_{\omega_T}(\alpha) := \lim_{t\to\infty} \tfrac{1}{t} \log \mathfrak{F}^{\mathrm{2tm}}_{\omega_T,t}(\alpha)$$

then exist and coincide. This is always the case for finite T, and when $\omega_{\infty,\mathrm{S}}$ is faithful we get

$$\boldsymbol{F}^{\mathrm{2tm}}_{\omega} = \boldsymbol{F}^{\mathrm{2tm}}_{\omega_\infty}.$$

In the 2TM approach, the existence and differentiability of the first, respectively second limit in (6.1) imply the corresponding restricted LDP by the Gärtner–Ellis Theorem; a full LDP requires its additional global hypotheses. Indeed, as in the classical case, consider the families of probability measures $(P^{\mathrm{2tm}}_{\omega_T,t})_{t>0}$, $T \in \mathbb{R}_+ \cup \{\infty\}$, defined by

$$P^{\mathrm{2tm}}_{\omega_T,t}(B) = Q^{\mathrm{2tm}}_{\omega_T,t}(tB),$$

for all Borel sets $B \subset \mathbb{R}$ and $t > 0$. They describe the statistics of the two-time measurement entropy production per unit time for a protocol initialized in the state ω_T, with time t between the two measurements. The following results are then immediate consequences of Proposition 5.3 and Theorem 5.1 respectively.

1. If $(P^{\mathrm{2tm}}_{\omega,t})_t$ satisfies a full LDP, *i.e.,* a quantum Evans–Searles fluctuation theorem holds, with rate function $\mathbb{I}$ and the system is TRI, then for all $s \in \mathbb{R}$,

$$\mathbb{I}(-s) = \mathbb{I}(s) + s, \tag{6.3}$$

   i.e., a quantum Evans–Searles fluctuation relation also holds. If the restricted LDP holds on $]-a, a[$ for some $a > 0$, then (6.3) holds for $s \in]-a, a[$.

2. Under the assumptions of Theorem 5.1, if the family $(P^{\mathrm{2tm}}_{\omega,t})_t$ satisfies a (full or restricted) LDP then each family $(P^{\mathrm{2tm}}_{\omega_T,t})_t$ with faithful $\omega_{T,\mathrm{S}}$ satisfies the same LDP, and with the *same rate function*. In particular, this applies at $T = \infty$ when $\omega_{\infty,\mathrm{S}}$ is faithful. Under these hypotheses, in the 2TM approach the validity of the Evans–Searles fluctuation theorem, with the associated Evans–Searles fluctuation relation when the system is TRI, automatically guarantees the validity of a quantum Gallavotti–Cohen fluctuation theorem and relation. This is in stark contrast with the classical situation.

6.3 PREF and EAST

While in the classical case PREF can be regarded as a deep dynamical property of the system exemplified by the exchange of limits (2.11), it essentially trivializes in the 2TM approach because of the first measurement's dominating effect in the thermodynamic limit. In a surprising turn, the parallel with the classical PREF is restored when considering the EAST functionals. Indeed, if Assumption **(AnV(ϑ))** holds with $\vartheta > \frac{1}{2}$, for any real $\alpha \in]-\vartheta, \vartheta[$ and finite T we actually have the following estimate, see [BBJ$^+$25a, Proposition 2.12],

$$\mathrm{e}^{-2|T|(\|\varsigma_\omega^{\mathrm{i}/2}(V)\|+\|V\|)}\mathfrak{F}^{\mathrm{2tm}}_{\omega,t}(\alpha) \leq \mathfrak{F}^{\mathrm{east}}_{\omega_T,t}(\alpha) \leq \mathrm{e}^{2|T|(\|\varsigma_\omega^{-\mathrm{i}/2}(V)\|+\|V\|)}\mathfrak{F}^{\mathrm{2tm}}_{\omega,t}(\alpha).$$

For any finite time T, assuming the existence of the first limit in (6.1) this gives that, for $\alpha \in]\vartheta_-, \vartheta_+[$ and *arbitrary* finite initial time T,

$$\boldsymbol{F}^{\mathrm{east}}_{\omega_T}(\alpha) = \boldsymbol{F}^{\mathrm{2tm}}_{\omega}(\alpha). \tag{6.4}$$

In other words, as mentioned in the introduction, in great generality the EAST protocol satisfies a weaker form of PREF, valid for each fixed finite waiting time T.

However, if one wants to go up to $T = \infty$, *i.e.,* consider the NESS ω_∞, under the hypotheses ensuring $\boldsymbol{F}^{\mathrm{2tm}}_{\omega} = \boldsymbol{F}^{\mathrm{2tm}}_{\omega_\infty}$, the second equality in (6.2) holds if and only if the exchange of limits, compare with (2.11),

$$\lim_{T\to\infty}\lim_{t\to\infty}\tfrac{1}{t}\log\mathfrak{F}^{\mathrm{east}}_{\omega_T,t}(\alpha) = \lim_{t\to\infty}\lim_{T\to\infty}\tfrac{1}{t}\log\mathfrak{F}^{\mathrm{east}}_{\omega_T,t}(\alpha)$$

is valid for $\alpha \in]\vartheta_-, \vartheta_+[$.

7 Quantum transfer operators approach to PREF

In this section, we briefly present the quantum transfer operators representation of the various entropic functionals, and we provide a characterization of the quantum PREF in terms of the spectral resonance of these operators. This approach builds on the quantum transfer operator construction of NESS developed in [JP02]; see also [JOPP10] for a pedagogical introduction to the topic. We then show how it can be applied to one of the paradigmatic models of quantum statistical mechanics, namely the spin-fermion model.

7.1 Liouvillean representation of entropic functionals

Again we suppose that Assumption **(AnV(ϑ))** holds for some $\vartheta > 1/2$. For all $\alpha \in \mathfrak{S}(\vartheta)$, we define the following families of so-called α-Liouvilleans:

$$\begin{aligned}\mathscr{L}_\alpha &= \mathscr{L}_{\mathrm{fr}} + V - J\varsigma_\omega^{-\mathrm{i}\tilde{\alpha}}(V)J,\\ \widehat{\mathscr{L}}_\alpha &= \mathscr{L}_{\mathrm{fr}} + \varsigma_\omega^{\mathrm{i}(1/4-\alpha/2)}(V) - J\varsigma_\omega^{-\mathrm{i}(1/4+\tilde{\alpha}/2)}(V)J.\end{aligned} \tag{7.1}$$

Here $\mathscr{L}_{\mathrm{fr}}$ is the standard Liouvillean of the non-interacting open system, J is the modular conjugation and ς_ω the modular group (see Section 3.4).

Remark. The $\mathscr{L}_\alpha$'s differ from the standard Liouvillean $\mathscr{L}$ only by a term in the commutant $\mathfrak{M}'$, namely

$$\mathscr{L}_\alpha = \mathscr{L} + J(V - \varsigma_\omega^{-\mathrm{i}\bar{\alpha}}(V))J,$$

and hence also implement the dynamics, *i.e.*, $\mathrm{e}^{\mathrm{i}t\mathscr{L}_\alpha} A \mathrm{e}^{-\mathrm{i}t\mathscr{L}_\alpha} = \tau^t(A)$ for all $A \in \mathfrak{M}$.

The primary reason for introducing these α-Liouvilleans is the following representation of the entropic functionals [BBJ+25a, Proposition 3.2 and Equations (5.11)–(5.12)].

Proposition 7.1. *For all $\alpha \in \mathrm{i}\mathbb{R}$ and $t, T \in \mathbb{R}$,*

(1) $\mathfrak{F}^{\mathrm{2tm}}_{\omega,t}(\alpha) = \langle \Omega, \mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega\rangle$.

(2) $\mathfrak{F}^{\mathrm{qpsc}}_{\omega_T,t}(\alpha) = \langle \Omega, \mathrm{e}^{\mathrm{i}T\mathscr{L}_{1/2}}[D\omega_{-t} : D\omega]_\alpha \Omega\rangle = \langle \Omega, \mathrm{e}^{\mathrm{i}T\mathscr{L}_{1/2}}\mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega\rangle$.

(3) $\mathfrak{F}^{\mathrm{east}}_{\omega_T,t}(\alpha) = \langle \Omega, \mathrm{e}^{\mathrm{i}T\mathscr{L}_{1/2}}[D\omega_{-t} : D\omega]^*_{\frac{\alpha}{2}}[D\omega_{-t} : D\omega]_{\frac{\alpha}{2}}\Omega\rangle = \langle \Omega, \mathrm{e}^{\mathrm{i}T\mathscr{L}_{1/2}}\mathrm{e}^{\mathrm{i}t\widehat{\mathscr{L}}_{1/2-\alpha}}\Omega\rangle$.

The identity in (1) analytically extends to the strip $|\mathrm{Re}(\alpha - \frac{1}{2})| < \vartheta$, the one in (2) extends to the strip $\frac{1}{2} - \vartheta < \mathrm{Re}(\alpha) < \vartheta$, while the one in (3) extends to the strip $1 - 2\vartheta < \mathrm{Re}(\alpha) < 2\vartheta$. We note that the first and last strips contain the real interval $[0,1]$.

Based on this representation, the study of the limits in (6.1) is linked to the large T, then t, behavior of the semigroups generated by the α-Liouvilleans. This behavior can be studied via spectral resonance theory, as we now briefly explain. Because $\mathscr{L}_\alpha$ is a bounded perturbation of the self-adjoint operator $\mathscr{L}_{\mathrm{fr}}$, its resolvent $z \mapsto (z - \mathscr{L}_\alpha)^{-1}$ is well-defined on some lower half-plane $\mathrm{Im}(z) < -m_\alpha$. Suppose that, between suitable vectors, it has a meromorphic continuation to $\mathrm{Im}(z) < \mu_\alpha$, with $\mu_\alpha > 0$, whose only singularity is a simple pole $\mathscr{E}(\alpha)$. Under the resolvent bounds stated in [BBJ+25a, Proposition 4.1], one obtains, for $\max\{0, \mathrm{Im}\,\mathscr{E}(\alpha)\} < \gamma < \mu_\alpha$,

$$\langle \phi, \mathrm{e}^{\mathrm{i}t\mathscr{L}_\alpha}\psi\rangle = \mathrm{e}^{\mathrm{i}t\mathscr{E}(\alpha)}\langle \phi, P_\alpha \psi\rangle + O\left(\mathrm{e}^{-\gamma t}\right), \tag{7.2}$$

where $\langle \phi, P_\alpha \psi\rangle$ denotes the residue of the continued scalar resolvent. If this coefficient is nonzero and $\mathscr{E}(\alpha)$ is purely imaginary, as for the real parameters relevant below, then

$$\lim_{t\to\infty} \frac{1}{t}\log\langle \phi, \mathrm{e}^{\mathrm{i}t\mathscr{L}_\alpha}\psi\rangle = \mathrm{i}\mathscr{E}(\alpha). \tag{7.3}$$

The analysis of the large T limit is motivated by the spectral theory of NESS developed in [JP02]. When the spectral approach applies to both limits, it gives that $\boldsymbol{F}^{\mathrm{2tm}}_{\omega}(\alpha)$ and $\boldsymbol{F}^{\mathrm{qpsc}}_{\omega_\infty}(\alpha)$ exist and equal $\mathrm{i}\mathscr{E}\left(\frac{1}{2} - \alpha\right)$, while $\boldsymbol{F}^{\mathrm{east}}_{\omega_\infty}(\alpha) = \mathrm{i}\hat{\mathscr{E}}\left(\frac{1}{2} - \alpha\right)$, where $\hat{\mathscr{E}}(\alpha)$ is the corresponding pole for $\widehat{\mathscr{L}}_\alpha$. The two families are closely related: on appropriate domains, their definitions give

$$\widehat{\mathscr{L}}_\alpha = \Delta_\omega^{\alpha/2-1/4}\mathscr{L}_\alpha\Delta_\omega^{-\alpha/2+1/4}.$$

The exact equality $\mathfrak{F}^{\mathrm{east}}_{\omega,t} = \mathfrak{F}^{\mathrm{2tm}}_{\omega,t}$ identifies their leading resonances once the corresponding expansions have been established with nonzero coefficients. Together with differentiability in α and the 2TM stability result, this yields a model-independent spectral criterion for the PREF.

The difficulty of this spectral approach comes from the fact that the meromorphic continuation is typically available only on a suitable subspace: one studies the map $z \mapsto D(z-\mathscr{L}_\alpha)^{-1}D$ for some bounded operator $D \geq 0$ with dense range. The relations (7.2)–(7.3) then apply to vectors $\phi, \psi \in \mathrm{Ran}(D)$. Such continuations are obtained via spectral deformation techniques initially developed in the theory of Schrödinger operators. This parallels the Ruelle transfer operator approach to classical dynamical systems, for which one has to consider suitable spaces of functions, e.g. Hölder continuous function spaces, to unveil the so-called Ruelle–Pollicott resonances.

Finally, we mention that the combination of the large T and t limits makes this approach quite technical. In [BBJ+25a, Section 4.3] a general axiomatic scheme is proposed that allows one to derive the PREF using this spectral resonance method. As expected given the information it yields, this scheme is technically involved to apply to concrete physically relevant models. In [BBJ+26] we apply it to generalized versions of the spin–fermion model, where a finite level system is coupled to several fermionic free reservoirs with interactions which are polynomial in the annihilation/creation operators. In the next section we show how this can be used in the context of the usual spin–fermion model.

7.2 Spin Fermion (simple version)

The spin–fermion model is a concrete example of an open quantum system with the structure described in Section 3.2. In its simplest version S is a two-level system, *i.e.*, $\mathscr{K}_\mathsf{S} = \mathbb{C}^2$, with Hamiltonian $H_\mathsf{S} = \sigma_z$. Each reservoir subsystem R_j is a free Fermi gas with single particle Hilbert space $\mathfrak{h}_j$, single particle Hamiltonian h_j and initial KMS state at inverse temperature β_j, see Example 2 in Section 3.1.

For each j, the interaction of S with R_j is of the form

$$V_j = \sigma_x \otimes \varphi_j(f_j),$$

where $\varphi_j(f) = \frac{1}{\sqrt{2}}(a_j(f) + a_j^*(f))$ is the Segal field operator and $f_j \in \mathfrak{h}_j$ is a form factor. The complete interaction is $V = \lambda(V_1 + V_2)$ where $\lambda \in \mathbb{R}$ is a coupling constant.

The following technical assumptions allow one to apply the Liouvillean approach presented in the previous section.

> **(SFM1)** The single particle spaces $\mathfrak{h}_j$ are of the form $\mathfrak{h}_j = L^2(\mathbb{R}_+, \mathrm{d}s) \otimes \mathfrak{H}$ for some Hilbert space $\mathfrak{H}$ equipped with a complex conjugation $\overline{\cdot}$, and h_j is the operator of multiplication by the variable $s \in \mathbb{R}_+$.

For $j = 1, 2$ we associate to the form factor f_j the function $\widetilde{f}_j \in L^2(\mathbb{R}, \mathrm{d}s) \otimes \mathfrak{H}$ defined as

$$\widetilde{f}_j(s) = \begin{cases} f_j(s) & \text{if } s \geq 0; \\ \overline{f_j(|s|)} & \text{if } s < 0. \end{cases}$$

and assume

(SFM2) For every $r > 0$, $\widetilde{f}_j$ is in the Hardy class $H^2(r)$ of all analytic functions

$$f : \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| < r\} \to \mathfrak{H}$$

such that

$$\|f\|^2_{H^2(r)} = \sup_{|\theta|<r} \int_{\mathbb{R}} \|f(s+\mathrm{i}\theta)\|^2_{\mathfrak{H}} \mathrm{d}s < \infty,$$

and, for all $r > 0$ and all $a > 0$ one has

$$\sup_{|\theta|<r} \int_{\mathbb{R}} \mathrm{e}^{a|s|} \|\widetilde{f}_j(s+\mathrm{i}\theta)\|^2_{\mathfrak{H}} \mathrm{d}s < \infty.$$

(SFM3) $\|\tilde{f}_j(2)\|^2_{\mathfrak{H}} > 0$.

(SFM4) $f_j = \overline{f_j}$.

Assumption **(SFM1)** ensures that the reservoirs are ergodic, while **(SFM2)** provides the regularity needed for the analytic deformation of the Liouvilleans. Assumption **(SFM3)** is the Fermi golden rule condition: it requires nonzero coupling to each reservoir at the spin transition energy 2. Together these assumptions allow one to apply the spectral approach for sufficiently small nonzero coupling. Finally, **(SFM4)** ensures that the system is TRI. Note that the generic assumptions **(Reg)** and **(AnV(ϑ))** translate respectively to $f_j \in Dom(h_j)$ and $f_j \in Dom(e^{\pm\beta_j\theta h_j})$, and thus follow from **(SFM2)**.

The following result was proven in [BBJ$^+$26].

Theorem 7.2. *Suppose that* **(SFM1)**–**(SFM4)** *hold. Then:*

(1) *There exists* $\Lambda > 0$ *such that for* $0 < |\lambda| < \Lambda$, **(NESS)** *holds and the restriction* $\omega_{\infty,\mathsf{S}}$ *of the NESS* ω_∞ *to* $\mathscr{O}_\mathsf{S}$ *is faithful.*

(2) *For any* $\vartheta > 0$ *there exists* $\Lambda > 0$ *such that the PREF holds on* $]-\vartheta, 1+\vartheta[$ *for* $0 < |\lambda| < \Lambda$. *Moreover, the function*

$$]-\vartheta, 1+\vartheta[\ni \alpha \mapsto \boldsymbol{F}(\alpha)$$

is real-analytic and satisfies the symmetry $\boldsymbol{F}(\alpha) = \boldsymbol{F}(1-\alpha)$. *It is identically vanishing on* $]-\vartheta, 1+\vartheta[$ *if* $\beta_1 = \beta_2$, *and otherwise strictly convex on this interval.*

(3) *For small* λ, *one has*

$$\boldsymbol{F}(\alpha) = \lambda^2 \boldsymbol{F}_2(\alpha) + O(\lambda^3),$$

where

$$\boldsymbol{F}_2(\alpha) = -\frac{\pi}{2}\Bigg(\sum_{j=1}^{2} \|\tilde{f}_j(2)\|^2_{\mathfrak{H}} - \sqrt{\sum_{j,k=1}^{2} \left(\tanh(\beta_j)\tanh(\beta_k) + \frac{\cosh((\beta_j-\beta_k)(1-2\alpha))}{\cosh(\beta_j)\cosh(\beta_k)} \right) \|\tilde{f}_j(2)\|^2_{\mathfrak{H}} \|\tilde{f}_k(2)\|^2_{\mathfrak{H}}} \Bigg).$$

Remark. The fact that $\omega_{\infty,S}$ is faithful guarantees that the stability Theorem 5.1 can be applied to the NESS ω_∞. Moreover, $-\boldsymbol{F}'(0) = \omega_\infty(\sigma)$ is the mean steady-state entropy production rate. It is positive when $\beta_1 \neq \beta_2$ and vanishes at equilibrium.

Data availability. No data is associated with this work.

Conflict of Interest. The Authors have no COI pertaining to this work.

References

[AI74] ARAKI, H. and ION, P. D. F.: On the equivalence of KMS and Gibbs conditions for states of quantum lattice systems. Commun. Math. Phys. **35**, 1–12 (1974), [DOI:10.1007/BF01646450].

[Ara77] ARAKI, H.: Relative entropy of states of von Neumann algebras II. Publ. Res. Inst. Math. Sci. **13**, 173–192 (1977), [DOI:10.2977/prims/1195190105].

[Ara76] ARAKI, H.: Relative entropy of states of von Neumann algebras. Publ. Res. Inst. Math. Sci. **11**, 809–833 (1975/76), [DOI:10.2977/prims/1195191148].

[BBJ+24] BENOIST, T., BRUNEAU, L., JAKŠIĆ, V., PANATI, A. and PILLET, C.-A.: A note on two-times measurement entropy production and modular theory. Lett. Math. Phys. **114**, 32 (2024), [DOI:10.1007/s11005-024-01777-0].

[BBJ+25a] _____: Entropic fluctuations in statistical mechanics II. Quantum dynamical systems. Commun. Math. Phys. **406**, 201 (2025), [DOI:10.1007/s00220-025-05360-z].

[BBJ+25b] _____: On the thermodynamic limit of two-times measurement entropy production. Rev. Math. Phys. (2025), [DOI:10.1142/S0129055X24610063].

[BBJ+26] _____: Entropic fluctuation theorems for the spin-fermion model. J. Math. Phys. **67**, 022101 (2026), [DOI:10.1063/5.0311182].

[BK77] BOCHKOV, G. N. and KUZOVLEV, Y. E.: General theory of thermal fluctuations in nonlinear systems. Sov. Phys. JETP **45**, 125–130 (1977).

[BK79] _____: Fluctuation-dissipation relations for nonequilibrium processes in open systems. Sov. Phys. JEPT **49**, 543–551 (1979).

[BK81] _____: Nonlinear fluctuation-dissipation relations and stochastic models in nonequilibrium thermodynamics II. Kinetic potential and variational principles for nonlinear irreversible processes. Physica A: Statistical Mechanics and its Applications **106**, 480–520 (1981), [DOI:10.1016/0378-4371(81)90123-0].

[BR81] BRATTELI, O. and ROBINSON, D. W.: *Operator Algebras and Quantum Statistical Mechanics II.* Texts and Monographs in Physics, Springer, New York, 1981, [DOI:10.1007/978-3-662-03444-6].

[BR87] ———: *Operator Algebras and Quantum Statistical Mechanics I.* Texts and Monographs in Physics, Springer, New York, 1987, [DOI:10.1007/978-3-662-02520-8].

[CJPS26] CUNEO, N., JAKŠIĆ, V., PILLET, C.-A. and SHIRIKYAN, A.: *What is a Fluctuation Theorem.* Springer Briefs in Mathematical Physics, vol. 54, Springer, 2026, [DOI:10.1007/978-3-032-02095-6].

[Cro98] CROOKS, G. E.: Nonequilibrium measurements of free energy differences for microscopically reversible Markovian systems. J. Stat. Phys. **90**, 1481–1487 (1998), [DOI:10.1023/A:1023208217925].

[DCSCR18] DE CHIARA, G., SOLINAS, P., CERISOLA, F. and RONCAGLIA, A. J.: Ancilla-assisted measurement of quantum work. In *Thermodynamics in the quantum regime* (Binder, F., Correa, L. A., Gogolin, C., Anders, J. and Adesso, G., eds.), Fundamental Theories of Physics, no. 195, Springer, p. 337–362, 2018, [DOI:10.1007/978-3-319-99046-0_14].

[DJP03] DEREZIŃSKI, J., JAKŠIĆ, V. and PILLET, C.-A.: Perturbation theory of W^*-dynamics, Liouvilleans and KMS-states. Rev. Math. Phys. **15**, 447–489 (2003), [DOI:10.1142/S0129055X03001679].

[DR09] DE ROECK, W.: Large deviation generating function for currents in the Pauli-Fierz model. Rev. Math. Phys. **21**, 549–585 (2009), [DOI:10.1142/S0129055X09003694].

[DZ00] DEMBO, A. and ZEITOUNI, O.: *Large Deviations Techniques and Applications.* Springer, Berlin, 2000, [DOI:10.1007/978-3-642-03311-7].

[ECM93] EVANS, D. J., COHEN, E. G. D. and MORRISS, G. P.: Probability of second law violations in shearing steady states. Phys. Rev. Lett. **71**, 2401–2404 (1993), [DOI:10.1103/PhysRevLett.71.2401].

[Ell06] ELLIS, R. S.: *Entropy, Large Deviations, and Statistical Mechanics.* Classics in Mathematics, Springer, Berlin, 2006, [DOI:10.1007/3-540-29060-5].

[ES94] EVANS, D. J. and SEARLES, D. J.: Equilibrium microstates which generate second law violating steady states. Phys. Rev. E **50**, 1645–1648 (1994), [DOI:10.1103/PhysRevE.50.1645].

[Gal97] GALLAVOTTI, G.: Dynamical ensembles equivalence in fluid mechanics. Phys. D **105**, 163–184 (1997), [DOI:10.1016/S0167-2789(96)00236-9].

[Haa96] HAAG, R.: *Local Quantum Physics.* Second ed., Texts and Monographs in Physics, Springer, Berlin, 1996, [DOI:10.1007/978-3-642-61458-3].

[Jar97] JARZYNSKI, C.: Nonequilibrium equality for free energy differences. Phys. Rev. Lett. **78**, 2690–2693 (1997), [DOI:10.1103/PhysRevLett.78.2690].

[JOPP10] JAKŠIĆ, V., OGATA, Y., PAUTRAT, Y. and PILLET, C.-A.: Entropic fluctuations in quantum statistical mechanics—An introduction. In *Quantum Theory from Small to Large Scales* (Fröhlich, J., Salmhofer, M., de Roeck, W., Mastropietro, V. and Cugliandolo, L., eds.), Lecture Notes of the Les Houches Summer School, vol. 95, Oxford University Press, Oxford, p. 213–410, 2010, [DOI:10.1093/acprof:oso/9780199652495.003.0004].

[JOPS12] JAKŠIĆ, V., OGATA, Y., PILLET, C.-A. and SEIRINGER, R.: Quantum hypothesis testing and non-equilibrium statistical mechanics. Rev. Math. Phys. **24**, 1230002, 67 (2012), [DOI:10.1142/S0129055X12300026].

[JP96a] JAKŠIĆ, V. and PILLET, C.-A.: On a model for quantum friction. II. Fermi's golden rule and dynamics at positive temperature. Commun. Math. Phys. **176**, 619–644 (1996), [DOI:10.1007/bf02099252].

[JP96b] JAKŠIĆ, V. and PILLET, C.-A.: On a model for quantum friction. III. Ergodic properties of the spin-boson system. Commun. Math. Phys. **178**, 627–651 (1996), [DOI:10.1007/bf02108818].

[JP01] JAKŠIĆ, V. and PILLET, C.-A.: On entropy production in quantum statistical mechanics. Commun. Math. Phys. **217**, 285–293 (2001), [DOI:10.1007/s002200000339].

[JP02] ———: Non-equilibrium steady states of finite quantum systems coupled to thermal reservoirs. Commun. Math. Phys. **226**, 131–162 (2002), [DOI:10.1007/s002200200602].

[JPRB11] JAKŠIĆ, V., PILLET, C.-A. and REY-BELLET, L.: Entropic fluctuations in statistical mechanics I. Classical dynamical systems. Nonlinearity **24**, 699–763 (2011), [DOI:10.1088/0951-7715/24/3/003].

[Kur00] KURCHAN, J.: A quantum fluctuation theorem. Unpublished, 2000, [DOI:10.48550/arXiv.cond-mat/0007360].

[OP93] OHYA, M. and PETZ, D.: *Quantum Entropy and its Use*. Texts and Monographs in Physics, Springer, Berlin, 1993, [DOI:10.1007/978-3-642-57997-4].

[Str81] STRĂTILĂ, C.: *Modular Theory in Operator Algebras*. Editura Academiei Republicii Socialiste România, Bucharest; Abacus Press, Tunbridge Wells, 1981.

[Tas00] TASAKI, H.: Jarzynski relations for quantum systems and some applications. Unpublished, 2000, [DOI:10.48550/arXiv.cond-mat/0009244].

[TM03] TASAKI, S. and MATSUI, T.: Fluctuation theorem, nonequilibrium steady states and MacLennan-Zubarev ensembles of a class of large quantum systems. In *Fundamental Aspects of Quantum Physics (Tokyo, 2001)*, QP–PQ: Quantum Probab. White Noise Anal., vol. 17, World Sci. Publ., River Edge, NJ, p. 100–119, 2003, [DOI:10.1142/9789812704412_0006].