Entropic Fluctuations in Statistical Mechanics II. Quantum Dynamical Systems

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Abstract. The celebrated Evans–Searles, respectively Gallavotti–Cohen, fluctuation theorem concerns certain universal statistical features of the entropy production rate of a classical system in a transient, respectively steady, state. In this paper, we consider and compare several possible extensions of these fluctuation theorems to quantum systems. In addition to the direct two-time measurement approach whose discussion is based on [LMP 114:32 (2024)], we discuss a variant where measurements are performed indirectly on an auxiliary system called ancilla, and which allows to retrieve non-trivial statistical information using ancilla state tomography. We also show that modular theory provides a way to extend the classical notion of phase space contraction rate to the quantum domain, which leads to a third extension of the fluctuation theorems. We further discuss the quantum version of the principle of regular entropic fluctuations, introduced in the classical context in [Nonlinearity 24, 699 (2011)]. Finally, we relate the statistical properties of these various notions of entropy production to spectral resonances of quantum transfer operators. The obtained results shed a new light on the nature of entropic fluctuations in quantum statistical mechanics.

Contents

1 Introduction

This paper is a sequel to [JPRB11], which was centered on two celebrated results of classical non-equilibrium statistical mechanics: the Evans–Searles and Gallavotti–Cohen *Fluctuation Theorems* [ES94, GC95a, GC95b]. The main contribution of [JPRB11] was to realize that these two physically and mathematically distinct results were intimately related by a so-called *Principle of Regular Entropic Fluctuations* (abbreviated PREF in the following). The main subject of the present work is the extension of these two fluctuation theorems to the quantum domain, and a discussion of the PREF in this context.

From the perspective of classical statistical mechanics, both fluctuation theorems involve a *Large Deviation Principle* (LDP) for the time-averaged entropy production observable (*i.e.,* the phase space contraction rate) in the large time limit. In the Evans–Searles case, the statistics is induced by a reference, initial, state of the system[1] that evolves towards a non-equilibrium steady state (NESS) in the large time limit. The Evans–Searles fluctuation theorem deals with a transient process, and for this reason is often called *Transient Fluctuation Theorem.* In the Gallavotti–Cohen case, the statistics is directly induced by the NESS, and thus pertains to a stationary process. Except in thermodynamically trivial situations, the reference state and the NESS are mutually singular measures, and hence both physically and mathematically the two fluctuation theorems are very different statements.

The starting point of the PREF is that, in spite of this difference, in all known non-trivial examples where both theorems hold the respective LDP rate functions are identical, and that this identity is equivalent to an exchange of limits in the derivation of the LDP. The justification of this exchange of limits is typically a deep dynamical problem whose validity was raised in [JPRB11] to a principle: the PREF.

The classical fluctuation theorems come with equally celebrated *Fluctuation Relations.* If the system is *Time-Reversal Invariant* (abbreviated TRI), the Evans–Searles fluctuation relation asserts that the rate function $\mathbb{I}$ governing the large deviations of entropy production in the reference state satisfies

$$\mathbb{I}(-s) = \mathbb{I}(s) + s \tag{1.1}$$

on its domain. In the non-equilibrium steady state, according to the Gallavotti–Cohen fluctuation relation, these large deviations are described by a rate function $\mathbb{I}_+$ satisfying the same relation

$$\mathbb{I}_+(-s) = \mathbb{I}_+(s) + s \tag{1.2}$$

[1] A state of a classical system is a Borel probability measure on its phase space.

on its domain. The general mechanism behind (1.1) is simple and this relation is an immediate consequence of the LDP and time-reversal invariance; see [CJPS17, Proposition 1.4], Proposition 2.6 below, and the comment after Theorem 2.14. In contrast, the only known general mechanism behind (1.2) is the PREF that gives the equality $\mathbb{I} = \mathbb{I}_+$, and so (1.2) is forced by (1.1). For further discussion of these points we refer the reader to [JPRB11].

Quantum fluctuation relations first appeared in the works [Kur00, Tas00, TM03]. The LDP aspect was not discussed in these works, and fluctuation relations were considered only for finite times (see Theorem 2.3(4) below). In [Kur00, Tas00], they were formulated in the context of finite quantum systems[2] and the main object of interest was the distribution of a random variable expressing the change of entropy (or entropy production) in a two-time measurement protocol. We will refer to this random variable as the *Two-Times Measurement Entropy Production* (abbreviated 2TMEP). In [TM03], the proposed fluctuation relation was formulated in the general algebraic framework of quantum dynamical systems, and was phrased in terms of the spectral measure of a suitable relative modular operator that provided a non-commutative extension of the classical phase contraction along the state trajectory. It turned out that these two proposals are identical and that they provide a natural basis for a quantum Evans–Searles fluctuation theorem and relation; see [JOPP10] for a pedagogical introduction to this topic and [DR09] for an early work on the subject. The quantum Evans–Searles fluctuation theorem and relation will be also briefly reviewed in Section 2.5 below.

For a long time, a starting point for a quantum Gallavotti–Cohen fluctuation theorem and quantum PREF in the same algebraic setting of quantum dynamical systems was missing. The two main obstacles were:

(a) The limiting procedure that would allow to define the 2TMEP with respect to NESS was unknown.

(b) Any potential resolution of (a) faces, in a more severe way, a problem already present in the Evans–Searles case: the 2TMEP of large quantum systems is not, even in principle, directly experimentally accessible.

The point (a) was recently resolved in [BBJ⁺23, BBJ⁺24b], and we will quickly review the respective results in Section 2.3. A perhaps surprising consequence of the results of [BBJ⁺23] is that the emerging quantum Gallavotti–Cohen fluctuation theorem will come with a degree of rigidity that makes it an immediate consequence of the quantum Evans–Searles fluctuation theorem. The same applies to the equality of the respective rate functions. This purely quantum phenomenon, due to the dominating decoherence effect of the first measurement, essentially trivializes the resulting quantum PREF.

In this paper the point (b) is resolved by the introduction of the *Entropic Ancilla State Tomography* (abbreviated EAST) which is, in principle, experimentally accessible. This resolves the observability problem of the 2TMEP with respect to *any state*, including the NESS, and at the same time allows for the introduction of a non-trivial quantum PREF that parallels the classical one in its nature. This comes at the cost of a definition that does not ensure an interpretation in terms of an entropy production random variable.

[2] By *finite quantum system* we mean a system whose observables are linear operators acting on a finite-dimensional Hilbert space.

The introduction of EAST and of the resulting new quantum PREF (which we will call *strong)* is the first main set of results of this paper. The second one is a characterization of this strong quantum PREF in terms of spectral resonances of a quantum transfer operator which we develop axiomatically. In the follow-up work [BBJ+24a], building on the techniques introduced in [JP96a, JP96b, JP02b], we illustrate this spectral theory of the strong PREF on the example of the Spin–Fermion model which is one of the paradigmatic models of open quantum systems (see the seminal works [Dav74, SL78]).

We finish this introduction with two general remarks.

- The presentation we have chosen does not follow the most general possible route, and we have attempted to strike a balance between emphasizing the mathematical structure and focusing on physically relevant settings like open quantum systems. The results formulated and proven in the specific setting of open quantum systems can be easily generalized. We leave these generalizations to the interested readers.

- The main technical tool of this work is Araki's relative modular operator theory [Ara76, Ara77, AM82], which is perfectly suited for non-commutative/quantum extensions of entropic notions in classical dynamical systems and probability theory. This theory is discussed and reviewed in many places in the literature[3], and for definiteness we adopt, as we did in [BBJ+23], the notation and conventions of [JOPS12, Section 6].

The paper is organized as follows. In Sections 2.1–2.2, we briefly review the notation and definitions of [BBJ+23] which will be reused in the sequel. For reason of space we can not enter into too many details, and we strongly encourage the reader to consult this reference. In Section 2.3 we introduce the two-time measurement entropy production and review the main result of [BBJ+23]. In Section 2.4 we introduce the entropic ancilla state tomography. The quantum Evans–Searles and Gallavotti–Cohen fluctuation theorems are discussed in Section 2.5, where we also introduce the quantum (weak and strong) principle of regular entropic fluctuations. The classical theory of entropic fluctuations based on the entropy production observable/phase space contraction rate is reviewed in Section 2.6. The quantum entropy production observable/phase space contraction rate is discussed in Section 2.7. The classical and quantum case are compared in Section 2.8. The quantum transfer operators are introduced in Section 3. The associated spectral resonance theory of the strong quantum PREF is described in Section 4. The proofs are given in Section 5.

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[3] See [JOPP10] for a pedagogical introduction to this topic.

2 Quantum dynamical systems

2.1 C^*-dynamical systems and their modular structure

In this and the following section we briefly review the mathematical description of quantum dynamical systems that will be used in the present work, and describe the structure of thermally driven open quantum systems which will serve as our paradigmatic examples. We follow the conventions and notations of [BBJ+23] and refer the reader to this paper for further details and references.

A C^**-quantum dynamical system* is a triple $(\mathcal{O},\tau,\omega)$ where:

- $\mathcal{O}$ is a C^*-algebra with a unit $\mathbb{1}$. Its elements $A\in\mathcal{O}$ describe the observables of the system.
- $\mathbb{R}\ni t\mapsto\tau^t$ is a C^*-dynamics, *i.e.,* a strongly continuous group of $*$-automorphisms of $\mathcal{O}$. It describes the Heisenberg time-evolution $A_t=\tau^t(A)$ of the system observables. The infinitesimal generator of a C^*-dynamics τ is a possibly unbounded $*$-derivation δ of $\mathcal{O}$. We use the convention $\tau^t=\mathrm{e}^{t\delta}$.
- ω is a state on $\mathcal{O}$, *i.e.,* an element of the closed convex subset $\mathcal{S}_{\mathcal{O}}$ of the dual space $\mathcal{O}^*$ consisting of linear functionals $\omega\in\mathcal{O}^*$ such that $\omega(A^*A)\geq 0$ for all $A\in\mathcal{O}$, and $\omega(\mathbb{1})=1$. We shall always equip $\mathcal{S}_{\mathcal{O}}$ with the weak*-topology. The number $\omega(A)$ is the quantum expectation value of the observable $A\in\mathcal{O}$, when the system is in the state ω. A state $\omega\in\mathcal{S}_{\mathcal{O}}$ is faithful whenever $\omega(A^*A)=0$ implies $A=0$. States evolve according to the Schrödinger picture $\omega_t=\omega\circ\tau^t$, so that $\omega_t(A)=\omega(A_t)$. ω is called τ-invariant whenever $\omega_t=\omega$ for all $t\in\mathbb{R}$.
- Thermal equilibrium of the system at inverse temperature $\beta\in\mathbb{R}^*$ is described by a state ω satisfying the (τ,β)-KMS boundary condition: for any $A,B\in\mathcal{O}$ the function $F_{A,B}(t)=\omega(A\tau^t(B))$ has an analytic extension to the complex strip $\{z\mid 0<\mathrm{sign}(\beta)\,\mathrm{Im}\,z<|\beta|\}$, which is bounded and continuous on its closure, and satisfies

  $$F_{A,B}(t+\mathrm{i}\beta)=\omega(\tau^t(B)A)$$

  for all $t\in\mathbb{R}$. Such states are said to be (τ,β)-KMS, and are τ-invariant.

Given such a C^*-quantum dynamical system, the GNS representation produces a triple $(\mathcal{H},\pi,\Omega)$ where $\mathcal{H}$ is a Hilbert space, $\pi:\mathcal{O}\to\mathcal{B}(\mathcal{H})$ a $*$-morphism from $\mathcal{O}$ to the bounded linear operators on $\mathcal{H}$, and $\Omega\in\mathcal{H}$ a unit vector such that $\omega(A)=\langle\Omega,\pi(A)\Omega\rangle$ for all $A\in\mathcal{O}$. Moreover, Ω is cyclic for $\pi(\mathcal{O})$, *i.e.,* $\pi(\mathcal{O})\Omega$ is a dense subspace of $\mathcal{H}$. The weak closure of the set $\pi(\mathcal{O})\subset\mathcal{B}(\mathcal{H})$ coincides with its bicommutant[4] $\mathfrak{M}=\pi(\mathcal{O})''$, and is the *enveloping von Neumann algebra* of $\mathcal{O}$ induced by ω. The state ω clearly extends to a state on $\mathfrak{M}$ which we denote by the same symbol. A density matrix ρ on $\mathcal{H}$ defines a state on $\mathfrak{M}$ by the familiar quantum mechanical rule $\mathfrak{M}\ni A\mapsto\mathrm{tr}(\rho A)$. Such states on $\mathfrak{M}$ are called *normal,* and their restriction to $\pi(\mathcal{O})$ induce states on $\mathcal{O}$ which are called ω-*normal.* The *folium* of ω is the set $\mathcal{N}$ of all ω-normal states on $\mathcal{O}$.

[4] We use the standard notation $\mathcal{A}'=\{B\in\mathcal{B}(\mathcal{H})\mid[A,B]=0 \text{ for all } A\in\mathcal{A}\}$ for the commutant of a subset $\mathcal{A}\subset\mathcal{B}(\mathcal{H})$.

We will always assume that the reference state ω is *modular*, namely that there exists a C^*-dynamics ς_ω on $\mathscr{O}$ such that ω is a $(\varsigma_\omega,-1)$-KMS state. If ω is modular, we will say that $(\mathscr{O},\tau,\omega)$ is a modular C^*-quantum dynamical system. The C^*-dynamics ς_ω is the *modular group* of ω, and is unique when it exists. The extension to $\mathfrak{M}$ of a modular state is faithful, *i.e.,* the map $\mathfrak{M}\ni A\mapsto A\Omega$ is injective. It follows from the Tomita–Takesaki theory, see *e.g.* [DJP03, JOPS12] and references therein, that the GNS Hilbert space $\mathscr{H}$ comes with a modular structure:

- A positive operator Δ_ω, called the *modular operator* of ω, which implements the modular group on $\pi(\mathscr{O})$,
$$\pi(\varsigma_\omega^\theta(A))=\Delta_\omega^{\mathrm{i}\theta}\pi(A)\Delta_\omega^{-\mathrm{i}\theta},$$
and thus allows us to extend this group to a W^*-dynamics on $\mathfrak{M}$, which we denote by the same symbol.
- The *modular conjugation* J, an anti-unitary involution satisfying
$$\mathfrak{M}'=J\mathfrak{M}J.$$
- The *natural cone*, a self-dual cone $\mathscr{H}_+\subset\mathscr{H}$ such that $\Delta_\omega^{\mathrm{i}\theta}\mathscr{H}_+=\mathscr{H}_+$ for all $\theta\in\mathbb{R}$ and $J\Psi=\Psi$ for all $\Psi\in\mathscr{H}_+$. Every state $\mu\in\mathscr{N}$ has a unique vector representative $\Omega_\mu\in\mathscr{H}_+$ satisfying
$$\mu(A)=\langle\Omega_\mu,\pi(A)\Omega_\mu\rangle$$
for all $A\in\mathscr{O}$. Moreover, Ω_μ is cyclic for $\pi(\mathscr{O})$ iff μ extends to a faithful state on $\mathfrak{M}$, in which case $(\mathscr{H},\pi,\Omega_\mu)$ is a GNS representation induced by μ.

A W^*-dynamics on the enveloping algebra $\mathfrak{M}$ is a group $\mathbb{R}\ni t\mapsto\varsigma^t$ of $*$-automorphisms of $\mathfrak{M}$ such that the function $\mathbb{R}\ni t\mapsto\mu\circ\varsigma^t(A)$ is continuous for all $\mu\in\mathscr{N}$ and $A\in\mathfrak{M}$. Given such a W^*-dynamics ς, there exists a unique self-adjoint operator $\mathscr{L}$ on the GNS space $\mathscr{H}$ satisfying
$$\mathrm{e}^{-\mathrm{i}t\mathscr{L}}\mathscr{H}_+=\mathscr{H}_+,\qquad\varsigma^t(A)=\mathrm{e}^{\mathrm{i}t\mathscr{L}}A\mathrm{e}^{-\mathrm{i}t\mathscr{L}},$$
for all $t\in\mathbb{R}$ and $A\in\mathfrak{M}$. In particular, $\mathrm{e}^{-\mathrm{i}t\mathscr{L}}\Omega_\mu$ is the vector representative of $\mu\circ\varsigma^t$ in the natural cone. The operator $\mathscr{L}$ is often called the *standard Liouvillean*. Note that $\mathscr{L}_\omega=\log\Delta_\omega$ is the standard Liouvillean of the modular group ς_ω.

The basic modular structure induced by the modular state ω is complemented with:

- To any pair (μ,ν) of elements of $\mathscr{N}$ is associated another positive operator $\Delta_{\mu|\nu}$ called *relative modular operator*. We will consider it only in the case where both μ and ν are faithful on $\mathfrak{M}$. Then $\mathfrak{M}\Omega_\nu$ is a core for $\Delta_{\mu|\nu}^{1/2}$ and one has
$$J\Delta_{\mu|\nu}^{1/2}A\Omega_\nu=A^*\Omega_\mu$$
for all $A\in\mathfrak{M}$. One checks, see *e.g.* [JOPP10], that
$$\mathrm{e}^{-\mathrm{i}t\mathscr{L}}\Delta_{\mu|\nu}\mathrm{e}^{\mathrm{i}t\mathscr{L}}=\Delta_{\mu\circ\tau^t|\nu\circ\tau^t}\tag{2.1}$$
where $\mathscr{L}$ is the standard Liouvillean of the W^*-dynamics τ. We note also that $\Delta_{\mu|\mu}=\Delta_\mu$, the modular operator of μ, and that for $\theta\in\mathbb{R}$ and $A\in\mathfrak{M}$ one has $\Delta_{\mu|\nu}^{\mathrm{i}\theta}A\Delta_{\mu|\nu}^{-\mathrm{i}\theta}=\Delta_\mu^{\mathrm{i}\theta}A\Delta_\mu^{-\mathrm{i}\theta}=\varsigma_\mu^\theta(A)$.

- The Araki–Connes cocycle of the pair (μ,ν) is the strongly continuous one parameter family of unitaries in $\mathfrak{M}$ given by[5]

$$[D\mu:D\nu]_\alpha = \Delta^{\alpha}_{\mu|\nu}\Delta^{-\alpha}_{\nu}, \qquad (\alpha \in \mathrm{i}\mathbb{R}). \tag{2.2}$$

  This family satisfies the multiplicative cocycle relation [AM82, Appendix C]

$$[D\mu:D\nu]_{\alpha+\beta} = [D\mu:D\nu]_\alpha \varsigma_\nu^{-\mathrm{i}\alpha}([D\mu:D\nu]_\beta), \tag{2.3}$$

  and the chain rule

$$[D\mu:D\nu]_\alpha [D\nu:D\omega]_\alpha = [D\mu:D\omega]_\alpha \tag{2.4}$$

  for $\alpha,\beta \in \mathrm{i}\mathbb{R}$ and $\mu,\nu,\omega \in \mathcal{N}$. It also intertwines the modular groups of μ and ν,

$$\varsigma_\mu^\theta(A)[D\mu:D\nu]_{\mathrm{i}\theta} = [D\mu:D\nu]_{\mathrm{i}\theta}\varsigma_\nu^\theta(A).$$

- Whenever μ and ν are faithful on $\mathfrak{M}$, their relative entropy is defined by

$$\mathrm{Ent}(\nu|\mu) = \langle \Omega_\nu, \log \Delta_{\mu|\nu}\Omega_\nu\rangle. \tag{2.5}$$

  It satisfies $\mathrm{Ent}(\nu|\mu) \le 0$ with equality if and only if $\mu = \nu$.

In the algebraic framework, a time-reversal of $(\mathcal{O},\tau)$ is an anti-linear involutive $*$-automorphism Θ of $\mathcal{O}$ such that $\Theta\circ\tau^t = \tau^{-t}\circ\Theta$ for all $t \in \mathbb{R}$. The quantum dynamical system $(\mathcal{O},\tau,\omega)$ is called *time-reversal invariant* (TRI for short) whenever such a Θ exists and satisfies $\omega\circ\Theta(A) = \omega(A^*)$ for all $A \in \mathcal{O}$.

Without further mention, all the C^*-quantum dynamical systems $(\mathcal{O},\tau,\omega)$ considered in this paper are assumed to satisfy the two basic regularity assumptions of [BBJ+23]:

(Reg1) The family $\{\pi\circ\tau^t \mid t \in \mathbb{R}\}$ extends to a W^*-dynamics on $\mathfrak{M}$ which we again denote by τ. We will denote by $\mathscr{L}$ its standard Liouvillean.

Note that, under this assumption, $\omega_t \in \mathcal{N}$ for any $t \in \mathbb{R}$.

(Reg2) For all $t \in \mathbb{R}$ and $\alpha \in \mathrm{i}\mathbb{R}$,

$$[D\omega_t : D\omega]_\alpha \in \pi(\mathcal{O}).$$

Whenever the meaning is clear within the context we denote $\pi^{-1}([D\omega_t : D\omega]_\alpha) \in \mathcal{O}$ by $[D\omega_t : D\omega]_\alpha$.

[5] Here, we depart from the traditional convention, denoting by $[D\mu:D\nu]_{\mathrm{i}t}$ what is usually written $[D\mu:D\nu]_t$.

2.2 Open quantum systems

An open quantum system is a small system S, described by a finite-dimensional Hilbert space $\mathcal{H}_\mathsf{S}$, coupled to M thermal reservoirs $\mathsf{R}_1,\ldots,\mathsf{R}_M$. In the algebraic framework, observables of the small system are elements of the finite-dimensional C^*-algebra $\mathcal{O}_\mathsf{S}=\mathcal{B}(\mathcal{H}_\mathsf{S})$. The dynamics τ_S is generated by a self-adjoint Hamiltonian $H_\mathsf{S}\in\mathcal{O}_\mathsf{S}$,

$$\tau_\mathsf{S}^t(A)=\mathrm{e}^{\mathrm{i}tH_\mathsf{S}}A\mathrm{e}^{-\mathrm{i}tH_\mathsf{S}}. \tag{2.6}$$

Each reservoir R_j is described by a C^*-quantum dynamical system $(\mathcal{O}_j,\tau_j,\omega_j)$, where ω_j is a (τ_j,β_j)-KMS state for some $\beta_j>0$. We denote by δ_j the generator of τ_j. The joint system $\mathsf{S}+\mathsf{R}_1+\cdots+\mathsf{R}_M$ is described by the C^*-algebra

$$\mathcal{O}=\mathcal{O}_\mathsf{S}\otimes\mathcal{O}_\mathsf{R}=\mathcal{O}_\mathsf{S}\otimes\left(\bigotimes_{j=1}^M\mathcal{O}_j\right).$$

The reference state of the joint system is the product state

$$\omega=\omega_\mathsf{S}\otimes\omega_\mathsf{R}=\omega_\mathsf{S}\otimes\left(\bigotimes_{j=1}^M\omega_j\right), \tag{2.7}$$

where ω_S is an arbitrary τ_S-invariant faithful state on $\mathcal{O}_\mathsf{S}$. The state ω is modular and its modular group ς_ω is generated by the $*$-derivation[6]

$$\delta_\omega=\delta_{\omega_\mathsf{S}}+\delta_{\omega_\mathsf{R}}=\mathrm{i}[\log\omega_\mathsf{S},\,\cdot\,]-\sum_{j=1}^M\beta_j\delta_j.$$

The decoupled or free joint dynamics

$$\tau_{\mathrm{fr}}^t=\tau_\mathsf{S}^t\otimes\tau_\mathsf{R}^t=\tau_\mathsf{S}^t\otimes\left(\bigotimes_{j=1}^M\tau_j^t\right)$$

is generated by

$$\delta_{\mathrm{fr}}=\mathrm{i}[H_\mathsf{S},\,\cdot\,]+\sum_{j=1}^M\delta_j,$$

and commutes with the modular group. Note that since $\varsigma_{\omega_j}^\theta=\tau_j^{-\beta_j\theta}$, τ_j satisfies Assumption **(Reg1)**, and that its standard Liouvillean is $\mathcal{L}_j=-\beta_j^{-1}\log\Delta_{\omega_j}$. One easily infers that τ_{fr} also satisfies Assumption **(Reg1)**, with the standard Liouvillean

$$\mathcal{L}_{\mathrm{fr}}=\sum_{j=1}^M\mathcal{L}_j+H_\mathsf{S}-JH_\mathsf{S}J.$$

The coupling between the small system and the reservoirs is described by a self-adjoint element $V\in\mathcal{O}$ of the form

$$V=\sum_{j=1}^M V_j,\qquad V_j=V_j^*\in\mathcal{O}_\mathsf{S}\otimes\mathcal{O}_j.$$

[6]Whenever the meaning is clear within the context we write A for $A\otimes\mathbb{1}$ and $\mathbb{1}\otimes A$, δ for $\delta\otimes\mathrm{Id}$ and $\mathrm{Id}\otimes\delta$, etc.

The coupled joint system $\mathsf{S}+\mathsf{R}_1+\cdots+\mathsf{R}_M$ is thus described by the C^*-quantum dynamical system $(\mathcal{O},\tau,\omega)$, where $\tau^t=\mathrm{e}^{t\delta}$ with

$$\delta=\delta_{\mathrm{fr}}+\mathrm{i}[V,\cdot\,].$$

Invoking time-dependent perturbation theory, the dynamics τ satisfies Assumption **(Reg1)**. Its standard Liouvillean is, see *e.g.* [DJ03] and references therein,

$$\mathscr{L}=\mathscr{L}_{\mathrm{fr}}+V-JVJ. \tag{2.8}$$

We note that the self-adjoint operator $\mathscr{L}_{\mathrm{fr}}+V$ also implements the coupled dynamics τ on $\mathfrak{M}$. However, it fails to preserve the natural cone and is sometimes called semi-standard Liouvillean associated to the local perturbation V.

By [BBJ+24b, Lemma 2.4], Assumption **(Reg2)** is also satisfied if $V\in\mathrm{Dom}(\delta_\omega)$ which is equivalent to $V_j\in\mathrm{Dom}(\delta_j)$ for all $j\in\{1,\dots,M\}$.

We conclude this section by recalling the definition of Nonequilibrium Steady State (NESS). This concept was originally introduced in [Rue00], and was studied in a number of follow-up works; an incomplete list of references is [HA00, Pil01, JP02a, JP02b, Rue02, AP03, FMU03, MO03, TM03, Oga04, TM05, AJPP06, JOP06b, JOP06a, JOP06c, JKP06, Tas06, AJPP07, JOP07, JP07, MMS07a, MMS07b, AS07, JOPP10, JLP13]. The NESSs of the dynamical system $(\mathcal{O},\tau,\omega)$ are defined as the weak*-limit points of the net

$$\left\{\frac{1}{T}\int_0^T\omega_t\mathrm{d}t\,\middle|\,T>0\right\} \tag{2.9}$$

as $T\to\infty$. The set of NESSs is always non-empty and any NESS is τ-invariant.

2.3 Two-time measurement entropy production

Consider a finite quantum system with Hamiltonian H. Let ω and ν be two faithful states of this system. To the first one we associate the *entropy observable* $S=-\log\omega$. The second state ν describes the state of the system at the instant t_i of a first measurement of S. After this first measurement, whose outcome we denote by $s_i\in\mathrm{sp}(S)$, the system evolves according to (2.6). At a later time t_f a second measurement of S is performed with outcome s_f. The increment $s=s_f-s_i$ is interpreted as the entropy produced in system in the time period $t=t_f-t_i$. As argued in [BBJ+23], the characteristic function of the law $Q_{\nu,t}$ of the random variable s relates to the modular structure of the system according to

$$\int_{\mathbb{R}}\mathrm{e}^{-\alpha s}\mathrm{d}Q_{\nu,t}(s)=\lim_{R\to\infty}\frac{1}{R}\int_0^R\nu\circ\varsigma_\omega^\theta([D\omega_{-t}:D\omega]_\alpha)\,\mathrm{d}\theta,\qquad(\alpha\in\mathrm{i}\mathbb{R}).$$

The interpretation of s as an entropy can be motivated by considering the case of an open system with finite reservoirs, where the state ω is given by (2.7) with

$$\omega_{\mathsf{S}}=\frac{\mathbb{1}}{\mathrm{tr}(\mathbb{1})},\qquad\omega_j=\frac{\mathrm{e}^{-\beta_jH_j}}{\mathrm{tr}(\mathrm{e}^{-\beta_jH_j})},$$

and where H_j denotes the Hamiltonian of the j^{th} reservoir. Then $S=\sum_j\beta_jH_j$ and hence

$$s=\sum_{j=1}^M\beta_j\Delta E_j,$$

where ΔE_j is the measured change in energy of the j^{th} reservoir. Thus, s can be interpreted as the entropy dumped in the reservoirs during the two-time measurement process.

In the following, we consider the general setting of a C^*-quantum dynamical system $(\mathcal{O}, \tau, \omega)$ with modular reference state ω and satisfying the two basic regularity assumptions **(Reg1)**, **(Reg2)**.

By [BBJ⁺23, Theorem 1.3], for all $\nu \in \mathcal{N}$, $t \in \mathbb{R}$ and $\alpha \in \mathrm{i}\mathbb{R}$, the limit

$$\mathfrak{F}_{\nu,t}^{2\mathrm{tm}}(\alpha) := \lim_{R\to\infty} \frac{1}{R} \int_0^R \nu \circ \varsigma_\omega^\theta \left([D\omega_{-t} : D\omega]_\alpha\right) \mathrm{d}\theta \tag{2.10}$$

exists, and there is unique Borel probability measure $Q_{\nu,t}^{2\mathrm{tm}}$ on $\mathbb{R}$ such that

$$\mathfrak{F}_{\nu,t}^{2\mathrm{tm}}(\alpha) = \int_{\mathbb{R}} \mathrm{e}^{-\alpha s} \mathrm{d}Q_{\nu,t}^{2\mathrm{tm}}(s).$$

Moreover, one also has that

$$\mathfrak{F}_{\nu,t}^{2\mathrm{tm}}(\alpha) = \lim_{R\to\infty} \frac{1}{R} \int_0^R \nu \circ \varsigma_\omega^\theta \left([D\omega_{-t} : D\omega]_{\frac{\alpha}{2}}^* [D\omega_{-t} : D\omega]_{\frac{\alpha}{2}}\right) \mathrm{d}\theta. \tag{2.11}$$

The measure $Q_{\nu,t}^{2\mathrm{tm}}$ gives the statistics of an idealized two-time measurement of the entropy production over a time period of length t in the system $(\mathcal{O}, \tau, \omega)$, the latter being in the state ν at the instant of the first measurement. The *thermodynamic limit* justification of this idealization was carried out in [BBJ⁺24b].

As we have already mentioned, in the special case $\nu = \omega$, the family $(Q_{\omega,t}^{2\mathrm{tm}})_{t\in\mathbb{R}}$ associated to finite quantum systems was introduced in [Kur00, Tas00] and was studied in detail in [JOPP10]. In the more general setting of algebraic quantum dynamical systems, it first appeared in [TM03]. To the best of our knowledge, the case of general $\nu \in \mathcal{N}$ was considered for the first time in [BBJ⁺23], where the following rigidity result was established [BBJ⁺23, Theorems 1.5 and 1.6].

Theorem 2.1 (1) *Suppose that the C^*-quantum dynamical system $(\mathcal{O}, \varsigma_\omega, \omega)$ is ergodic. Then for any $\nu \in \mathcal{N}$ and $t > 0$,*

$$Q_{\nu,t}^{2\mathrm{tm}} = Q_{\omega,t}^{2\mathrm{tm}}. \tag{2.12}$$

(2) *Suppose that the open quantum system $(\mathcal{O}, \tau, \omega)$ is such that each reservoir subsystem $(\mathcal{O}_j, \tau_j, \omega_j)$ is ergodic. Let $\nu \in \mathcal{N}$, and denote by ν_{S} its restriction to $\mathcal{O}_{\mathrm{S}}$. Then, for all $\alpha \in \mathrm{i}\mathbb{R}$,*

$$\mathfrak{F}_{\nu,t}^{2\mathrm{tm}}(\alpha) = \nu_{\mathrm{S}} \otimes \omega_{\mathrm{R}}\left([D\omega_{-t} : D\omega]_\alpha\right) = \langle \Omega_{\nu_{\mathrm{S}}\otimes\omega_{\mathrm{R}}}, \Delta_{\omega_{-t}|\omega}^\alpha \Omega_{\nu_{\mathrm{S}}\otimes\omega_{\mathrm{R}}} \rangle, \tag{2.13}$$

and in particular, $Q_{\nu,t}^{2\mathrm{tm}}$ is the spectral measure of $-\log \Delta_{\omega_{-t}|\omega}$ for the vector $\Omega_{\nu_{\mathrm{S}}\otimes\omega_{\mathrm{R}}}$. Moreover, if ν_{S} is faithful, then, for any $t \in \mathbb{R}$,[7]

$$\dim(\mathcal{H}_{\mathrm{S}}) \min \mathrm{sp}(\nu_{\mathrm{S}}) \le \frac{\mathrm{d}Q_{\nu,t}^{2\mathrm{tm}}}{\mathrm{d}Q_{\omega,t}^{2\mathrm{tm}}} \le \dim(\mathcal{H}_{\mathrm{S}}). \tag{2.14}$$

[7] $\mathrm{sp}(A)$ denotes the spectrum of a linear operator A.

Remark 1. In the open quantum system case (2), the presence of the small system prevents the validity of (2.12). However, Inequalities (2.14) force the same LDP for all $\nu \in \mathscr{N}$ (see Remark after Proposition 2.6). Intuitively, (2.14) expresses the fact that, in the large-time limit, the dynamics of the system is completely dominated by the reservoirs.

Remark 2. Recall that, by **(Reg1)**, $\omega_T \in \mathscr{N}$ for all $T \in \mathbb{R}$, so that the previous theorem applies to $\nu = \omega_T$. $Q^{2\mathrm{tm}}_{\omega_T,t}$ can also be interpreted as corresponding to a measurement protocol where the system, initially in state ω, is measured first at time T and then at time $T+t$. In this case, the equality $Q^{2\mathrm{tm}}_{\omega_T,t} = Q^{2\mathrm{tm}}_{\omega,t}$ and the Inequalities (2.14) can be interpreted as the memory erasing effect due to the decoherence induced by the first measurement on the reservoirs.

Denote by $\mathscr{P}(\mathbb{R})$ the set of all Borel probability measures on $\mathbb{R}$ equipped with the weak topology. By [Tak55, Lemma 2.1] and [Fel60, Theorem 1.1], the folium $\mathscr{N}$ is dense in $\mathscr{S}_{\mathscr{O}}$. Hence, under the assumptions of Theorem 2.1, Part (1) or (2), the map

$$\mathscr{N} \ni \nu \mapsto Q^{2\mathrm{tm}}_{\nu,t} \in \mathscr{P}(\mathbb{R})$$

uniquely extends to a continuous map

$$\mathscr{S}_{\mathscr{O}} \ni \nu \mapsto Q^{2\mathrm{tm}}_{\nu,t} \in \mathscr{P}(\mathbb{R}).$$

If Part (1) holds then obviously $Q^{2\mathrm{tm}}_{\nu,t} = Q^{2\mathrm{tm}}_{\omega,t}$ for all $\nu \in \mathscr{S}_{\mathscr{O}}$. In the case of open quantum systems $Q^{2\mathrm{tm}}_{\nu,t}$ is again the spectral measure of $-\log \Delta_{\omega_{-t}|\omega}$ for the vector $\Omega_{\nu_{\mathrm{S}} \otimes \omega_{\mathrm{R}}}$, and the estimates (2.14) hold if ν_{S} is faithful. This defines the 2TMEP of $(\mathscr{O}, \tau, \omega)$ with respect to any initial state $\nu \in \mathscr{S}_{\mathscr{O}}$, and applies, in particular, to a NESS ω_+ as defined in (2.9). Under additional hypothesis, $Q^{2\mathrm{tm}}_{\omega_+,t}$ can be obtained as the weak limit of $Q^{2\mathrm{tm}}_{\omega_T,t}$ as $T \to \infty$, see Assumption **(NESS)** below.

Thus, this resolves the obstacle (a) mentioned in the introduction. However, this achievement has been obtained in the context of idealized measurements. Indeed, the ergodicity assumptions of Theorem 2.1 require the system under consideration, or the reservoirs in the case of an open system, to be infinitely extended. This means in particular that the energies supposed to be measured are infinite[8]. In the spirit of statistical mechanics, such idealized measurements can be understood as approximating those made on a large but finite system. We refer the reader to [BBJ+23] for further discussion and to [BBJ+24b] for a mathematical justification of this *thermodynamic limit*.

2.4 Entropic ancilla state tomography

To provide another interpretation of the probability measure $Q^{2\mathrm{tm}}_{\omega,t}$, and overcome obstacle (b), we shall couple the system $(\mathscr{O}, \tau, \omega)$ to an auxiliary finite quantum system, called the *ancilla*, and replace the two idealized measurements of the entropic observable S with a specific sequence of projective measurements performed on the ancilla, a procedure often called tomography of the ancilla state. We shall see that, under appropriate choice of the coupling to a single qubit, one can relate the state of this ancillary qubit to the functional

$$\mathfrak{F}^{\mathrm{ancilla}}_{\nu,t}(\alpha) := \nu\left([D\omega_{-t} : D\omega]^*_{\frac{\alpha}{2}} [D\omega_{-t} : D\omega]_{\frac{\alpha}{2}}\right), \tag{2.15}$$

[8] On a more technical level, the quantity S is no more an observable.

where ν is an arbitrary initial state of the system.[9] It follows from (2.11) that $\mathfrak{F}^{2\mathrm{tm}}_{\omega,t} = \mathfrak{F}^{\mathrm{ancilla}}_{\omega,t}$, but for an arbitrary initial state ν the functionals $\mathfrak{F}^{2\mathrm{tm}}_{\nu,t}$ and $\mathfrak{F}^{\mathrm{ancilla}}_{\nu,t}$ will be generally distinct. However, when ν is chosen to be the NESS, the asymptotic relation between $\mathfrak{F}^{2\mathrm{tm}}_{\nu,t}$ and $\mathfrak{F}^{\mathrm{ancilla}}_{\nu,t}$, in the limit $t \to \infty$, is part of the PREF, see Definition 2.10. From the physics perspective the main point is that the ancilla's state, and hence the functional $\mathfrak{F}^{\mathrm{ancilla}}_{\nu,t}$, are experimentally accessible through state tomography. We refer the interested reader to [DCH+13, CBK+13, JCM+16, RCP14a, CRP15, GPM14, MDCP13] for related theoretical studies in the physics literature and to [AZU+14, BSM+14, BSS+15, PBH+19] for experimental implementations.

To elucidate the definition of $\mathfrak{F}^{\mathrm{ancilla}}_{\nu,t}$, let us consider a finite quantum dynamical system $(\mathscr{O}, \tau, \omega)$ where:

- $\mathscr{O} = L(\mathscr{K})$ for a finite-dimensional Hilbert space $\mathscr{K}$.
- τ is the C^*-dynamics generated by the self-adjoint Hamiltonian $H \in \mathscr{O}$.
- ω is a faithful density matrix on $\mathscr{O}$.

In this case it follows that

$$[D\omega_{-t} : D\omega]_{\frac{\alpha}{2}} = \omega_{-t}^{\alpha/2}\omega^{-\alpha/2},$$

and Definition (2.15) becomes

$$\mathfrak{F}^{\mathrm{ancilla}}_{\nu,t}(\alpha) = \mathrm{tr}(\nu\omega^{-\alpha/2}\omega_{-t}^{\alpha}\omega^{-\alpha/2}).$$

The ancilla's Hilbert space is $\mathbb{C}^2$ and its initial state is a density matrix

$$\rho = \begin{bmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{bmatrix},$$

which is assumed not to commute with the Pauli matrix σ_z. The Hilbert space of the coupled system is $\widehat{\mathscr{K}} = \mathscr{K} \otimes \mathbb{C}^2$ and its initial state is $\widehat{\nu} = \nu \otimes \rho$, where ν is a density matrix on $\mathscr{K}$. The coupling between the system and the ancilla is given by the Hamiltonian

$$\widehat{H}_\alpha = \mathrm{e}^{\frac{\alpha}{2}\log\omega\otimes\sigma_z}(H \otimes \mathbb{1})\,\mathrm{e}^{-\frac{\alpha}{2}\log\omega\otimes\sigma_z},$$

parametrized by $\alpha \in \mathrm{i}\mathbb{R}$. A simple calculation gives that the ancilla's state at time t is given by

$$\rho_t = \mathrm{tr}_{\mathscr{K}}(\mathrm{e}^{-\mathrm{i}t\widehat{H}_\alpha}\widehat{\nu}\mathrm{e}^{\mathrm{i}t\widehat{H}_\alpha}) = \begin{bmatrix} \rho_{++} & \mathfrak{F}^{\mathrm{ancilla}}_{\nu,t}(\alpha)\rho_{+-} \\ \overline{\mathfrak{F}^{\mathrm{ancilla}}_{\nu,t}(\alpha)}\rho_{-+} & \rho_{--} \end{bmatrix}. \tag{2.16}$$

Note that in the special case $H = H_{\mathrm{fr}} + V$ with $[\omega, H_{\mathrm{fr}}] = 0$[10], one has $\widehat{H}_\alpha = H \otimes \mathbb{1} + \widehat{W}_\alpha$ where

$$\widehat{W}_\alpha = \tfrac{1}{2}W_\alpha \otimes (\mathbb{1} + \sigma_z) + \tfrac{1}{2}W_{-\alpha} \otimes (\mathbb{1} - \sigma_z), \tag{2.17}$$

[9] Note that, for $\nu \neq \omega$, $\mathfrak{F}^{\mathrm{ancilla}}_{\nu,t}$ is not necessarily the Fourier transform of a Borel probability measure.

[10] This will be the case in finite open quantum systems.

with

$$W_\alpha = \varsigma_\omega^{-\mathrm{i}\alpha/2}(V) - V. \tag{2.18}$$

To describe the general case, we denote by $\mathrm{Mat}_2(\mathbb{C})$ the algebra of complex 2×2 matrices and consider an open quantum system $(\mathscr{O},\tau,\omega)$ with coupling V. The algebra of observables of this system coupled to a qubit is $\widehat{\mathscr{O}} := \mathscr{O}\otimes \mathrm{Mat}_2(\mathbb{C})$ which we will often identify with the algebra $\mathrm{Mat}_2(\mathscr{O})$ of 2×2 matrices with entries in $\mathscr{O}$, see [BR87, Section 2.7.2]. The decoupled dynamics on $\widehat{\mathscr{O}}$ is given by $\widehat{\tau}^t = \tau^t \otimes \mathrm{Id}$ while W_α and $\widehat{W}_\alpha$ are defined by (2.17) and (2.18) (note that they are self-adjoint). Let

$$\widehat{\tau}_\alpha^t = \mathrm{e}^{t\widehat{\delta}_\alpha}, \qquad \widehat{\delta}_\alpha = \delta\otimes\mathrm{Id} + \mathrm{i}[\widehat{W}_\alpha, \cdot\,]$$

be the perturbation of $\widehat{\tau}$ by $\widehat{W}_\alpha$ and let ρ be as above. For $\nu\in\mathscr{S}_{\mathscr{O}}$ we set $\widehat{\nu} = \nu\otimes\rho$, $\widehat{\nu}_t = \widehat{\nu}\circ\widehat{\tau}^t$, and $\rho_t = \widehat{\nu}_t|_{\mathrm{Mat}_2(\mathbb{C})}$. We then have:

Proposition 2.2 *Suppose that $V\in\mathrm{Dom}(\delta_\omega)$. Then, for all $\alpha\in\mathrm{i}\mathbb{R}$ and $\nu\in\mathscr{S}_{\mathscr{O}}$, the ancilla's state at time $t\in\mathbb{R}$ is given by the right-hand side of* (2.16).

The proof is given in Section 5.1.

Beyond open quantum systems, the definition (2.15) remains useful in the study of the general mathematical structure of non-equilibrium quantum statistical mechanics, and we will make use of it in that context.

The identity $\mathfrak{F}_{\omega,t}^{\mathrm{2tm}} = \mathfrak{F}_{\omega,t}^{\mathrm{ancilla}}$ is of considerable theoretical and practical importance. The two-time measurement entropy production protocol is always introduced in the context of finite quantum systems (or, slightly more generally, confined quantum system with possibly infinite discrete energy spectra). This identity is also of experimental relevance. Indeed, the ancilla technique has been used to access the two time measurement distribution of work [RCP14b, CMM+17, DCSCR18].

Thanks to its connection with modular theory, the two-time measurement statistics has a thermodynamic limit [BBJ+24b] which, by the identity $\mathfrak{F}_{\omega,t}^{\mathrm{2tm}} = \mathfrak{F}_{\omega,t}^{\mathrm{ancilla}}$, is experimentally accessible through ancilla state tomography. This relation makes it possible to interpret the results of entropic ancilla state tomography in terms of energy transfers in open quantum systems with large reservoirs.

2.5 The quantum principles of regular entropic fluctuations

We recall the basic properties of $Q_{\omega,t}^{\mathrm{2tm}}$; see [TM03, Theorem 7] and [BBJ+23, Theorem 1.4].

Proposition 2.3 (1) $\int_{\mathbb{R}} s\,\mathrm{d}Q_{\omega,t}^{\mathrm{2tm}}(s) = -\mathrm{Ent}(\omega_t|\omega)$. *In particular,*

$$\int_{\mathbb{R}} s\,\mathrm{d}Q_{\omega,t}^{\mathrm{2tm}}(s) \geq 0,$$

with equality iff $\omega = \omega_t$.

(2) *The map $\mathrm{i}\mathbb{R}\ni\alpha\mapsto\mathfrak{F}_{\omega,t}^{\mathrm{2tm}}(\alpha)$ has an analytic extension to the vertical strip $0<\mathrm{Re}\,\alpha<1$ which is bounded and continuous on its closure.*

In the remaining statements we assume that $(\mathscr{O},\tau,\omega)$ is time-reversal invariant.

(3) *For any* α *satisfying* $0 \le \operatorname{Re}\alpha \le 1$,

$$\mathfrak{F}^{\rm 2tm}_{\omega,t}(\alpha) = \overline{\mathfrak{F}^{\rm 2tm}_{\omega,t}(1-\overline{\alpha})}.$$

(4) *Let* $\mathfrak{r}:\mathbb{R}\to\mathbb{R}$ *be the reflection at* 0, $\mathfrak{r}(s)=-s$, *and* $\overline{Q}^{\rm 2tm}_{\omega,t} = Q^{\rm 2tm}_{\omega,t}\circ\mathfrak{r}$. *Then the measures* $Q^{\rm 2tm}_{\omega,t}$ *and* $\overline{Q}^{\rm 2tm}_{\omega,t}$ *are mutually absolutely continuous and*

$$\frac{{\rm d}\overline{Q}^{\rm 2tm}_{\omega,t}}{{\rm d}Q^{\rm 2tm}_{\omega,t}}(s) = {\rm e}^{-s}. \tag{2.19}$$

We consider the family $(P^{\rm 2tm}_{\omega,t})_{t>0}\subset\mathscr{P}(\mathbb{R})$ defined by

$$P^{\rm 2tm}_{\omega,t}(B) = Q^{\rm 2tm}_{\omega,t}(tB),$$

for all Borel sets $B\subset\mathbb{R}$ and $t>0$. It describes the statistics of the two-time measurement entropy production *per unit time* of $(\mathcal{O},\tau,\omega)$ with respect to ω over the time interval $[0,t]$. The relation (2.19) has an important consequence for the Large Deviation Principle (LDP) satisfied by $(P^{\rm 2tm}_{\omega,t})_{t>0}$. Before describing it, we give a short general overview of LDP that is suited for our purposes.

Definition 2.4 *The family* $(P_t)_{t>0}\subset\mathscr{P}(\mathbb{R})$ *satisfies a* ***full LDP*** *if there exists a lower-semicontinuous function* $\mathbb{I}:\mathbb{R}\to[0,\infty]$, *called the* ***rate function***, *such that for any Borel set* $B\subset\mathbb{R}$,

$$-\inf_{s\in{\rm int}(B)}\mathbb{I}(s) \le \liminf_{t\to\infty}\tfrac{1}{t}\log P_t(B) \le \limsup_{t\to\infty}\tfrac{1}{t}\log P_t(B) \le -\inf_{s\in{\rm cl}(B)}\mathbb{I}(s), \tag{2.20}$$

where ${\rm int}(B)$ / ${\rm cl}(B)$ *denotes the interior/closure of* B.

If (2.20) *holds only for Borel sets* $B\subseteq]a,b[$, *where* $]a,b[\neq\mathbb{R}$, *we then say that a* ***local LDP*** *holds for* $(P_t)_{t>0}$ *on the interval* $]a,b[$.

Remark. The lower-semicontinuity assumption ensures that the rate function $\mathbb{I}$ is unique whenever it exists.

The celebrated Gärtner–Ellis theorem gives an important criterion that ensures the validity of the LDP.

Theorem 2.5 *Let* $I=]\vartheta_-,\vartheta_+[\subset\mathbb{R}$ *be an open interval containing* 0, *and suppose that the limit*

$$F(\alpha) = \lim_{t\to\infty}\frac{1}{t}\log\int_{\mathbb{R}} e^{-\alpha s}{\rm d}P_t(s)$$

exists, is finite for $\alpha\in I$, *and that the function* $F:I\to\mathbb{R}$ *is differentiable. If* $I=\mathbb{R}$, *then the full LDP holds with the rate function*

$$\mathbb{I}(s) = \sup_{-\alpha\in I}(s\alpha - F(-\alpha)). \tag{2.21}$$

Otherwise, the local LDP with rate function (2.21) *holds for any Borel* $B\subset]a,b[$ *where*

$$a = \lim_{\alpha\downarrow\vartheta_-}F'(\alpha), \qquad b = \lim_{\alpha\uparrow\vartheta_+}F'(\alpha). \tag{2.22}$$

Remark. By Hölder's inequality, F is convex. Thus, F' is increasing on I and the limits (2.22) always exist. It may happen that $a = -\infty$ and $b = \infty$ (even for bounded I), in which case the full LDP again holds. Otherwise, if either a or b is finite, the Gärtner–Ellis theorem yields only a local LDP. For more information about LDP we refer the reader to [DZ00, Ell06].

Returning to the family $(P^{2\text{tm}}_{\omega,t})_{t>0}$, a consequence of the relation (2.19) is:

Proposition 2.6 *Suppose that the system $(\mathcal{O},\tau,\omega)$ is time-reversal invariant and that the full LDP holds for the family $(P^{2\text{tm}}_{\omega,t})_{t>0}$ with rate function $\mathbb{I}$. Then for all $s\in\mathbb{R}$,*

$$\mathbb{I}(-s) = \mathbb{I}(s) + s. \tag{2.23}$$

If the local LDP holds on $]-a,a[$ for some $a>0$, then (2.23) holds for $s\in]-a,a[$.

Remark. Under the hypothesis of Theorem 2.1, one has obviously

$$\lim_{t\to\infty}\frac{1}{t}\log\mathfrak{F}^{2\text{tm}}_{\omega,t}(\alpha) = \lim_{t\to\infty}\frac{1}{t}\log\mathfrak{F}^{2\text{tm}}_{\nu,t}(\alpha),$$

for all $\nu\in\mathcal{N}$. Therefore, whenever Theorem 2.5 implies the LDP for $(P^{2\text{tm}}_{\omega,t})_{t>0}$, the *same* LDP must hold for $(P^{2\text{tm}}_{\nu,t})_{t>0}$, with the *same* rate function. Notice that this holds in particular for $\nu=\omega_T$.

Proof. We follow [CJPS17] and prove the result in the full LDP case. The local LDP case is identical. We abbreviate $Q^{2\text{tm}}_{\omega,t}$ and $P^{2\text{tm}}_{\omega,t}$ with Q_t and P_t. Relation (2.19) gives that for any Borel set $B\subset\mathbb{R}$ we have

$$Q_t(B) \le \mathrm{e}^{\sup B}Q_t(-B).$$

Replacing B with tB, the LDP gives

$$-\inf_{u\in\mathrm{int}(B)}\mathbb{I}(u) \le \liminf_{t\to\infty}\tfrac{1}{t}\log P_t(B) \le \limsup_{t\to\infty}\tfrac{1}{t}\log\left(\mathrm{e}^{t\sup B}P_t(-B)\right) \le \sup B - \inf_{u\in\mathrm{cl}(B)}\mathbb{I}(-u).$$

Taking $B =]s-\epsilon, s+\epsilon[$ we derive

$$\inf_{|u+s|<2\epsilon}\mathbb{I}(u) \le \inf_{|u+s|\le\epsilon} I(u) \le s+\epsilon+\inf_{|u-s|<\epsilon}\mathbb{I}(u). \tag{2.24}$$

Since the function $\mathbb{I}$ is lower semicontinuous,

$$\mathbb{I}(s) = \lim_{\epsilon\downarrow 0}\inf_{|u-s|<\epsilon}\mathbb{I}(u),$$

and (2.24) gives that $\mathbb{I}(-s)\le s+\mathbb{I}(s)$ for any $s\in\mathbb{R}$. Replacing s with $-s$ and combining the two inequalities we derive (2.23). □

Due to the obvious parallel with the foundational works of [ES94] in classical statistical mechanics[11], if the full/local LDP holds for $(P^{2\text{tm}}_{\omega,t})_{t>0}$, we will say that the full/local quantum Evans–Searles fluctuation theorem holds for $(\mathcal{O},\tau,\omega)$. The relations (2.19) and (2.23) are sometimes called quantum

[11] See [JPRB11] and Section 2.6 below.

Evans–Searles fluctuation relation. We emphasize that (2.19) is an immediate consequence of time-reversal invariance, and that (2.23) follows from (2.19) and the LDP.

The equally celebrated Gallavotti–Cohen fluctuation theorem [GC95a, GC95b] refers to the statistics of entropy production with respect to the non-equilibrium steady state reached in the long-time limit. In spite of a formal similarity, it is conceptually and technically a very different statement than the Evans–Searles theorem that refers to the statistics of the entropy production with respect to the reference (initial) state of the system. The two theorems are related by an exchange of limits. The validity of this exchange of limits is a deep dynamical problem that has been lifted to the principle of regular entropic fluctuations in [JPRB11]. We will review these points in Section 2.6 and 2.8; for an in depth discussion see [JPRB11].

Returning to quantum statistical mechanics, we make the following assumption concerning the NESS defined in (2.9):

(NESS) For all $A \in \mathcal{O}$, the limit

$$\lim_{t\to\infty} \omega\circ\tau^t(A) = \omega_+(A), \tag{2.25}$$

exists, so that ω_+ is the unique NESS of the system $(\mathcal{O},\tau,\omega)$. Moreover, for all $t>0$ the weak limit

$$Q^{2\mathrm{tm}}_{\omega_+,t} := \lim_{T\to\infty} Q^{2\mathrm{tm}}_{\omega_T,t} \tag{2.26}$$

exists.[12]

The family $(P^{2\mathrm{tm}}_{\omega_+,t})_{t>0}$ is defined by $P^{2\mathrm{tm}}_{\omega_+,t}(B) = Q^{2\mathrm{tm}}_{\omega_+,t}(tB)$. In parallel with the classical theory of entropic fluctuations [JPRB11], see also Section 2.6, the following definitions are natural.

Definition 2.7 *Suppose that* **(NESS)** *holds. We say that the* ***full weak quantum Gallavotti–Cohen theorem*** *holds for* $(\mathcal{O},\tau,\omega)$ *if* $(P^{2\mathrm{tm}}_{\omega_+,t})_{t>0}$ *satisfies the full LDP. The* ***local weak quantum Gallavotti–Cohen theorem*** *holds if the LDP holds on some finite interval* $]-a,a[$, $a>0$.

Definition 2.8 *Suppose that* **(NESS)** *holds. We say that the* ***full weak quantum PREF*** *holds for* $(\mathcal{O},\tau,\omega)$ *if the families* $(P^{2\mathrm{tm}}_{\omega,t})_{t>0}$ *and* $(P^{2\mathrm{tm}}_{\omega_+,t})_{t>0}$ *both satisfy the full LDP with the same rate function. The* ***local weak quantum PREF*** *holds if they satisfy local LDP on the same finite interval* $]-a,a[$, $a>0$, *with the same rate function.*

In the context of 2TTM, the PREF essentially trivializes. Indeed, in the directly coupled case, Theorem 2.1(1) implies immediately $Q_{\omega_+,t} = Q_{\omega,t}$, while in the open quantum system case (2), it suffices to notice that the function $F(\alpha)$ in Theorem 2.5 is the same whenever computed with respect to ω or ω_+ (see remark after Proposition 2.6). Therefore, the LDP for $(P^{2\mathrm{tm}}_{\omega,t})_{t>0}$ implies the *same* LDP for $(P^{2\mathrm{tm}}_{\omega_+,t})_{t>0}$.

[12] Recall that (2.25) ⇒ (2.26) under the assumptions of Theorem 2.1.

Again this trivialization can be ultimately understood as a consequence of the first measurement's dominating effect in the thermodynamic limit. This rigidity has no classical analog. In the classical case, the equality of the rate functions is not only non-trivial, but it can be regarded as a deep dynamical property of the system.

However, in a surprising turn, the parallel with the classical PREF is restored when following another route to the quantum extension of entropic functionals, given by EAST, which we now describe. In order to open this route, we need to introduce an additional regularity assumption which will allow to extend the entropic functionals to a complex domain.

To $\vartheta > 0$ we associate the vertical strip

$$\mathfrak{S}(\vartheta) = \{z \in \mathbb{C} \mid |\mathrm{Re}\, z| < \vartheta\},$$

and the assumption

(AnC(ϑ)) For any $t \in \mathbb{R}$ the function

$$\mathrm{i}\mathbb{R} \ni \alpha \mapsto [D\omega_t : D\omega]_\alpha \in \mathcal{O}$$

has an analytic extension to the strip $\mathfrak{S}(\vartheta)$.

The cocycle relation (2.3) gives

$$[D\omega_t : D\omega]_{\alpha_1+\alpha_2} = [D\omega_t : D\omega]_{\alpha_1} \varsigma_\omega^{-\mathrm{i}\alpha_1}([D\omega_t : D\omega]_{\alpha_2}),$$

which holds for $\alpha_1, \alpha_2 \in \mathrm{i}\mathbb{R}$. Assuming **(AnC($\vartheta$))**, this last relation extends by analytic continuation to $\alpha_1 \in \mathrm{i}\mathbb{R}$ and $\alpha_2 \in \mathfrak{S}(\vartheta)$. This gives that if **(AnC(ϑ))** holds, then for all α in the sub-strip $\mathfrak{S}(\vartheta')$, $0 < \vartheta' < \vartheta$,

$$\|[D\omega_s : D\omega]_\alpha\| \le \sup_{|\gamma|<\vartheta'} \|[D\omega_s : D\omega]_\gamma\|. \tag{2.27}$$

The next proposition is an immediate consequence of this bound and Vitali's convergence theorem [Tit39, Theorem 5.21] applied to Relation (2.10).

Proposition 2.9 *Suppose that Assumptions* **(NESS)** *and* **(AnC(ϑ))** *hold. Then for any $T > 0$ and $t \in \mathbb{R}$ the map*

$$\mathrm{i}\mathbb{R} \ni \alpha \mapsto \mathfrak{F}^{\mathrm{2tm}}_{\omega_T,t}(\alpha)$$

has an analytic continuation to the strip $\mathfrak{S}(\vartheta)$ such that, for any α in this strip, the limit

$$\mathfrak{F}^{\mathrm{2tm}}_{\omega_+,t}(\alpha) := \lim_{T\to\infty} \mathfrak{F}^{\mathrm{2tm}}_{\omega_T,t}(\alpha)$$

exists and is finite. Moreover, the function $\alpha \mapsto \mathfrak{F}^{\mathrm{2tm}}_{\omega_+,t}$ is analytic on $\mathfrak{S}(\vartheta)$, and for any α in this strip,

$$\mathfrak{F}^{\mathrm{2tm}}_{\omega_+,t}(\alpha) = \int_{\mathbb{R}} \mathrm{e}^{-\alpha s} \mathrm{d}Q^{\mathrm{2tm}}_{\omega_+,t}(s). \tag{2.28}$$

Remark. By (2.28), $\mathfrak{F}^{\rm 2tm}_{\omega_+,t}$ obviously extends analytically to the half-plane $\mathrm{Re}(\alpha)>0$, but we will not make use of that fact.

It follows directly from **(NESS)** and **(AnC(ϑ))** that for all $t>0$ the function $\mathrm{i}\mathbb{R}\ni\alpha\mapsto\mathfrak{F}^{\rm ancilla}_{\omega_+,t}(\alpha)$ has an analytic continuation to the strip $\mathfrak{S}(2\vartheta)$, and that for α in this strip,

$$\mathfrak{F}^{\rm ancilla}_{\omega_+,t}(\alpha)=\omega_+\left([D\omega_{-t}:D\omega]^*_{\frac{\alpha}{2}}[D\omega_{-t}:D\omega])_{\frac{\alpha}{2}}\right).$$

Definition 2.10 *We say that $(\mathscr{O},\tau,\omega)$ satisfies the **strong quantum PREF** on the interval $]\vartheta_-,\vartheta_+[$ containing 0 if Assumptions **(NESS)** and **(AnC(ϑ))** hold, with $\vartheta>\max\{|\vartheta_-|,\vartheta_+\}$, and the limits*

$$\boldsymbol{F}^{\rm 2tm}_{\omega}(\alpha):=\lim_{t\to\infty}\frac{1}{t}\log\mathfrak{F}^{\rm 2tm}_{\omega,t}(\alpha),$$
$$\boldsymbol{F}^{\rm 2tm}_{\omega_+}(\alpha):=\lim_{t\to\infty}\frac{1}{t}\log\mathfrak{F}^{\rm 2tm}_{\omega_+,t}(\alpha),$$
$$\boldsymbol{F}^{\rm ancilla}_{\omega_+}(\alpha):=\lim_{t\to\infty}\frac{1}{t}\log\mathfrak{F}^{\rm ancilla}_{\omega_+,t}(\alpha),$$

exist for all $\alpha\in]\vartheta_-,\vartheta_+[$, and define differentiable functions on this interval satisfying

$$\boldsymbol{F}^{\rm 2tm}_{\omega}=\boldsymbol{F}^{\rm 2tm}_{\omega_+}=\boldsymbol{F}^{\rm ancilla}_{\omega_+}.\tag{2.29}$$

Remark. Setting

$$a=\lim_{\alpha\downarrow\vartheta_-}\partial_\alpha\boldsymbol{F}^{\rm 2tm}_{\omega}(\alpha),\qquad b=\lim_{\alpha\uparrow\vartheta_+}\partial_\alpha\boldsymbol{F}^{\rm 2tm}_{\omega}(\alpha),$$

and invoking the Gärtner–Ellis theorem, the strong quantum PREF implies that the families $(P^{\rm 2tm}_{\omega,t})_{t>0}$ and $(P^{\rm 2tm}_{\omega_+,t})_{t>0}$ both satisfy a LDP on the interval $]a,b[$, with the same rate function. If either $]\vartheta_-,\vartheta_+[=\mathbb{R}$ or $]a,b[=\mathbb{R}$, we say that the full strong quantum PREF holds, otherwise that the local strong quantum PREF holds. Obviously, the local/full strong quantum PREF implies the local/full weak quantum PREF.

If the assumptions of Theorem 2.1 are satisfied then (2.26) and the first equality in (2.29) automatically hold. We shall see that under the following regularity assumption, which will also play an important role in our discussion of quantum transfer operators, the second relation in (2.29) can be reduced to an exchange of limits.

(AnV(ϑ)) $(\mathscr{O},\tau,\omega)$ describes an open quantum system where the connecting perturbation V is such that the map

$$\mathrm{i}\mathbb{R}\ni\theta\mapsto\varsigma_\omega^{-\mathrm{i}\theta}(V)\in\mathscr{O}$$

has an analytic extension to the strip $\mathfrak{S}(\vartheta)$.

Proposition 2.11 *Suppose that* **(AnV(ϑ))** *holds. Then* **(AnC(ϑ))** *holds, and for any $t \in \mathbb{R}$ and $\alpha \in \mathfrak{S}(\vartheta)$*

$$\|[D\omega_t : D\omega]_\alpha\| \le \mathrm{e}^{|t|(\|\varsigma_\omega^{-\mathrm{i}\,\mathrm{Re}\,\alpha}(V)\|+\|V\|)}. \tag{2.30}$$

Proposition 2.12 *Suppose that* **(AnV(ϑ))** *holds for some $\vartheta > \frac{1}{2}$ and set*

$$C_T := \mathrm{e}^{2|T|(\|\varsigma_\omega^{-\mathrm{i}/2}(V)\|+\|V\|)},$$
$$D_T := \mathrm{e}^{-2|T|(\|\varsigma_\omega^{\mathrm{i}/2}(V)\|+\|V\|)}.$$

Then for any $\alpha \in]-\vartheta, \vartheta[$,

$$D_T \mathfrak{F}_{\omega,t}^{2\mathrm{tm}}(\alpha) \le \mathfrak{F}_{\omega_T,t}^{\mathrm{ancilla}}(\alpha) \le C_T \mathfrak{F}_{\omega,t}^{2\mathrm{tm}}(\alpha). \tag{2.31}$$

Remark 1. Assuming the existence of $\boldsymbol{F}_\omega^{2\mathrm{tm}}$, the estimate (2.31) gives that for $\alpha \in]-\vartheta, \vartheta[$,

$$\boldsymbol{F}_\omega^{2\mathrm{tm}}(\alpha) = \lim_{T\to\infty}\lim_{t\to\infty} \frac{1}{t}\log \mathfrak{F}_{\omega_T,t}^{\mathrm{ancilla}}(\alpha).$$

It follows that the second relation in (2.29) holds iff the exchange of limits

$$\lim_{T\to\infty}\lim_{t\to\infty} \frac{1}{t}\log \mathfrak{F}_{\omega_T,t}^{\mathrm{ancilla}}(\alpha) = \lim_{t\to\infty}\lim_{T\to\infty} \frac{1}{t}\log \mathfrak{F}_{\omega_T,t}^{\mathrm{ancilla}}(\alpha) \tag{2.32}$$

is valid for $\alpha \in]-\vartheta, \vartheta[$.

Remark 2. Under the assumptions of Proposition 2.12, for $\alpha \in]-\vartheta, \vartheta[$ we also have the estimates

$$D_T \mathfrak{F}_{\omega,t}^{2\mathrm{tm}}(\alpha) \le \mathfrak{F}_{\omega_T,t}^{2\mathrm{tm}}(\alpha) \le C_T \mathfrak{F}_{\omega,t}^{2\mathrm{tm}}(\alpha). \tag{2.33}$$

Theorem 2.1 of course provides much stronger estimates with T-independent constants, but they hold only under ergodicity assumptions. The estimates (2.33) only require the regularity assumption **(AnV(ϑ))**. In particular, they hold for finite quantum systems to which Theorem 2.1 cannot be applied.

Proposition 2.11 is proven in Section 5.2 while Proposition 2.12 and Estimate (2.33) are proven in Section 5.3.

The validity of an exchange of limits plays an equally crucial role in the theory of classical PREF, see (2.39). For that and other comparison reasons, we review briefly the classical theory before returning to the quantum case.

2.6 Entropy production and entropic fluctuations in classical dynamical systems

This section follows [JPRB11]. Since this material was not discussed in [BBJ+23] we provide some details, referring the reader to [JPRB11] for a complete exposition and proofs.

We start with a pair $(\mathscr{X}, \phi)$, where $\mathscr{X}$ is a compact metric space and $\phi = \{\phi^t \mid t \in \mathbb{R}\}$ is a group of homeomorphisms of $\mathscr{X}$ such that the map

$$\mathbb{R} \times \mathscr{X} \ni (t, x) \mapsto \phi^t(x) \in \mathscr{X}$$

is continuous. We denote by $C(\mathscr{X})$ the vector space of all continuous complex-valued functions on $\mathscr{X}$ and equip it with the sup norm $\|f\|_\infty := \sup_{x\in\mathscr{X}}|f(x)|$. Observables are functions $f \in C(\mathscr{X})$, and they evolve in time as $f \mapsto f_t = f\circ\phi^t$. States are Borel probability measures on $\mathscr{X}$, and we write $\nu(f) = \int_{\mathscr{X}} f\mathrm{d}\nu$. They evolve in time as $\nu \mapsto \nu_t = \nu\circ\phi^{-t}$. A state ν is ϕ-invariant if $\nu_t = \nu$ for all t. The relative entropy of two states ν and μ is defined by

$$\mathrm{Ent}(\nu|\mu) = \begin{cases} \int_{\mathscr{X}} \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\right)\mathrm{d}\nu & \text{if } \mu \ll \nu; \\ -\infty & \text{otherwise.} \end{cases}$$

Its basic property is $\mathrm{Ent}(\nu|\mu) \le 0$ with equality iff $\nu = \mu$.

A time-reversal of $(\mathscr{X},\phi)$ is an involutive homeomorphism $\iota : \mathscr{X} \to \mathscr{X}$ such that

$$\iota\circ\phi^t = \phi^{-t}\circ\iota$$

for all $t \in \mathbb{R}$. Given such a time-reversal map ι, a state ν is called time-reversal invariant (TRI) if $\nu\circ\iota = \nu$.

Our starting point is a classical dynamical system $(\mathscr{X},\phi,\omega)$ where, to avoid triviality, the reference (initial) state ω is supposed not to be ϕ-invariant. The system is TRI if ω is TRI for some time-reversal of $(\mathscr{X},\phi)$.

We set the regularity assumptions. The first one is:

(Cl1) For all $t \in \mathbb{R}$ the measures ω and ω_t are mutually absolutely continuous.

This allows us to set

$$\Delta_{\omega_t|\omega} := \frac{\mathrm{d}\omega_t}{\mathrm{d}\omega}, \qquad \ell_{\omega_t|\omega} := \log\Delta_{\omega_t|\omega}, \qquad c^t := \ell_{\omega_t|\omega}\circ\phi^t,$$

for $t \in \mathbb{R}$. The following property is immediate and will play a central role.

Proposition 2.13 *The family $(c^t)_{t\in\mathbb{R}}$ is an additive ϕ-cocycle, i.e.,*

$$c^{t+s} = c^t + c^s\circ\phi^t$$

holds for all $t,s \in \mathbb{R}$. Moreover one has $\mathrm{Ent}(\omega_t|\omega) = -\omega(c^t)$ for all $t \in \mathbb{R}$.

We denote by $Q_{\omega,t}$ the law of c^t w.r.t. ω and set

$$\mathfrak{F}_{\omega,t}(\alpha) := \int_{\mathbb{R}} \mathrm{e}^{-\alpha s}\mathrm{d}Q_{\omega,t}(s), \qquad (\alpha \in \mathbb{C}).$$

The next two assumptions will allow us to define the entropy production observable.

(Cl2) $c^t \in C(\mathscr{X})$ for all $t \in \mathbb{R}$.

(Cl3) The map $\mathbb{R} \ni t \mapsto c^t \in C(\mathscr{X})$ is differentiable at $t = 0$.

Note that **(Cl1)** is the classical analog of Assumption **(Reg1)**, that **(Cl2)** is the classical analog of Assumption **(Reg2)**, and that **(Cl3)** corresponds to the condition $V \in \mathrm{Dom}(\delta_\omega)$.

The *entropy production* observable (or *phase space contraction rate*) of (X, ϕ, ω) is defined by

$$\sigma = \frac{\mathrm{d}}{\mathrm{d}t} c^t|_{t=0}.$$

Theorem 2.14 *Assume that* **(Cl1)**–**(Cl3)** *hold. Then*

(1)
$$c^t = \int_0^t \sigma_s \mathrm{d}s,$$

and
$$\mathrm{Ent}(\omega_t|\omega) = -\int_0^t \omega_s(\sigma)\mathrm{d}s.$$

(2) $\omega(\sigma) = 0$.

In the remaining statements we also assume that $(\mathscr{X}, \phi, \omega)$ *is* TRI *w.r.t. the time-reversal* ι.

(3) $c^t \circ \iota = c^{-t}$ *for all* $t \in \mathbb{R}$, *and* $\sigma \circ \iota = -\sigma$.

(4) *For all* $t \in \mathbb{R}$ *and* $\alpha \in \mathbb{C}$,
$$\mathfrak{F}_{\omega,t}(\alpha) = \overline{\mathfrak{F}_{\omega,t}(1-\overline{\alpha})}.$$

(5) *Let* $\mathfrak{r} : \mathbb{R} \to \mathbb{R}$ *be the reflection* $\mathfrak{r}(s) = -s$ *and* $\overline{Q}_{\omega,t} = Q_{\omega,t} \circ \mathfrak{r}$. *Then the measures* $Q_{\omega,t}$ *and* $\overline{Q}_{\omega,t}$ *are mutually absolutely continuous and*
$$\frac{\mathrm{d}\overline{Q}_{\omega,t}}{\mathrm{d}Q_{\omega,t}}(s) = \mathrm{e}^{-s}. \tag{2.34}$$

For the proof we refer the reader to [JPRB11].

We set
$$P_{\omega,t}(B) = Q_{\omega,t}(tB).$$

If the full/local LDP holds for $(P_{\omega,t})_{t>0}$, we will say that the *full/local classical Evans–Searles fluctuation theorem* holds for $(\mathscr{X}, \phi, \omega)$. The statement and the proof of Proposition 2.6 directly extend to the family $(P_{\omega,t})_{t>0}$; see [CJPS17]. The relation (2.34) and the induced symmetry (2.23) of the LDP rate function for $(P_{\omega,t})_{t>0}$ constitute the *classical Evans–Searles fluctuation relation.*

Until the end of this section we assume that **(Cl1)**–**(Cl3)** hold.

Going beyond the reference state, for any state ν on $\mathcal{X}$ we denote by $Q_{\nu,t}$ the law of c^t w.r.t. ν. Let

$$\mathfrak{F}_{\nu,t}(\alpha) := \int_{\mathbb{R}} e^{-\alpha s} dQ_{\nu,t}(s), \qquad (\alpha \in \mathbb{C}),$$

and $P_{\nu,t}(B) = Q_{\nu,t}(tB)$. Since

$$\mathfrak{F}_{\nu,t}(\alpha) = \int_{\mathcal{X}} e^{-\alpha c^t(x)} d\nu(x), \tag{2.35}$$

if $\nu_n \to \nu$ weakly then $Q_{\nu_n,t} \to Q_{\nu,t}$ weakly.

Of particular interest is the case where ν is a NESS of $(\mathcal{X},\phi,\omega)$. A NESS is a weak-limit point of the net

$$\left\{ \frac{1}{T}\int_0^T \omega_t dt \,\middle|\, T>0 \right\}$$

as $T \uparrow \infty$. The set of NESS is non-empty and any NESS is ϕ-invariant. Moreover, for any NESS ω_+ one has

$$\omega_+(\sigma) \geq 0.$$

Until the end of this section we also assume the following classical analogue of **(NESS)**.

> **(Cl4)** The weak limit $\lim_{t\to\infty} \omega_t = \omega_+$ exists so that, in particular, ω_+ is the unique NESS of the system $(\mathcal{X},\phi,\omega)$.

If the full/local LDP holds for $(P_{\omega_+,t})_{t>0}$, we say that the *full/local classical Gallavotti–Cohen fluctuation theorem* holds for $(\mathcal{X},\phi,\omega)$. If in addition the respective rate function $\mathbb{I}_+$ satisfies

$$\mathbb{I}_+(-s) = \mathbb{I}_+(s) + s \tag{2.36}$$

on its domain, we say that the *classical Gallavotti–Cohen fluctuation relation* holds. Unlike in the Evans–Searles case, (2.36) is not forced by the LDP and time-reversal. The classical PREF and the related exchange of limits argument that we discuss next is the only known general mechanism that ensures its validity.

Definition 2.15 *We say that the* ***full weak classical PREF*** *holds for* $(\mathcal{X},\phi,\omega)$ *if the families* $(P_{\omega,t})_{t>0}$ *and* $(P_{\omega_+,t})_{t>0}$ *both satisfy a full LDP with the same rate function. The* ***local weak classical PREF*** *holds if they satisfy a local LDP on the same interval* $]-a,a[$, $a>0$, *with the same rate function.*

Definition 2.16 *We say that* $(\mathcal{X},\phi,\omega)$ *satisfies the* ***strong classical PREF*** *on an open interval* $]\vartheta_-,\vartheta_+[$ *containing* 0 *if the limits*

$$\begin{aligned} \boldsymbol{F}_{\omega}(\alpha) &:= \lim_{t\to\infty} \tfrac{1}{t}\log \mathfrak{F}_{\omega,t}(\alpha), \\ \boldsymbol{F}_{\omega_+}(\alpha) &:= \lim_{t\to\infty} \tfrac{1}{t}\log \mathfrak{F}_{\omega_+,t}(\alpha), \end{aligned} \tag{2.37}$$

exist for all $\alpha \in]\vartheta_-,\vartheta_+[$, *and define differentiable functions on this interval satisfying*

$$\boldsymbol{F}_{\omega} = \boldsymbol{F}_{\omega_+}.$$

The remark after Definition 2.10 applies to the strong classical PREF as well, and we adopt the parallel terminology of local/full strong classical PREF.

The equality of the rate function in the full weak classical PREF is related to an exchange of limits. Indeed, suppose that the families $(P_{\omega,t})_{t>0}$ and $(P_{\omega_+,t})_{t>0}$ satisfy full LDP. Then, by Varadhan's Lemma, the limits

$$\boldsymbol{F}_\omega(\alpha) = \lim_{t\to\infty} \tfrac{1}{t}\log \mathfrak{F}_{\omega,t}(\alpha), \quad \text{and} \quad \boldsymbol{F}_{\omega_+}(\alpha) = \lim_{t\to\infty} \tfrac{1}{t}\log \mathfrak{F}_{\omega_+,t}(\alpha)$$

exist, and are obviously finite. The respective LDP rate functions satisfy

$$\mathbb{I}(s) = \sup_{\alpha\in\mathbb{R}} \left(s\alpha - \boldsymbol{F}_\omega(-\alpha)\right), \qquad \mathbb{I}_+(s) = \sup_{\alpha\in\mathbb{R}} \left(s\alpha - \boldsymbol{F}_{\omega_+}(-\alpha)\right). \tag{2.38}$$

By the basic property of the Legendre transform, $\mathbb{I} = \mathbb{I}_+$ iff $\boldsymbol{F}_\omega = \boldsymbol{F}_{\omega_+}$. Note also that

$$\mathfrak{F}_{\omega_T,t}(\alpha) = \int_{\mathcal{X}} \mathrm{e}^{-\alpha c^t(x)} \mathrm{d}\omega_T(x) = \int_{\mathcal{X}} \mathrm{e}^{-\alpha c^t(x)} \mathrm{e}^{\int_0^T \sigma_{-s}(x)\mathrm{d}s} \mathrm{d}\omega(x),$$

and so for all $\alpha \in \mathbb{R}$,

$$C_T^{-1}\mathfrak{F}_{\omega,t}(\alpha) \le \mathfrak{F}_{\omega_T,t}(\alpha) \le C_T \mathfrak{F}_{\omega,t}(\alpha),$$

where $C_T = \mathrm{e}^{T\|\sigma\|_\infty}$. Thus, $\boldsymbol{F}_\omega = \boldsymbol{F}_{\omega_+}$ iff

$$\lim_{T\to\infty}\lim_{t\to\infty} \tfrac{1}{t}\log \mathfrak{F}_{\omega_T,t}(\alpha) = \lim_{t\to\infty}\lim_{T\to\infty} \tfrac{1}{t}\log \mathfrak{F}_{\omega_T,t}(\alpha) \tag{2.39}$$

holds for all $\alpha \in \mathbb{R}$.

In the case of the local weak PREF one cannot invoke Varadhan's lemma. However, the strong PREF postulates the existence of the limits (2.37) and the relations (2.38) remain valid with suprema taken over $\alpha \in]\vartheta_-, \vartheta_+[$. Again, $\mathbb{I} = \mathbb{I}_+$ iff $\boldsymbol{F}_\omega = \boldsymbol{F}_{\omega_+}$ iff (2.39) holds.

The results of this section simplify in the case of discrete time dynamical systems where $\phi = (\phi^t)_{t\in\mathbb{Z}}$. While Assumptions **(Cl1)** and **(Cl2)** remain in force, **(Cl3)** is dropped and the entropy production observable is defined by

$$\sigma := \ell_{\omega_1|\omega} \circ \phi^1.$$

Theorem 2.14 extends directly to the discrete setting, with simplified proofs and time integrals replaced with sums; see [JPRB11]. The same applies to PREFs and the exchange of limits argument.

A celebrated example of this discrete time setting are Anosov diffeomorphisms of compact Riemannian manifolds, see the seminal papers [GC95a, GC95b]. In the context of the classical PREF this example has been discussed in [JPRB11, CJPS17], and the full strong classical PREF holds with $\boldsymbol{F}_{\omega_+} = \boldsymbol{F}_\omega$ equal to the topological pressure of a suitable Hölder continuous function on $\mathcal{X}$. For Anosov diffeomorphisms the map $\mathbb{R} \ni \alpha \mapsto \boldsymbol{F}_\omega(\alpha)$ is real-analytic.

2.7 Quantum phase space contraction

The extension of the notions and results of the previous section to quantum dynamical systems $(\mathcal{O}, \tau, \omega)$ relies on the non-commutative version of the Radon–Nikodym derivative $\frac{\mathrm{d}\omega_t}{\mathrm{d}\omega}$ which, as suggested by our notation, is related to the relative modular operators $\Delta_{\omega_t|\omega}$ and the Araki–Connes family $([D\omega_t : D\omega]_\alpha)_{t\in\mathbb{R},\alpha\in\mathrm{i}\mathbb{R}}$. Besides the general cocycle relation (2.3), the latter also satisfies the following multiplicative τ-cocycle relation

Proposition 2.17 *For all $t, s \in \mathbb{R}$ and $\alpha \in \mathrm{i}\mathbb{R}$,*

$$[D\omega_{t+s} : D\omega]_\alpha = \tau^{-t}([D\omega_s : D\omega]_\alpha)[D\omega_t : D\omega]_\alpha. \tag{2.40}$$

Since we lack a convenient reference, the proof of this result is given in Section 5.4.

To proceed, we make two additional regularity assumptions.

(Qu1) For all $t \in \mathbb{R}$, the map $\mathbb{R} \ni \theta \mapsto [D\omega_t : D\omega]_{\mathrm{i}\theta} \in \mathcal{O}$ is differentiable at $\theta = 0$.

Let

$$\ell_{\omega_t|\omega} := \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}\theta} [D\omega_t : D\omega]_{\mathrm{i}\theta}\Big|_{\theta=0}, \qquad c^t := \tau^t(\ell_{\omega_t|\omega}), \tag{2.41}$$

and note that $\ell_{\omega_t|\omega}$, and consequently c^t, are self-adjoint elements of $\mathcal{O}$.

(Qu2) The map $\mathbb{R} \ni t \mapsto c^t \in \mathcal{O}$ is differentiable at $t = 0$.

Let

$$\sigma = \frac{\mathrm{d}}{\mathrm{d}t} \ell_{\omega_t|\omega}\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} c^t\Big|_{t=0}.$$

The next result has been known to workers in the field for a long time; see for example [JOPS12, Section 7.2]. Since its proof has not appeared in print, we will provide it in Section 5.5.

Theorem 2.18 *Suppose that* **(Qu1)** *holds.*

(1) *The family $(c_t)_{t\in\mathbb{R}}$ is an additive τ-cocycle: for any $s, t \in \mathbb{R}$, one has*

$$c^{t+s} = c^t + \tau^t(c^s). \tag{2.42}$$

(2) $\log \Delta_{\omega_t|\omega} = \log \Delta_\omega + \ell_{\omega_t|\omega}$ *and* $\mathrm{Ent}(\omega_t|\omega) = -\omega(c^t)$ *hold for all* $t \in \mathbb{R}$.

In the remaining statements we assume that **(Qu2)** *also holds.*

(3)

$$c^t = \int_0^t \sigma_s \mathrm{d}s,$$

and

$$\mathrm{Ent}(\omega_t|\omega) = -\int_0^t \omega_s(\sigma) \mathrm{d}s. \tag{2.43}$$

(4) $\omega(\sigma) = 0$.

(5) *If $(\mathcal{O}, \tau, \omega)$ is* TRI *with time-reversal map Θ, then $\Theta(c^t) = c^{-t}$ and $\Theta(\sigma) = -\sigma$.*

Remark 1. σ is called the *entropy production observable* of $(\mathscr{O},\tau,\omega)$ and (2.43) is the *entropy balance equation.* The entropy balance equation has a long history in mathematical physics. It goes back at least to [PW78][13], and was re-introduced independently in the literature several times since then; see [OHI88, Oji89, Oji91]. A basic consequence of the entropy balance equation and the sign of relative entropy is that $\omega_+(\sigma) \geq 0$ for any NESS ω_+.

Remark 2. If $(\mathscr{O},\tau,\omega)$ is an open quantum system with $V \in \mathrm{Dom}(\delta_\omega)$, it follows from [BBJ+24b, Lemma 2.4] that

$$\log \Delta_{\omega_t|\omega} = \log \Delta_\omega + \int_0^t \tau^{-s}(\delta_\omega(V))\mathrm{d}s. \tag{2.44}$$

It is thus easy to check that Assumptions **(Qu1)** and **(Qu2)** hold, and that the entropy production observable is given by $\sigma = \delta_\omega(V)$.

Remark 3. In further parallel with the classical case and following [BK77] one can take the spectral measure $Q^{\mathrm{naive}}_{\omega,t}$ for ω and c^t as a possible candidate for formulation of a quantum Evans–Searls fluctuation theorem. However, for this choice, which is in the literature sometimes called the "naive" or "direct" quantization of the classical $Q_{\omega,t}$, in the TRI case the finite time Evans–Searles fluctuation relation (2.34) *fails*, see *e.g.* [JOPP10, Section 3.3]. It is precisely this failure that motivated the early searches [Kur00, Tas00, TM03] for alternative candidates for a quantum Evans–Searles fluctuation theorem and relation.

Motivated by Proposition 2.17 and Theorem 2.18[14] we consider the map

$$\mathrm{i}\mathbb{R} \ni \alpha \mapsto [D\omega_{-t} : D\omega]_\alpha \in \mathscr{O}$$

as a characterization of the *quantum phase space contraction* of $(\mathscr{O},\tau,\omega)$ at time t, and set[15]

$$\mathfrak{F}^{\mathrm{qpsc}}_{\nu,t}(\alpha) := \nu([D\omega_{-t} : D\omega]_\alpha) \tag{2.45}$$

for $\nu \in \mathscr{S}_{\mathscr{O}}$. When $\nu = \omega$ this functional is linked to the 2TMEP and the EAST protocols by the identities

$$\mathfrak{F}^{\mathrm{qpsc}}_{\omega,t} = \mathfrak{F}^{\mathrm{2tm}}_{\omega,t} = \mathfrak{F}^{\mathrm{ancilla}}_{\omega,t}. \tag{2.46}$$

These identities are broken as soon as ω is replaced by some other state ν.

With the introduction of $\mathfrak{F}^{\mathrm{qpsc}}_{\nu,t}$ it appears natural to add to the strong quantum PREF of Definition 2.10 the requirement that also the limit

$$\boldsymbol{F}^{\mathrm{qpsc}}_{\omega_+}(\alpha) := \lim_{t\to\infty} \frac{1}{t} \log \mathfrak{F}^{\mathrm{qpsc}}_{\omega_+,t}(\alpha)$$

exists for $\alpha \in]\vartheta_-,\vartheta_+[$, and that

$$\boldsymbol{F}^{\mathrm{2tm}}_{\omega} = \boldsymbol{F}^{\mathrm{qpsc}}_{\omega_+}$$

on this interval. We will refer to such PREF as strong + qpsc quantum PREF.[16]

[13] See the Remark on page 281. Another pioneering work on the subject is [SL78].
[14] Compare them with Proposition 2.13 and Theorem 2.14. See also the next section.
[15] Compare with (2.35).
[16] In the sequel, whenever the meaning is clear within the context, we will simply write PREF for any of its variants.

2.8 Comparison of the classical and quantum cases

We start with the algebraic description of the classical setting of Section 2.6. Let $\mathcal{O} = C(\mathcal{X})$ and $\tau^t(f) = f \circ \phi^t$. Obviously, $\mathcal{O}$ is a commutative C^*-algebra and τ a C^*-dynamics on $\mathcal{O}$. If ι is a time-reversal of $(\mathcal{X},\phi)$, $\Theta(f) = \overline{f} \circ \iota$ is the corresponding time-reversal of $(\mathcal{O},\tau)$. The Banach space dual $\mathcal{O}^*$ is identified with the vector space of all Borel complex measures on $\mathcal{X}$ equipped with total variation norm. $\mathcal{S}_{\mathcal{O}}$ coincides with the set of Borel probability measures on $\mathcal{X}$ equipped with topology of weak convergence.

Fixing a reference state leads to a triple $(\mathcal{O},\tau,\omega)$. The GNS representation $(\mathcal{H},\pi,\Omega)$ of $\mathcal{O}$ associated to ω is given by $\mathcal{H} = L^2(\mathcal{X},\mathrm{d}\omega)$, $\pi(f)g = fg$, and $\Omega = \mathbb{1}$, the constant function $\mathbb{1}(x) = 1$ for $x \in \mathcal{X}$. We then have

$$\mathfrak{M} = \pi(\mathcal{O})'' = L^\infty(\mathcal{X},\mathrm{d}\omega).$$

The natural cone is $\mathcal{H}_+ = \{g \in \mathcal{H} \mid g \geq 0\}$ and the modular conjugation is $J(g) = \overline{g}$. The state ω is automatically faithful, and a state ν is ω-normal iff $\nu \ll \omega$. In this case, the unique representative of ν in $\mathcal{H}_+$ is the vector

$$\Omega_\nu = \left(\frac{\mathrm{d}\nu}{\mathrm{d}\omega}\right)^{1/2}.$$

If $\nu \ll \rho$, then

$$\Delta_{\nu|\rho}(g) = \frac{\mathrm{d}\nu}{\mathrm{d}\rho} g$$

is the relative modular operator of the pair (ν,ρ). Note that $\Delta_\omega = \mathbb{1}$ and $\log \Delta_\omega = 0$. The triviality of Δ_ω is the crucial difference between the classical (commutative) and the quantum (non-commutative) case.

The triviality of the classical Δ_ω gives that in the classical setting of Section 2.6, Proposition 2.17 reduces to Proposition 2.13, and similarly Theorem 2.18 reduces to Theorem 2.14(1)-(2)-(3) while Proposition 2.3 and Identities (2.46) give Theorem 2.14(4)-(5). Thus, the quantum phase space construction discussed in Section 2.7 is a natural non-commutative extension of the classical theory described in Section 2.6. But it is not the only possible one: the functionals $\mathfrak{F}^{\mathrm{2tm}}_{\nu,t}$ and $\mathfrak{F}^{\mathrm{ancilla}}_{\nu,t}$ are also non-commutative extensions of the classical $\mathfrak{F}_{\nu,t}$, and in fact one can construct an entire host of mathematically natural non-commutative extensions of $\mathfrak{F}_{\nu,t}$; see [JOPP10, Section 3.3]. Some of these non-commutative extensions have direct quantum mechanical interpretations, like $\mathfrak{F}^{\mathrm{2tm}}_{\nu,t}$ and $\mathfrak{F}^{\mathrm{ancilla}}_{\nu,t}$, and some do not, or such interpretations are not known at the moment.[17]

The diversity of non-commutative/quantum theory of entropic fluctuations stems from the rich KMS structure associated to ω via Δ_ω. Our focus here is on the three non-commutative extensions described by $\mathfrak{F}^{\mathrm{2tm}}_{\nu,t}$, $\mathfrak{F}^{\mathrm{ancilla}}_{\nu,t}$ and $\mathfrak{F}^{\mathrm{qpsc}}_{\nu,t}$, and their relation to what one can reasonably call the quantum extension of the classical Evans–Searles/Gallavotti–Cohen fluctuation theorem and the classical PREF. Other routes are possible, and we will discuss some of them in a forthcoming review paper. The emerging picture is that there is no unique theory of entropy production and entropic fluctuations in quantum statistical mechanics and that distinct approaches, with distinct physical interpretations, are linked to the richness of the modular structure.

[17] For example, if $\nu \neq \omega$, then to the best of our knowledge, the functional $\mathfrak{F}^{\mathrm{qpsc}}_{\nu,t}$ does not have a quantum mechanical interpretation.

Returning to the results presented so far, we make the following remarks.

1. The rigidity result of Theorem 2.1 is due to the ergodicity of the modular group(s)[18] and has no classical analog. To illustrate this, suppose that in the classical setting of Section 2.6 one has $Q_{\omega_T,t} = Q_{\omega,t}$ for all $t, T \geq 0$. This implies $\omega(c^t \circ \phi^T) = \omega(c^t)$ and the cocycle relation of Proposition 2.13 gives

$$\omega(c^{t+T}) - \omega(c^T) = \omega(c^t).$$

Dividing this identity with t and taking $t \to 0$ gives that, for all $T > 0$, $\omega(\sigma_T) = \omega(\sigma) = 0$. This in turns implies that $\omega(c^t) = 0$ for all $t > 0$, and the second assertion in Proposition 2.13 gives that $\mathrm{Ent}(\omega_t|\omega) = 0$, or equivalently that $\omega_t = \omega$, for all $t > 0$. Thus, in the classical setting the stability result of Theorem 2.1(1) is possible only if all entropic quantities are identically equal to zero.[19]

2. The weak classical PREF relies on two independent ingredients. The first is the validity of the LDP for the families $(P_{\omega,t})_{t>0}$ and $(P_{\omega_+,t})_{t>0}$, and the second is the equality of the respective rate functions. In the case of the full weak classical PREF, the rate functions are equal iff the exchange of limits (2.39) holds. The validity of this exchange of limits is a strong ergodic type dynamical property of $(\mathscr{X}, \phi, \omega)$ which must be checked on a case by case basis and which typically depends on the fine details of the dynamics. The strong classical PREF goes further in the sense that it rests the validity of the LDP on the Gärtner–Ellis theorem. Its naturalness partly stems from the interpretation of $\boldsymbol{F}_\omega$ and $\boldsymbol{F}_{\omega_+}$ as spectral resonances of classical transfer operators ; see [JPRB11] and Section 4.4 below.

The passage to the non-commutative/quantum theory comes with a number of surprises that do not have classical analog. The first of them is the rigidity result of Theorem 2.1 that essentially trivializes a very important aspect of the PREF and gives that very generally the quantum Evans–Searles and Gallavotti–Cohen fluctuation theorem are mathematically equivalent statement: one holds iff the other holds. What is classically a fine model dependent dynamical property of the system, in the quantum case follows (in a rather strong form) from a modular ergodicity assumption that holds in paradigmatic models of open quantum systems. The novel physical and mathematical aspect of the strong quantum PREF concerns the ancilla part. The relation

$$\boldsymbol{F}_\omega^{\mathrm{2tm}} = \boldsymbol{F}_{\omega_+}^{\mathrm{ancilla}} \tag{2.47}$$

is a fine model dependent quantum dynamical property that can be seen as a non-trivial counterpart of the classical PREF relation $\boldsymbol{F}_\omega = \boldsymbol{F}_{\omega_+}$. The exchange of limits characterization of (2.47), Relation (2.32), parallels in its depth the exchange of limits (2.39). We again emphasize that under the assumptions of Theorem 2.1, the exchange of limits

$$\lim_{T\to\infty}\lim_{t\to\infty}\frac{1}{t}\log\mathfrak{F}_{\omega_T,t}^{\mathrm{2tm}}(\alpha) = \lim_{t\to\infty}\lim_{T\to\infty}\frac{1}{t}\log\mathfrak{F}_{\omega_T,t}^{\mathrm{2tm}}(\alpha)$$

is a triviality.

3. The identities

$$\mathfrak{F}_{\omega,t}^{\mathrm{2tm}} = \mathfrak{F}_{\omega,t}^{\mathrm{ancilla}} = \mathfrak{F}_{\omega,t}^{\mathrm{qpsc}} \tag{2.48}$$

[18] See [BBJ+23] for a discussion of this point.

[19] Needless to say, the ergodicity assumption of Theorem 2.1(1) is also never satisfied in the classical case since the modular group is trivial.

enrich the quantum Evans–Searles fluctuation theorem by providing additional physical and mathematical interpretations of the 2TMEP. The identities (2.48) are broken if ω is replaced by some other state ν. However, while $\mathfrak{F}_{\nu,t}^{2\mathrm{tm}}$ and $\mathfrak{F}_{\nu,t}^{\mathrm{ancilla}}$ have quantum mechanical interpretations for any state ν, such an interpretation is lacking for $\mathfrak{F}_{\nu,t}^{\mathrm{qpsc}}$ if $\nu \neq \omega$. Thus, although the identities (2.48) are restored by the strong + qpsc quantum PREF on the LDP scale for $\nu = \omega_+$ in the sense that

$$\boldsymbol{F}_{\omega}^{2\mathrm{tm}} = \boldsymbol{F}_{\omega_+}^{2\mathrm{tm}} = \boldsymbol{F}_{\omega_+}^{\mathrm{ancilla}} = \boldsymbol{F}_{\omega_+}^{\mathrm{qpsc}}, \tag{2.49}$$

the last term $\boldsymbol{F}_{\omega_+}^{\mathrm{qpsc}}$ lacks a physical interpretation. The first equality in (2.49) is immediate under the assumptions of Theorem 2.1, while the validity of $\boldsymbol{F}_{\omega}^{2\mathrm{tm}} = \boldsymbol{F}_{\omega_+}^{\mathrm{qpsc}}$ is mathematically subtle but, we emphasize, without physical interpretation. It is precisely the ancilla part in (2.49) that makes the strong + qpsc quantum PREF both mathematically and physically as deep as its classical counterpart.

Due to its mathematical naturalness, we feel that the quantum phase space contraction should be studied as an integral part of quantum theory of entropic fluctuations.

4. After this introduction, in the next two sections we will describe quantum transfer operators and the spectral resonance theory of the strong quantum PREF. These two sections are a non-commutative extension of the classical theory described in [JPRB11].They build on the quantum transfer operator construction of NESS developed in [JP02b]; see also [JOPP10] for a pedagogical introduction to the topic. For comparison purposes, we will briefly review the classical theory in Section 4.4. The transfer operator connection supports the naturalness of the strong PREFs.

3 Quantum transfer operators

We continue with the same setting: $(\mathcal{O}, \tau, \omega)$ is a modular C^*-quantum dynamical system whose modular structure is described in Section 2.1. In particular, $(\mathcal{H}, \pi, \Omega)$ denotes the GNS representation of $\mathcal{O}$ associated to ω, and we write A for $\pi(A)$ and τ for $\pi \circ \tau$ whenever the meaning is clear within the context. $\mathfrak{M} = \pi(\mathcal{O})''$ and $\mathcal{L}$ is the standard Liouvillean of τ in the representation π.

Throughout this section Assumptions **(AnV(ϑ))** and **(AnC(ϑ))** play an important role. Recall that, by Proposition 2.9, **(AnV(ϑ))** $\Rightarrow$ **(AnC(ϑ))**.

3.1 One-parameter families of Liouvillians

Throughout this section we assume that **(AnC(ϑ))** holds. For $\alpha \in \mathfrak{S}(\vartheta)$ we define the family $U_\alpha = (U_\alpha^t)_{t\in\mathbb{R}}$ of maps on the GNS Hilbert space $\mathcal{H}$ by

$$U_\alpha^t := \mathrm{e}^{\mathrm{i}t\mathcal{L}} J [D\omega_t : D\omega]_{\overline{\alpha}} J. \tag{3.1}$$

The following proposition is proved in Section 5.6.

Proposition 3.1 (1) *For $\alpha \in \mathfrak{S}(\vartheta)$, U_α is a C_0-group of bounded operators on $\mathcal{H}$.*

(2) *For $\alpha \in \mathrm{i}\mathbb{R}$, U_α is a strongly continuous unitary group.*

(3) *For all* $A \in \mathfrak{M}$, $t \in \mathbb{R}$ *and* $\alpha \in \mathfrak{S}(\vartheta)$

$$U_\alpha^t A U_{-\overline{\alpha}}^{t*} = \tau^t(A).$$

(4) *For all* $t \in \mathbb{R}$ *and* $\alpha \in \mathfrak{S}(\vartheta)$, $U_\alpha^{t*} = U_{-\overline{\alpha}}^{-t}$.

We denote by $\mathscr{L}_\alpha$ *the generator of* U_α *with the convention* $U_\alpha^t = \mathrm{e}^{\mathrm{i}t\mathscr{L}_\alpha}$.

(5) *Suppose that* **(AnV(ϑ))** *holds. Then, for all* $\alpha \in \mathfrak{S}(\vartheta)$,

$$\begin{aligned} \mathscr{L}_\alpha &= \mathscr{L} + JVJ - J\varsigma_\omega^{-\mathrm{i}\overline{\alpha}}(V)J \\ &= \mathscr{L}_{\mathrm{fr}} + V - J\varsigma_\omega^{-\mathrm{i}\overline{\alpha}}(V)J. \end{aligned} \tag{3.2}$$

Remark 1. It follows from (1) that there exist constants M_α and m_α such that for $t \in \mathbb{R}$,

$$\|U_\alpha^t\| \le M_\alpha \mathrm{e}^{m_\alpha |t|}.$$

If **(AnV(ϑ))** holds then, by Proposition 2.11, one can take $M_\alpha = 1$ and

$$m_\alpha = \|\varsigma_\omega^{-\mathrm{i}\,\mathrm{Re}\,\alpha}(V)\| + \|V\|.$$

Remark 2. For later comparison with the classical case, we note that the first formula in (3.2) can be written as

$$\mathscr{L}_\alpha = \mathscr{L} - \mathrm{i}\alpha \int_0^1 J\varsigma_\omega^{-\mathrm{i}s\overline{\alpha}}(\sigma)J\mathrm{d}s.$$

This formula is valid under more general conditions than **(AnV(ϑ))**. We leave this topic to an interested reader.

We will refer to the U_α^t's as the Quantum Transfer Operators, and to $\mathscr{L}_\alpha$ as the α-Liouvillian of $(\mathscr{O}, \tau, \omega)$. Obviously, $\mathscr{L}_0 = \mathscr{L}$, $\mathscr{L}_\alpha$ is self-adjoint for $\alpha \in \mathrm{i}\mathbb{R}$, and more generally $\mathscr{L}_\alpha^* = \mathscr{L}_{-\overline{\alpha}}$. Writing $\alpha = \frac{1}{2} - \frac{1}{p}$ with $p \in [1, \infty]$, one can show that $\mathscr{L}_\alpha$ isometrically implements the dynamics τ in the Araki–Masuda non-commutative $L^p(\mathfrak{M}, \omega)$-space (see [JOPP10, Section 3.4]). For this reason $\mathscr{L}_\alpha$ is also called the L^p-Liouvillean.

If $\vartheta > 1/2$ and $\alpha \in \mathrm{i}\mathbb{R}$, we also set

$$\widehat{\mathscr{L}}_\alpha = \mathscr{L}_{\mathrm{fr}} + \varsigma_\omega^{\mathrm{i}\alpha/2}(V) - J\varsigma_\omega^{-\mathrm{i}(1-\overline{\alpha})/2}(V)J. \tag{3.3}$$

The naturalness of $\widehat{\mathscr{L}}_\alpha$ stems from its connection with EAST which we describe in the next section. Note that

$$\widehat{\mathscr{L}}_\alpha = \Delta_\omega^{-\alpha/2} \mathscr{L}_{1/2-\alpha} \Delta_\omega^{\alpha/2}. \tag{3.4}$$

3.2 Liouvillian representation of entropic functionals

The representations described below, and proved in Section 5.7, are the primary reasons for introducing the α-Liouvilleans in the context of this work.

Proposition 3.2 *Suppose that* **(AnV(ϑ))** *holds for some* $\vartheta > \frac{1}{2}$*. Then, for* $\alpha \in \mathrm{i}\mathbb{R}$ *and* $t, T \in \mathbb{R}$*:*

(1)
$$\mathfrak{F}^{\rm 2tm}_{\omega,t}(\alpha) = \langle \Omega, \mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega\rangle.$$

(2)
$$\mathfrak{F}^{\rm qpsc}_{\omega_T,t}(\alpha) = \langle \Omega, \mathrm{e}^{\mathrm{i}T\mathscr{L}_{1/2}}\mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega\rangle.$$

(3)
$$\mathfrak{F}^{\rm ancilla}_{\omega_T,t}(\alpha) = \langle \Omega, \mathrm{e}^{\mathrm{i}T\mathscr{L}_{1/2}}\mathrm{e}^{\mathrm{i}t\widehat{\mathscr{L}}_{\alpha}}\Omega\rangle.$$

(4) *The relation in part (1) analytically extends to the strip* $|\mathrm{Re}(\alpha-\frac{1}{2})| < \vartheta$*, the one in part (2) extends to the strip* $\frac{1}{2}-\vartheta < \mathrm{Re}(\alpha) < \vartheta$*, while the one in part (3) extends to the strip* $1-2\vartheta < \mathrm{Re}(\alpha) < 2\vartheta$*. We note that the first and last strip contain the real interval* $[0,1]$*.*

4 Spectral resonance theory of PREF

4.1 Prologue

The exponential asymptotics of a C_0-group is closely linked to the complex resonances of its generator. We start with a discussion of two general results elucidating this relation.

Let $U = (U^t)_{t\in\mathbb{R}}$ be a C_0-group on a Hilbert space $\mathfrak{H}$ and $M, m > 0$ be such that

$$\|U^t\| \le M\mathrm{e}^{m|t|} \tag{4.1}$$

for all $t \in \mathbb{R}$. We denote by L the generator of U with the convention $U^t = \mathrm{e}^{-\mathrm{i}tL}$. The estimate (4.1) gives that $\mathrm{sp}(L) \subset \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| \le m\}$. Moreover, for $\mathrm{Im}\, z > m$ one has

$$(z-L)^{-1} = \frac{1}{\mathrm{i}}\int_0^\infty \mathrm{e}^{\mathrm{i}zt}\mathrm{e}^{-\mathrm{i}tL}\mathrm{d}t,$$

which gives that

$$\|(z-L)^{-1}\| \le \frac{M}{|\mathrm{Im}\, z| - m}.$$

Proposition 4.1 *Let* $\phi, \psi \in \mathfrak{H}$*. Suppose that, for some* $\mu > 0$*, the map*

$$z \mapsto f(z) = \langle \phi, (z-L)^{-1}\psi\rangle,$$

originally defined for $\mathrm{Im}\, z > m$*, has a meromorphic continuation to the half-plane* $\mathrm{Im}\, z > -\mu$ *such that its only singularity in this half-plane is a pole of order* N *at* $\mathfrak{r}$*. Suppose also that for some* $r > 0$ *and for* $j = 0, 1,$[20]

$$\sup_{y>-\mu}\int_{|x|>r} |(\partial^j f)(x+\mathrm{i}y)|^{2-j}\mathrm{d}x < \infty. \tag{4.2}$$

[20] Here and in the following, ∂ denotes the Wirtinger derivative of a function of a complex variable.

Then, for all γ *satisfying* $\max(-\operatorname{Im}\mathfrak{r}, 0) < \gamma < \mu$, *we have*

$$\langle \phi, \mathrm{e}^{-\mathrm{i}tL}\psi\rangle = \mathrm{e}^{-\mathrm{i}t\mathfrak{r}}p(t) + O(\mathrm{e}^{-\gamma t})$$

as $t \uparrow \infty$, *where* p *is a polynomial of degree* $N-1$ *such that* $p(0)$ *is the residue of* f *at* $\mathfrak{r}$. *In particular, if* $N = 1$ *then* p *is constant equal to this residue.*

The number $\mathfrak{r}$ is called spectral resonance of the generator L.

The above proposition has a converse:

Proposition 4.2 *Let* $\phi, \psi \in \mathfrak{H}$ *be such that for some* $\mathfrak{r} \in \mathbb{C}$, $\gamma > \max(0, -\operatorname{Im}\mathfrak{r})$ *and some polynomial* p *of degree* $N - 1 \geq 0$ *one has*

$$R(t) = \langle \phi, \mathrm{e}^{-\mathrm{i}tL}\psi\rangle - \mathrm{e}^{-\mathrm{i}t\mathfrak{r}}p(t) = O(\mathrm{e}^{-\gamma t})$$

as $t \uparrow \infty$. *Then the function*

$$z \mapsto f(z) = \langle \phi, (z - L)^{-1}\psi\rangle$$

has a meromorphic continuation from the half-plane $\operatorname{Im} z > m$ *to the half-plane* $\operatorname{Im} z > -\gamma$, *and its only singularity there is a pole of order* N *at* $\mathfrak{r}$. *Moreover, for some* $r > 0$ *and any* $0 < \mu < \gamma$ *the estimate*

$$\sup_{y>-\mu} \int_{|x|>r} |(\partial^j f)(x + \mathrm{i}y)|^{2-j}\mathrm{d}x < \infty \tag{4.3}$$

holds for $j = 0$. *If* R *is twice differentiable, with derivatives satisfying*

$$|R^{(k)}(t)| = O(\mathrm{e}^{-\gamma t}), \qquad (k = 1, 2),$$

as $t \uparrow \infty$, *then the estimate* (4.3) *also holds for* $j = 1$.

Special cases of the above two results were formulated in [JPRB11, Section 5.4]. We provide proofs in Sections 5.8 and 5.9. The results have obvious analogs in the case of convention $U^t = \mathrm{e}^{\mathrm{i}tL}$, where the meromorphic continuation of the resolvent is taken from the half-plane $\operatorname{Im} z < -m$ to $\operatorname{Im} z < \mu$ for some $\mu > 0$. Alternatively, with this convention one can simply apply Propositions 4.1 and 4.2 to $-L$, and this is the way we will proceed.

Proposition 3.2 and 4.1 offer a spectral resonance route to the verification of the strong PREF. The route is however somewhat indirect, and we start by reviewing the spectral theory of NESS developed in [JP02b].

4.2 Spectral theory of NESS

Throughout this section we assume that Assumption **(AnV(ϑ))** holds with $\vartheta > 1/2$. Central to the spectral theory of NESS is the $\frac{1}{2}$-Liouvillean $\mathscr{L}_{1/2}$. Note that

$$\mathscr{L}_{1/2} = \mathscr{L}_{\mathrm{fr}} + V - J\Delta^{1/2}V\Delta^{-1/2}J,$$

hence $\mathscr{L}_{1/2}\Omega = 0$. Proposition 3.1(3)-(4) give that, for $A \in \mathscr{O}$,

$$\omega \circ \tau^t(A) = \langle \Omega, \mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2}}A\mathrm{e}^{-\mathrm{i}t\mathscr{L}_{1/2}}\Omega\rangle = \langle \Omega, \mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2}}A\Omega\rangle. \tag{4.4}$$

The next assumption sets an abstract spectral deformation scheme that leads to the spectral theory of NESS.

(Deform1) There exists a bounded operator $D \geq 0$ on $\mathscr{H}$ such that $\operatorname{Ran} D$ is dense in $\mathscr{H}$, $D\Omega = \Omega$, and that the following holds:

(a) The set $\mathscr{O}_D = \{A \in \mathscr{O} \mid A\Omega \in \operatorname{Dom}(D^{-1})\}$ is dense in $\mathscr{O}$.

(b) The map
$$z \mapsto F(z) = D(z + \mathscr{L}_{1/2})^{-1} D \in \mathscr{B}(\mathscr{H}), \tag{4.5}$$
originally defined for $\operatorname{Im} z > m_{1/2}$, has a meromorphic continuation to a half-plane $\operatorname{Im} z > -\mu$ for some $\mu > 0$ such that its only singularity in this half-plane is a simple pole at zero with residue $\mathscr{R}_{1/2}$.

(c) For some $r > 0$ and $j = 0, 1$,
$$\sup_{y > -\mu} \int_{|x|>r} \|\partial^j F(x + \mathrm{i}y)\|^{2-j} \mathrm{d}x < \infty.$$

Note that since $\Omega \in \ker \mathscr{L}_{1/2}$, the singularity at 0 of the map (4.5) is forced by the relation
$$\langle \Omega, (z + \mathscr{L}_{1/2})^{-1} \Omega \rangle = \frac{1}{z}.$$

An immediate consequence of Proposition 4.1 and (4.4) is the following result of [JP02b]:

Theorem 4.3 *Suppose that* **(Deform1)** *holds. Then the limit*
$$\omega_+(A) = \lim_{t \to \infty} \omega \circ \tau^t(A) \tag{4.6}$$
exists for all $A \in \mathscr{O}$. For $A \in \mathscr{O}_D$ the convergence is exponentially fast, i.e.,
$$|\omega_+(A) - \omega \circ \tau^t(A)| = O(\mathrm{e}^{-\gamma t}) \tag{4.7}$$
as $t \uparrow \infty$ for any $0 < \gamma < \mu$, and
$$\omega_+(A) = \langle \Omega, \mathscr{R}_{1/2} D^{-1} A\Omega \rangle. \tag{4.8}$$

Proof. Note that by **(Deform1)**(b+c) Proposition 4.1 holds for all $\phi, \psi \in \operatorname{Ran} D$ with $L = -\mathscr{L}_{1/2}$, $\mathfrak{r} = 0$, and constant polynomial
$$p = \langle D^{-1}\phi, \mathscr{R}_{1/2} D^{-1} \psi \rangle.$$
By **(Deform1)**(a), $A\Omega = DD^{-1}A\Omega \in \operatorname{Ran} D$ for $A \in \mathscr{O}_D$, and so (4.4) and Proposition 4.1 yield (4.7) and (4.8). Since $\mathscr{O}_D$ is dense in $\mathscr{O}$, (4.7) $\Rightarrow$ (4.6). □

4.3 Spectral theory of PREF

We continue to assume that **(AnV(ϑ))** holds with $\vartheta > 1/2$. For $\zeta > 0$ let
$$B(\vartheta, \zeta) := \{z \in \mathbb{C} \mid |\operatorname{Re} z| < \vartheta, \text{ and } |\operatorname{Im} z| < \zeta\}.$$

We strengthen **(Deform1)** to

(Deform2) Assumption **(Deform1)** holds and there exists $\zeta > 0$ such that:

(a) For all $\alpha \in B(\vartheta,\zeta)$ and all $t \in \mathbb{R}$, $[D\omega_t : D\omega]_\alpha \in \mathscr{O}_D$.

(b) For all $\alpha \in B(\vartheta,\zeta)$ and all $t \in \mathbb{R}$, $[D\omega_t : D\omega]^*_{\frac{\alpha}{2}}[D\omega_t : D\omega]_{\frac{\alpha}{2}} \in \mathscr{O}_D$.

(c) For all $\alpha \in B(\vartheta,\zeta)$ there exists $\mu_\alpha > 0$ such that the map

$$z \mapsto F_\alpha(z) = D(z + \mathscr{L}_\alpha)^{-1}D \in \mathscr{B}(\mathscr{H}),$$

originally defined on the half-plane $\operatorname{Im} z > m_\alpha$, has a meromorphic continuation to the half-plane $\operatorname{Im} z > -\mu_\alpha$ whose only singularity in this half-plane is a pole at $\mathscr{E}(\alpha)$ with residue $\mathscr{R}_\alpha$.

(d) For all $\alpha \in B(\vartheta,\zeta)$ there exists $r_\alpha > 0$ such that, for $j = 0,1$ and all $\phi,\psi \in \mathscr{H}$,

$$\sup_{y>-\mu_\alpha} \int_{|x|>r_\alpha} |\langle \phi, (\partial^j F_\alpha)(x + \mathrm{i}y)\psi\rangle|^{2-j}\mathrm{d}x < \infty.$$

(e) $\mathscr{R}^*_{1/2}\Omega \in \operatorname{Dom}(D^{-2})$.

(f) For all $\alpha \in B(\vartheta,\zeta)$, $D^{-1}\mathscr{R}^*_{1/2}\Omega \in \operatorname{Dom}(\Delta^{\alpha/2})$ and $\Delta^{\alpha/2}D^{-1}\mathscr{R}^*_{1/2}\Omega \in \operatorname{Dom}(D^{-1})$.

(g) For all $\alpha \in B(\vartheta,\zeta)$,

$$\langle \Omega, \mathscr{R}_\alpha\Omega\rangle \neq 0, \quad \langle D^{-1}\mathscr{R}^*_{1/2}\Omega, \mathscr{R}_\alpha\Omega\rangle \neq 0, \quad \langle D^{-1}\Delta^{-\overline{\alpha}/2}D^{-1}\mathscr{R}^*_{1/2}\Omega, \mathscr{R}_\alpha\Omega\rangle \neq 0.$$

Remark. Note that (f) with $\alpha = 0$ reduces to (e). We have separated the two assumptions because of their roles in the proof, and because of possible alternative axiomatic schemes in which (f) is bypassed; see Remark 3 at the end of this section.

Theorem 4.4 *Under Assumption* **(Deform2)** *the following hold:*

(1) $\mathscr{E}(\alpha) \in \mathrm{i}\mathbb{R}$ *for* $\alpha \in]-\vartheta,\vartheta[$.

(2) *The limits*

$$\begin{aligned} \boldsymbol{F}^{\mathrm{2tm}}_{\omega}(\alpha) &= \lim_{t\to\infty} \tfrac{1}{t}\log \mathfrak{F}^{\mathrm{2tm}}_{\omega,t}(\alpha), \\ \boldsymbol{F}^{\mathrm{ancilla}}_{\omega_+}(\alpha) &= \lim_{t\to\infty} \tfrac{1}{t}\log \mathfrak{F}^{\mathrm{ancilla}}_{\omega_+,t}(\alpha), \\ \boldsymbol{F}^{\mathrm{qpsc}}_{\omega_+}(\alpha) &= \lim_{t\to\infty} \tfrac{1}{t}\log \mathfrak{F}^{\mathrm{qpsc}}_{\omega_+,t}(\alpha), \end{aligned} \tag{4.9}$$

exist for $\alpha \in]-\vartheta + \frac{1}{2},\vartheta[$[21]*, and for* α *in this interval,*

$$\boldsymbol{F}^{\mathrm{2tm}}_{\omega}(\alpha) = \boldsymbol{F}^{\mathrm{ancilla}}_{\omega_+}(\alpha) = \boldsymbol{F}^{\mathrm{qpsc}}_{\omega_+}(\alpha) = -\mathrm{i}\mathscr{E}\left(\tfrac{1}{2} - \alpha\right).$$

[21] The first limit actually exists for $\alpha \in]-\vartheta + \frac{1}{2},\vartheta + \frac{1}{2}[$.

(3) *If in addition the assumptions of Theorem 2.1 are satisfied*[22]*, then also*

$$\boldsymbol{F}^{2\mathrm{tm}}_{\omega_+}(\alpha) = \lim_{t\to\infty}\frac{1}{t}\log\mathfrak{F}^{2\mathrm{tm}}_{\omega_+,t}(\alpha) \tag{4.10}$$

exists for $\alpha \in]-\vartheta+\frac{1}{2},\vartheta[$ *and*

$$\boldsymbol{F}^{2\mathrm{tm}}_{\omega_+}(\alpha) = -\mathrm{i}\mathscr{E}\left(\tfrac{1}{2}-\alpha\right). \tag{4.11}$$

Remark. The proof gives that (4.9) holds for all $\alpha\in\mathbb{C}$ such that $-\vartheta+\frac{1}{2}<\mathrm{Re}\,\alpha<\vartheta$ and $|\mathrm{Im}\,\alpha|<\zeta$.

We emphasize that (4.10) and (4.11) follow from Theorem 2.1 and the induced equality $\boldsymbol{F}^{2\mathrm{tm}}_{\omega}=\boldsymbol{F}^{2\mathrm{tm}}_{\omega_+}$, and do not require a proof.

To establish strong + qpsc PREF, Theorem 4.4 needs to be complemented with a study of the regularity of $\mathscr{E}$ on $(-\vartheta,\vartheta)$, and indeed the axiomatic scheme **(Deform2)** can be further strengthened to also yield the analyticity of $\mathscr{E}$ on $B(\vartheta,\zeta)$.

(Deform3) Assumption **(Deform2)** holds for some $\zeta>0$ such that any $\alpha_0\in B(\vartheta,\zeta)$ has a neighborhood $U\subset B(\vartheta,\zeta)$ with the following properties:

(a) For $\alpha\in U$, Conditions **(Deform2)**(c+d) hold with constants $r_\alpha=r_U$ and $\mu_\alpha=\mu_U$ which do not depend on α.

(b)
$$\max_{j\in\{0,1\}}\sup_{\alpha\in U}\sup_{y>-\mu_U}\int_{|x|>r_U}|\langle\phi,(\partial^j F_\alpha)(x+\mathrm{i}y)\psi\rangle|^{2-j}\,\mathrm{d}x<\infty.$$

(c) For any $\epsilon>0$ there exists $C_\epsilon>0$ such that

$$\sup_{\alpha\in U}\sup_{\substack{\mathrm{Im}\,z>-\mu\\|z-\mathscr{E}(\alpha)|>\epsilon}}\|F_\alpha(z)\|<C_\epsilon.$$

(d) $\inf_{\alpha\in U_0}|\langle\Omega,\mathscr{R}_\alpha\Omega\rangle|>0$.

Theorem 4.5 *Suppose that* **(Deform3)** *holds. Then, in addition to the conclusions of Theorem 4.4, the map*

$$B(\vartheta,\zeta)\ni\alpha\mapsto\mathscr{E}(\alpha)$$

is analytic. In particular strong + qpsc PREF holds on $]\frac{1}{2}-\vartheta,\vartheta[$.

We finish with four remarks.

Remark 1. As already mentioned, the spectral resonance approach to the PREF described in this section stems from Propositions 3.2 and 4.1, and the axiomatic schemes **(Deform1)**-**(Deform3)** are natural in view of the spectral deformation techniques developed in the theory of Schrödinger operators [AC71, BC71, Sim73, HP83, Hun90], see also [CFKS87, Chapter 8], [RS78, Sections XII.6 and XII.10].

[22] Recall also the remark after this theorem; for Part (3) we require that $\omega_{+\mathrm{S}}>0$.

The schemes can be simplified in particular settings where additional structural information is available, like the Spin–Fermion model, or adjusted to the settings where they cannot be directly applied, like the Spin–Boson model where V is unbounded.

Remark 2. Since $\mathscr{L}_\alpha$ is an analytic family of type A, see *e.g.* [Kat66, RS78], for $\alpha \in \mathfrak{S}(\vartheta)$, it is natural to expect that in concrete models for which **(Deform2)** holds, the proof also gives that $\mathscr{E}$ is analytic on $B(\vartheta,\zeta)$. This is indeed the case for Spin–Fermion systems which we study in a continuation of this work [BBJ+24a].

Remark 3. Proposition 3.2 (3) offers a parallel spectral resonance approach to the ancilla part of PREF. It starts by formulating a variant of **(Deform2)** for the family $\widehat{\mathscr{L}}_\alpha$:

(Deform2A) Assumption **(Deform1)** holds and there exists $\zeta > 0$ such that:

(a) For all $\alpha \in B(\vartheta,\zeta)$ and all t,

$$[D\omega_t : D\omega]^*_{\frac{\overline{\alpha}}{2}} [D\omega_t : D\omega]_{\frac{\alpha}{2}} \in \mathscr{O}_D.$$

(b) For all $\alpha \in B(\vartheta,\zeta)$ there exists $\mu_\alpha > 0$ such that the function

$$z \mapsto \widehat{F}_\alpha(z) = D(z + \widehat{\mathscr{L}}_\alpha)^{-1} D \in \mathscr{B}(\mathscr{H}),$$

originally defined for $\operatorname{Im} z > \widehat{m}_\alpha$, has a meromorphic continuation to the half-plane $\operatorname{Im} z > -\mu_\alpha$ such that its only singularity in this half-plane is a pole at $\widehat{\mathscr{E}}(\alpha)$ with residue $\widehat{\mathscr{R}}_\alpha$.

(c) For all $\alpha \in B(\vartheta,\zeta)$ there exists $r_\alpha > 0$ such that for $j = 0,1$ and all $\phi,\psi \in \mathscr{H}$,

$$\sup_{y > -\mu_\alpha} \int_{|x|>r_\alpha} |\langle \phi, (\partial^j \widehat{F}_\alpha)(x + \mathrm{i}y)\psi\rangle|^{2-j} \mathrm{d}x < \infty.$$

(d) $\widehat{\mathscr{R}}^*_{1/2}\Omega \in \operatorname{Dom}(D^{-2})$.

(e) $\langle D^{-1}\widehat{\mathscr{R}}^*_{1/2}\Omega, \widehat{\mathscr{R}}_\alpha\Omega\rangle \neq 0$.

Proposition 3.2 and **(Deform2A)** yield that for $\alpha \in]-\vartheta,\vartheta[$

$$\lim_{t\to\infty} \frac{1}{t} \log \mathfrak{F}^{\text{ancilla}}_{\omega_+,t}(\alpha) = -\mathrm{i}\widehat{\mathscr{E}}(\alpha),$$

with exactly the same proof as in Theorem 4.4. One then uses (3.4) to show that $\widehat{\mathscr{E}}(\alpha) = \mathscr{E}(\frac{1}{2} - \alpha)$. This eliminates the need to verify **(Deform2)**(f) and the third relation in **(Deform2)**(g), and potentially offers a technically simpler route for the verification of the ancilla part of the PREF. This again will be the case for the Spin–Fermion model.

Remark 4. In summary, **(Deform1)**–**(Deform3)** should be viewed as an adjustable frame for the study of PREF in concrete models. As one may expect given the information they yield, it is technically involved to verify these axiomatic schemes in physically relevant models. There is a strong parallel between the proposed spectral resonance approach to PREF and the use of transfer operators in classical dynamical systems, and we will comment more on this point in the next two sections.

4.4 Classical dynamical systems

We return to the setting of Section 2.6 and assume that Assumptions **(Cl1)**, **(Cl2)** and **(Cl3)** hold. We will work in the classical algebraic setting described in Section 2.8 and slightly adjust the presentation of the classical transfer operator theory of [JPRB11] for easier comparison with general Liouvilleans introduced here.

By Assumption **(Cl1)** the Koopman map $U^t_{\rm Koop} f = f_t$ extends from $C(\mathscr{X})$ to a C_0-group of bounded operators on the GNS Hilbert space $\mathscr{H} = L^2(\mathscr{X}, \mathrm{d}\omega)$ which we denote by the same symbol. The standard Liouvillean $\mathscr{L}$ of the classical system $(\mathscr{O}, \tau, \omega)$ is the generator of the C_0-group

$$U^t_{\rm Liouv} f = U^t_{\rm Koop}\pi\left(\frac{\mathrm{d}\omega_t}{\mathrm{d}\omega}\right)^{-\frac{1}{2}} f = \pi(\mathrm{e}^{-\frac{1}{2}\int_0^t \sigma_s \mathrm{d}s}) U^t_{\rm Koop} f,$$

with the usual convention $U^t_{\rm Liouv} = \mathrm{e}^{\mathrm{i}t\mathscr{L}}$. One easily checks that $U_{\rm Liouv}$ is a unitary group on $\mathscr{H}$ that preserves the positive cone $\mathscr{H}_+$ and implements the classical flow,

$$U^t_{\rm Liouv}\pi(f) U^{t*}_{\rm Liouv} = \pi(f_t).$$

For $\alpha \in \mathbb{C}$ we define maps $U^t_\alpha : \mathscr{H} \to \mathscr{H}$ by

$$U^t_\alpha f := \mathrm{e}^{\mathrm{i}t\mathscr{L}} J\pi\left(\frac{\mathrm{d}\omega_t}{\mathrm{d}\omega}\right)^{\overline{\alpha}} Jf,$$

which is of course the commutative case of (3.1). Note that

$$U^t_\alpha f = \pi\left(\mathrm{e}^{(\alpha - 1/2)\int_0^t \sigma_s \mathrm{d}s}\right) U^t_{\rm Koop} f.$$

Proposition 4.6 (1) *$U_\alpha := (U^t_\alpha)_{t\in\mathbb{R}}$ is a C_0-group of bounded operators on $\mathscr{H}$. If $\alpha \in \mathrm{i}\mathbb{R}$, then U_α is a unitary group.*

(2) *For all $f \in \mathscr{O}$, $t \in \mathbb{R}$ and $\alpha \in \mathbb{C}$,*

$$U^t_\alpha \pi(f) U^{t*}_{-\overline{\alpha}} = \pi(f_t).$$

(3) *For all $t \in \mathbb{R}$ and $\alpha \in \mathbb{C}$, $U^{t*}_\alpha = U^{-t}_{-\overline{\alpha}}$.*

(4) *$U_0 = U_{\rm Liouv}$ and $U_{1/2} = U_{\rm Koop}$.*

We denote by $\mathscr{L}_\alpha$ the generator of U_α with the convention $U^t_\alpha = \mathrm{e}^{\mathrm{i}t\mathscr{L}_\alpha}$. We will refer to $\mathscr{L}_\alpha$ as the classical α-Liouvillean. Obviously,

$$\mathscr{L}_\alpha = \mathscr{L} + \mathrm{i}\alpha\pi(\sigma).$$

For any $\alpha \in \mathbb{C}$ we have the following representations:

$$\mathfrak{F}_{\omega,t}(\alpha) = \int_{\mathscr{X}} \mathrm{e}^{-\alpha\int_0^t \sigma_s \mathrm{d}s} \mathrm{d}\omega = \langle \Omega, \mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega\rangle,$$

$$\mathfrak{F}_{\omega_T,t}(\alpha) = \int_{\mathscr{X}} \mathrm{e}^{-\alpha\int_0^t \sigma_s \mathrm{d}s} \mathrm{d}\omega_T = \langle \Omega, \mathrm{e}^{\mathrm{i}T\mathscr{L}_{1/2}} \mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega\rangle.$$

It follows that the axiomatic spectral approach to NESS and PREF, described in Section 4.2 and 4.3, applies directly to the classical setting.

The results of this section are easily adapted to the discrete time dynamical system; see [JPRB11, Sections 10 and 11] where the reader can also find non-trivial examples to which this spectral approach applies.

4.5 Remarks

Remark 1. In the context of finite quantum systems, the α-Liouvilleans were introduced in [JOPP10, Section 3.4], where also Part (1) of Proposition 3.2 is established. Moreover, in the same reference α-Liouvilleans are linked to Araki–Masuda non-commutative L_p spaces [AM82]. Although we believe that this link to modular theory is central to the understanding of the results of this work, for reasons of space we postpone its further discussion.

Remark 2. An early discussion of $\mathfrak{F}^{2\mathrm{tm}}_{\omega,t}$ can be found in [DR09] in the context of the Spin–Boson model. In the same reference it is proven that for some $\vartheta > 0$ and $\alpha \in (-\vartheta, \vartheta)$ the limit

$$\lim_{t\to\infty} \frac{1}{t} \log \mathfrak{F}^{2\mathrm{tm}}_{\omega,t}(\alpha) \tag{4.12}$$

exists and is real-analytic on $(-\vartheta, \vartheta)$. The proof is based on a dynamical perturbative variant of complex spectral deformation technique applied to a different class of transfer operators. Although we consider this work pioneering, the transfer operators used there are not algebraically natural and do not respect the fabric of the modular structure of quantum entropic fluctuations which is, in our opinion, central for the topic. In the context of the Spin–Fermion model, the limit (4.12) has been informally discussed at the end of [JOPP10, Section 5.4]. This discussion is in the spirit of [JP02b] and the axiomatic scheme **(Deform1)–(Deform2)**.

Remark 3. Returning to the classical dynamical system setting, in our opinion the closest parallel to our spectral resonance theory described in Sections 3 and 4 is Ruelle's abstract presentation of statistical mechanics on Smale spaces [Rue04, Section 7]. The parallel runs deep and concerns motivation/axioms selection/role of transfer operators and even certain technical aspects of the proof. The strong abstract chaoticity assumptions of Ruelle are matched by the strong dynamical assumptions inherent to **(Deform1)–(Deform3)**. Ruelle's axiomatization is motivated by the example of Anosov diffeomorphisms, and ours by the Spin–Boson and Spin–Fermion models. The hyperbolic structure of Anosov diffeomorphisms corresponds to strong dispersion of free quantum reservoirs and the resulting dynamical Fermi Golden Rule.[23]

5 Proofs

5.1 Proof of Proposition 2.2

We identify $\widehat{\mathcal{O}} = \mathcal{O} \otimes \mathrm{Mat}_2(\mathbb{C})$ with the C^*-algebra $\mathrm{Mat}_2(\mathcal{O})$ of all 2×2 matrices with entries in $\mathcal{O}$. Recall that $\mathcal{L}_{\mathrm{fr}}$ is the standard Liouvillean of the free dynamics τ_{fr}. Then, $(\mathcal{L}_{\mathrm{fr}} + V) \otimes \mathrm{Id} + \widehat{W}_\alpha$ is the semi-

[23] To learn more about these points, an interested reader can compare the results/presentation of [JPRB11] and of this work.

standard Liouvillean of $\widehat{\tau}_\alpha$, and hence

$$U_\alpha^t = \mathrm{e}^{\mathrm{i}t((\mathscr{L}_{\mathrm{fr}}+V)\otimes\mathrm{Id}+\widehat{W}_\alpha)} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V+W_\alpha)} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V+W_{-\alpha})} \end{bmatrix}$$

satisfies

$$\widehat{\tau}_\alpha^t(A) = U_\alpha^t A U_\alpha^{t*}$$

for $A \in \widehat{\mathscr{O}}$. An elementary calculation shows that, for $A \in \mathrm{Mat}_2(\mathbb{C})$ and $\widehat{\nu} = \nu \otimes \rho$,

$$\widehat{\nu} \circ \widehat{\tau}_\alpha^t(\mathbb{1} \otimes A) = \sum_{r,s\in\{\pm\}} \rho_{rs} A_{sr} \nu\left(\mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V+W_{s\alpha})}\mathrm{e}^{-\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V+W_{r\alpha})}\right),$$

and so it suffices to show that

$$\nu\left(\mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V+W_{-\alpha})}\mathrm{e}^{-\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V+W_\alpha)}\right) = \mathfrak{F}_{\nu,t}^{\mathrm{ancilla}}(\alpha).$$

Writing

$$\mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V+W_{-\alpha})} = \Delta_\omega^{-\alpha/2}\mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V)}\Delta_\omega^{\alpha/2} = \left[\Delta_\omega^{-\alpha/2}\mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V)}\Delta_\omega^{\alpha/2}\mathrm{e}^{-\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V)}\right]\mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V)},$$

we note that, since $\mathscr{L}_{\mathrm{fr}}$ and $\mathscr{L}_\omega = \log\Delta_\omega$ commute and $\sigma = \delta_\omega(V) = \mathrm{i}[\mathscr{L}_\omega, V] \in \mathscr{O}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V)}\mathscr{L}_\omega\mathrm{e}^{-\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V)} = -\mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V)}\mathrm{i}[\mathscr{L}_\omega, V]\mathrm{e}^{-\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V)} = -\tau^t(\sigma).$$

Thus

$$\mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V+W_{-\alpha})} = \left[\Delta_\omega^{-\alpha/2}\mathrm{e}^{\alpha(\mathscr{L}_\omega-\int_0^t\tau^s(\sigma)\mathrm{d}s)/2}\right]\mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V)}, \tag{5.1}$$

and since $\overline{\alpha} = -\alpha$, invoking Theorem 2.18(2) and recalling Relation (2.44), (5.1) gives

$$\mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V+W_{-\alpha})} = [D\omega_{-t} : D\omega]_{\frac{\overline{\alpha}}{2}}^*\mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V)}.$$

Reversing the sign of α we get

$$\mathrm{e}^{-\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V+W_\alpha)} = \left([D\omega_{-t} : D\omega]_{\frac{\overline{\alpha}}{2}}^*\mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V)}\right)^* = \mathrm{e}^{-\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V)}[D\omega_{-t} : D\omega]_{\frac{\alpha}{2}}, \tag{5.2}$$

and the result follows.

5.2 Proof of Proposition 2.11

The relation (5.2) gives that, for $\alpha \in \mathrm{i}\mathbb{R}$,

$$[D\omega_t : D\omega]_\alpha = \mathrm{e}^{-\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V)}\mathrm{e}^{\mathrm{i}t(\mathscr{L}_{\mathrm{fr}}+V+W_{2\alpha})}, \tag{5.3}$$

where $W_{2\alpha} = \varsigma_\omega^{-\mathrm{i}\alpha}(V) - V$. Writing the Dyson expansion of the right-hand side of the previous relation, we get

$$[D\omega_t : D\omega]_\alpha = \mathbb{1} + \sum_{n\geq1}(\mathrm{i}t)^n \int\limits_{0\leq\theta_1\leq\cdots\leq\theta_n\leq1} \tau^{-t\theta_n}(W_{2\alpha})\cdots\tau^{-t\theta_1}(W_{2\alpha})\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n. \tag{5.4}$$

Assumption **(AnV(ϑ))** gives that the function

$$\mathrm{i}\mathbb{R} \ni \alpha \mapsto W_{2\alpha} \in \mathcal{O}$$

has an analytic extension to the strip $\mathfrak{S}(\vartheta)$. It follows from the right-hand side of Relation (5.4) that the same holds for the map

$$\mathrm{i}\mathbb{R} \ni \alpha \mapsto [D\omega_t : D\omega]_\alpha \in \mathcal{O}.$$

The expansion (5.4) also yields the estimate (2.30).

5.3 Proof of Proposition 2.12

The unitarity of the Araki–Connes cocycle yields that, for $\alpha \in \mathrm{i}\mathbb{R}$ and $t \in \mathbb{R}$,

$$[D\omega_t : D\omega]^*_{-\overline{\alpha}}[D\omega_t : D\omega]_\alpha = [D\omega_t : D\omega]^*_\alpha[D\omega_t : D\omega]_\alpha = \mathbb{1}.$$

Under Assumption **(AnV(ϑ))**, Proposition 2.11 ensures that **(AnC(ϑ))** holds so that we may analytically continue the left-hand side of this identity and conclude that, for $\alpha \in \mathfrak{S}(\vartheta)$,

$$[D\omega_t : D\omega]^{-1}_\alpha = [D\omega_t : D\omega]^*_{-\overline{\alpha}}.$$

In particular, since $\vartheta > 1/2$ by assumption, we have

$$B := [D\omega_T : D\omega]_{\frac{1}{2}} \in \mathcal{O}, \quad \text{and} \quad B^{-1} = [D\omega_T : D\omega]^*_{-\frac{1}{2}}.$$

Further, the estimate (2.30) yields

$$D_T = \mathrm{e}^{-2|T|(\|\sigma_\omega^{\mathrm{i}/2}(V)\|+\|V\|)} \le \frac{1}{\|B^{-1}\|^2} \le B^*B \le \|B\|^2 \le \mathrm{e}^{2|T|(\|\sigma_\omega^{-\mathrm{i}/2}(V)+\|V\|)} = C_T. \tag{5.5}$$

The identities

$$\Delta^{1/2}_{\omega_T|\omega}\Omega = \Omega_{\omega_T},$$
$$[D\omega_T : D\omega]_\alpha\Omega = \Delta^\alpha_{\omega_T|\omega}\Omega, \qquad \alpha \in \mathrm{i}\mathbb{R},$$

and analytic continuation give that

$$B\Omega = \Omega_{\omega_T},$$

and hence

$$\Omega_{\omega_T} = J\Omega_{\omega_T} = JB\Omega = JBJ\Omega = B'\Omega$$

where $B' = JBJ \in \mathfrak{M}'$. For $\alpha \in]-\vartheta, \vartheta[$ and $\theta \in \mathbb{R}$, we set

$$A(\alpha,\theta) := \varsigma^\theta_\omega\left([D\omega_t : D\omega]^*_{\frac{\overline{\alpha}}{2}}[D\omega_t : D\omega]_{\frac{\alpha}{2}}\right).$$

Noticing that $A(\alpha,\theta)$ is a self-adjoint and strictly positive element of $\mathcal{O}$, we derive

$$\omega_T \circ \varsigma^\theta_\omega\left([D\omega_t : D\omega]^*_{\frac{\overline{\alpha}}{2}}[D\omega_t : D\omega]_{\frac{\alpha}{2}}\right) = \langle B'\Omega, A(\alpha,\theta)B'\Omega\rangle = \langle A(\alpha,\theta)^{\frac{1}{2}}\Omega, JB^*BJA(\alpha,\theta)^{\frac{1}{2}}\Omega\rangle$$

and hence, invoking the inequalities (5.5),

$$D_T\omega\left([D\omega_t:D\omega]_{\frac{\alpha}{2}}^*[D\omega_t:D\omega]_{\frac{\alpha}{2}}\right)\le\omega_T\circ\varsigma_\omega^\theta\left([D\omega_t:D\omega]_{\frac{\alpha}{2}}^*[D\omega_t:D\omega]_{\frac{\alpha}{2}}\right)$$
$$\le C_T\omega\left([D\omega_t:D\omega]_{\frac{\alpha}{2}}^*[D\omega_t:D\omega]_{\frac{\alpha}{2}}\right).$$

Setting $\theta=0$ yields the inequalities (2.31). Invoking Formula (2.11), we get the inequalities (2.33) by averaging over θ.

5.4 Proof of Proposition 2.17

By the chain rule (2.4), we have

$$[D\omega_{t+s}:D\omega]_\alpha=[D\omega_{t+s}:D\omega_t]_\alpha[D\omega_t:D\omega]_\alpha$$

for any $s,t\in\mathbb{R}$ and $\alpha\in\mathrm{i}\mathbb{R}$. Invoking Definition (2.2) and the covariance relation (2.1), we further derive

$$\begin{aligned}[D\omega_{t+s}:D\omega_t]_\alpha&=\Delta^\alpha_{\omega_s\circ\tau^t|\omega\circ\tau^t}\Delta^{-\alpha}_{\omega\circ\tau^t}\\&=\mathrm{e}^{-\mathrm{i}t\mathscr{L}}\Delta^\alpha_{\omega_s|\omega}\Delta^{-\alpha}_\omega\mathrm{e}^{\mathrm{i}t\mathscr{L}}\\&=\mathrm{e}^{-\mathrm{i}t\mathscr{L}}[D\omega_s:D\omega]_\alpha\mathrm{e}^{\mathrm{i}t\mathscr{L}}\\&=\tau^{-t}([D\omega_s:D\omega]_\alpha),\end{aligned}$$

from what the result follows.

5.5 Proof of Theorem 2.18

(1) follows from **(Qu1)** by differentiating the cocycle relation (2.40) at $\alpha=0$.

(2) Let $\Phi\in\mathrm{Dom}(\log\Delta_{\omega_t|\omega})$ and $\Psi\in\mathrm{Dom}(\log\Delta_\omega)$. Differentiating the identity

$$\langle\Phi,[D\omega_t:D\omega]_{\mathrm{i}\theta}\Psi\rangle=\langle\mathrm{e}^{-\mathrm{i}\theta\log\Delta_{\omega_t|\omega}}\Phi,\mathrm{e}^{-\mathrm{i}\theta\log\Delta_\omega}\Psi\rangle$$

w.r.t. θ at $\theta=0$ gives

$$\langle\log\Delta_{\omega_t|\omega}\Phi,\Psi\rangle=\langle\Phi,(\log\Delta_\omega+\ell_{\omega_t|\omega})\Psi\rangle,$$

and the first assertion follows. To prove the second one we observe that, starting with (2.5) and invoking the covariance relation (2.1), we get

$$\mathrm{Ent}(\omega_t|\omega)=\langle\Omega_{\omega_t},\log\Delta_{\omega|\omega_t}\Omega_{\omega_t}\rangle=\langle\mathrm{e}^{-\mathrm{i}t\mathscr{L}}\Omega,\log\Delta_{\omega|\omega_t}\mathrm{e}^{-\mathrm{i}t\mathscr{L}}\Omega\rangle=\langle\Omega,\log\Delta_{\omega_{-t}|\omega}\Omega\rangle.$$

By the first assertion and Definition (2.41), we further get

$$\mathrm{Ent}(\omega_t|\omega)=\langle\Omega,(\log\Delta_\omega+\ell_{\omega_{-t}|\omega})\Omega\rangle=\langle\Omega,\tau^t(c^{-t})\Omega\rangle.$$

Finally, (2.42) gives that $\tau^t(c^{-t}) = c^t$, and the second assertion follows.

(3) Assumption **(Qu2)** and Part (1) give

$$\frac{\mathrm{d}}{\mathrm{d}t}c^t = \frac{\mathrm{d}}{\mathrm{d}s}c^{t+s}\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\left(c^t + \tau^t(c^s)\right)\Big|_{s=0} = \tau^t(\sigma), \tag{5.6}$$

from which the first assertion immediately follows. Identity (5.6) combined with the second assertion of Part (2) give the entropy balance equation (2.43).

(4) We have

$$1 = \langle \Omega_{\omega_t}, \Omega_{\omega_t}\rangle = \langle \Delta_{\omega_t|\omega}^{1/2}\Omega, \Delta_{\omega_t|\omega}^{1/2}\Omega\rangle = \langle \mathrm{e}^{(\log\Delta_\omega + \ell_{\omega_t|\omega})/2}\Omega, \mathrm{e}^{(\log\Delta_\omega + \ell_{\omega_t|\omega})/2}\Omega\rangle. \tag{5.7}$$

Araki's perturbation theory of the KMS-structure gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{(\log\Delta_\omega + \ell_{\omega_t|\omega})/2}\Omega\Big|_{t=0} = \int_0^{1/2}\Delta_\omega^s\sigma\Omega\mathrm{d}s,$$

see *e.g.* [JP14, Relation (7.3)]. Differentiating (5.7) w.r.t. t at $t = 0$ thus yields

$$0 = 2\,\mathrm{Re}\int_0^{1/2}\langle \Delta_\omega\Omega, \Delta_\omega^s\sigma\Omega\rangle\mathrm{d}s = \omega(\sigma).$$

(5) By [BBJ⁺23, Proposition 2.2], there exists an anti-unitary operator U on the GNS Hilbert space $\mathscr{H}$ such that $\Theta(A) = UAU^*$ for all $A \in \mathscr{O}$ and $U\Delta_{\omega_t|\omega}U^* = \Delta_{\omega_{-t}|\omega}$ for all $t \in \mathbb{R}$. It follows that for $\alpha \in \mathrm{i}\mathbb{R}$ and $t \in \mathbb{R}$,

$$\Theta\left([D\omega_t : D\omega]_\alpha\right) = [D\omega_{-t} : D\omega]_{-\alpha}.$$

Differentiating this identity at $\alpha = 0$ gives $\Theta(\ell_{\omega_t|\omega}) = \ell_{\omega_{-t}|\omega}$ and hence

$$\Theta(c^t) = \Theta\circ\tau^t(\ell_{\omega_t|\omega}) = \tau^{-t}\circ\Theta(\ell_{\omega_t|\omega}) = \tau^{-t}(\ell_{\omega_{-t}|\omega}) = c^{-t}.$$

Differentiating now at $t = 0$ gives $\Theta(\sigma) = -\sigma$.

5.6 Proof of Proposition 3.1

(1) First we note that **(AnC(ϑ))** and analytic continuation give that (2.40) holds for all $\alpha \in \mathfrak{S}(\vartheta)$. We then have

$$\begin{aligned} U_\alpha^{t+s} &= \mathrm{e}^{\mathrm{i}(t+s)\mathscr{L}}J[D\omega_{t+s} : D\omega]_{\overline{\alpha}}J \\ &= \mathrm{e}^{\mathrm{i}(t+s)\mathscr{L}}J\mathrm{e}^{-\mathrm{i}s\mathscr{L}}[D\omega_t : D\omega]_{\overline{\alpha}}\mathrm{e}^{\mathrm{i}s\mathscr{L}}[D\omega_s : D\omega]_{\overline{\alpha}}J \\ &= \left(\mathrm{e}^{\mathrm{i}t\mathscr{L}}J[D\omega_t : D\omega]_{\overline{\alpha}}J\right)\left(\mathrm{e}^{\mathrm{i}s\mathscr{L}}J[D\omega_s : D\omega]_{\overline{\alpha}}J\right) \\ &= U_\alpha^t U_\alpha^s, \end{aligned}$$

where we used that, for any $s \in \mathbb{R}$, $J\mathrm{e}^{-\mathrm{i}s\mathscr{L}} = \mathrm{e}^{-\mathrm{i}s\mathscr{L}}J$. This yields the group property. To prove that this group has the C_0-property, invoking [Dav80, Proposition 1.18] it suffices to show that, for all $\Psi \in \mathscr{H}$,

$$\lim_{t\to 0} U_\alpha^t \Psi = \Psi. \tag{5.8}$$

By Vitali's convergence theorem, combined with **(AnC(ϑ))** and the bound (2.27), it is sufficient to prove (5.8) for $\alpha \in \mathrm{i}\mathbb{R}$. Using the definition of the Araki–Connes cocycle we can write

$$U_\alpha^t - I = \mathrm{e}^{\mathrm{i}t\mathscr{L}} J\Delta_{\omega_t|\omega}^{-\alpha}\Delta_\omega^\alpha J - I = \mathrm{e}^{\mathrm{i}t\mathscr{L}} J\Delta_{\omega_t|\omega}^{-\alpha}(\Delta_\omega^\alpha - \Delta_{\omega_t|\omega}^\alpha)J + (\mathrm{e}^{\mathrm{i}t\mathscr{L}} - I),$$

which leads to the bound

$$\|(U_\alpha^t - I)\Psi\| \le \|(\Delta_{\omega_t|\omega}^\alpha - \Delta_\omega^\alpha)J\Psi\| + \|(\mathrm{e}^{\mathrm{i}t\mathscr{L}} - I)\Psi\|,$$

so that it suffices to show that

$$\mathrm{s}-\lim_{t\to 0}\Delta_{\omega_t|\omega}^\alpha = \Delta_\omega^\alpha. \tag{5.9}$$

By a well known inequality [BR87, Theorem 2.5.31(b)], $\|\omega_t - \omega\| \le 2\|(\mathrm{e}^{-\mathrm{i}t\mathscr{L}} - I)\Omega\| \to 0$ as $t \to 0$, and it follows from [Ara77, Lemma 4.1] that $\Delta_{\omega_t|\omega}^{1/2} \to \Delta_\omega^{1/2}$ in the strong resolvent sense. Applying the well known result [RS80, Theorem VIII.20(b)] to the bounded continuous function $[0,\infty[\ni x \mapsto x^{\mathrm{i}t}$ yields (5.9), thus establishing (5.8).

(2) For $\alpha \in \mathrm{i}\mathbb{R}$ the operator U_α^t is the product of two unitaries, and hence is unitary.

(3) By analyticity, it suffices to prove the statement for $\alpha \in \mathrm{i}\mathbb{R}$. Then

$$U_\alpha^t A U_{-\overline{\alpha}}^{t*} = U_\alpha^t A U_\alpha^{t*} = \mathrm{e}^{\mathrm{i}t\mathscr{L}} J[D\omega_t : D\omega]_{\overline{\alpha}} JAJ[D\omega_t : D\omega]_{\overline{\alpha}}^* J\mathrm{e}^{-\mathrm{i}t\mathscr{L}},$$

and the statement then follows from the fact that $JAJ \in \mathfrak{M}'$ commutes with the unitary $[D\omega_t : D\omega]_{\overline{\alpha}} \in \mathfrak{M}$.

(4) Again, by analyticity it suffices to prove the statement for $\alpha \in \mathrm{i}\mathbb{R}$ in which case $-\overline{\alpha} = \alpha$. The identity (2.40) gives

$$\mathbb{1} = \tau^{-t}([D\omega_{-t} : D\omega]_{\overline{\alpha}})[D\omega_{-t} : D\omega]_{\overline{\alpha}},$$

and so

$$[D\omega_t : D\omega]_{\overline{\alpha}}^* \mathrm{e}^{-\mathrm{i}t\mathscr{L}} = \mathrm{e}^{-\mathrm{i}t\mathscr{L}}[D\omega_{-t} : D\omega]_{\overline{\alpha}}. \tag{5.10}$$

Since $J\mathrm{e}^{-\mathrm{i}t\mathscr{L}} = \mathrm{e}^{-\mathrm{i}t\mathscr{L}}J$, Relation (5.10) gives

$$J[D\omega_t : D\omega]_{\overline{\alpha}}^* J\mathrm{e}^{-\mathrm{i}t\mathscr{L}} = \mathrm{e}^{-\mathrm{i}t\mathscr{L}} J[D\omega_t : D\omega]_{\overline{\alpha}} J,$$

and the statement follows.

(5) The first identity in (3.2) follows from Relation (5.3). The second one follows from the first and Relation (2.8).

5.7 Proof of Proposition 3.2

(1) We will prove a stronger statement, namely that

$$\mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega = [D\omega_{-t} : D\omega]_\alpha \Omega. \tag{5.11}$$

Recalling that $\Delta_{\omega_t|\omega}^{1/2}\Omega = \Omega_{\omega_t}$, the definitions of the transfer operator and the Araki–Connes cocycle give

$$\begin{aligned}\mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega = U^t_{1/2-\alpha}\Omega = \mathrm{e}^{\mathrm{i}t\mathscr{L}} J[D\omega_t : D\omega]_{\frac{1}{2}-\overline{\alpha}} J\Omega \\ = \mathrm{e}^{\mathrm{i}t\mathscr{L}} J\Delta_{\omega_t|\omega}^{1/2-\overline{\alpha}}\Delta_\omega^{-1/2+\overline{\alpha}} J\Omega = \mathrm{e}^{\mathrm{i}t\mathscr{L}} J\Delta_{\omega_t|\omega}^{1/2-\overline{\alpha}}\Omega = \mathrm{e}^{\mathrm{i}t\mathscr{L}} J\Delta_{\omega_t|\omega}^{-\overline{\alpha}}\Omega_{\omega_t}.\end{aligned}$$

Using the fact that $\Omega_{\omega_t} = \mathrm{e}^{-\mathrm{i}t\mathscr{L}}\Omega$ and Relation (2.1), we get

$$\begin{aligned}\mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega = \mathrm{e}^{\mathrm{i}t\mathscr{L}} J\Delta_{\omega_t|\omega}^{-\overline{\alpha}}\mathrm{e}^{-\mathrm{i}t\mathscr{L}}\Omega = J\mathrm{e}^{\mathrm{i}t\mathscr{L}}\Delta_{\omega_t|\omega}^{-\overline{\alpha}}\mathrm{e}^{-\mathrm{i}t\mathscr{L}}\Omega \\ = J\Delta_{\omega|\omega_{-t}}^{-\overline{\alpha}} J\Omega = \Delta_{\omega_{-t}|\omega}^{\alpha}\Omega = [D\omega_{-t} : D\omega]_\alpha\Omega,\end{aligned}$$

and hence

$$\mathfrak{F}^{\mathrm{2tm}}_{\omega,t}(\alpha) = \omega([D\omega_{-t} : D\omega]_\alpha) = \langle \Omega, [D\omega_{-t} : D\omega]_\alpha\Omega\rangle = \langle \Omega, \mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega\rangle.$$

(2) Relations (4.4) and (5.11) yield

$$\mathfrak{F}^{\mathrm{qpsc}}_{\omega_T,t}(\alpha) = \omega_T([D\omega_{-t} : D\omega]_\alpha) = \langle \Omega, \mathrm{e}^{\mathrm{i}T\mathscr{L}_{1/2}}[D\omega_{-t} : D\omega]_\alpha\Omega\rangle = \langle \Omega, \mathrm{e}^{\mathrm{i}T\mathscr{L}_{1/2}}\mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega\rangle.$$

(3) It follows from (5.11) that

$$\Delta_\omega^{-\alpha/2}\mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega = \Delta_\omega^{-\alpha/2}\Delta^\alpha_{\omega_{-t}|\omega}\Delta_\omega^{-\alpha/2}\Omega = [D\omega_{-t} : D\omega]^*_{\frac{\alpha}{2}}[D\omega_{-t} : D\omega]_{\frac{\alpha}{2}}\Omega, \tag{5.12}$$

where we used that $\overline{\alpha} = -\alpha$ for $\alpha \in \mathrm{i}\mathbb{R}$. Relations (4.4) and (5.12) further yield

$$\begin{aligned}\mathfrak{F}^{\mathrm{ancilla}}_{\omega_T,t}(\alpha) = \omega_T\left([D\omega_{-t} : D\omega]^*_{\frac{\alpha}{2}}[D\omega_{-t} : D\omega]_{\frac{\alpha}{2}}\right) \\ = \langle \Omega, \mathrm{e}^{\mathrm{i}T\mathscr{L}_{1/2}}\Delta_\omega^{-\alpha/2}\mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega\rangle = \langle \Omega, \mathrm{e}^{\mathrm{i}T\mathscr{L}_{1/2}}\Delta_\omega^{-\alpha/2}\mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Delta_\omega^{\alpha/2}\Omega\rangle,\end{aligned}$$

and the desired identity follows immediately from Relation (3.4).

(4) The stated analytic extensions all follow from Assumption **(AnV(ϑ))** and Definitions (2.15), (2.45), (3.2) and (3.3).

5.8 Proof of Proposition 4.1

The argument follows the proof of Paley–Wiener's theorem [Rud87, Theorem 19.2]. We provide the details.

Recall that m, M are given by (4.1). For $R \geq R_0 = |\mathfrak{r}| + r + 1$ and $a > \gamma + m$, let Γ_R be the rectangle with vertices $\{\pm R - \mathrm{i}\gamma, \pm R + \mathrm{i}a\}$, and note that $\mathfrak{r}$ is inside Γ_R. The only singularity of the function

$$z \mapsto f(z) = \langle \phi, (z - L)^{-1}\psi \rangle$$

inside Γ_R is a pole of order $N \geq 1$ at $\mathfrak{r}$ and so

$$\frac{1}{2\pi i}\oint_{\Gamma_R} \mathrm{e}^{-\mathrm{i}tz} f(z)\mathrm{d}z = p(t)\mathrm{e}^{-\mathrm{i}t\mathfrak{r}},$$

where p is a polynomial of degree $N-1$ such that $p(0)$ is the residue of f at $\mathfrak{r}$. Note that, in particular, p is constant and equals to this residue if $N = 1$. We analyze separately the integrals over the four sides of the rectangle Γ_R as $R \to \infty$ along a well-chosen sequence, assuming that $t \geq 1$.

Consider first the integral over the bottom side of Γ_R. An integration by parts gives

$$I_1(R, t) = \int_{-R}^{R} \mathrm{e}^{-\mathrm{i}t(x-\mathrm{i}\gamma)} f(x - \mathrm{i}\gamma)\mathrm{d}x = \frac{\mathrm{e}^{-\gamma t}}{\mathrm{i}t}\left[\int_{-R}^{R} \mathrm{e}^{-\mathrm{i}tx} f'(x - \mathrm{i}\gamma)\mathrm{d}x + B\right] \tag{5.13}$$

where

$$B = \mathrm{e}^{\mathrm{i}Rt} f(-R - \mathrm{i}\gamma) - \mathrm{e}^{-\mathrm{i}Rt} f(R - \mathrm{i}\gamma).$$

Assumption (4.2) for $j = 1$ gives that both the integral and the boundary terms B in (5.13) are uniformly bounded with respect to $R \geq R_0$ and $t \geq 1$. More precisely, one has the bound

$$\sup_{R \geq R_0} |I_1(R, t)| \leq K\mathrm{e}^{-\gamma t}, \tag{5.14}$$

where

$$K = 2\int\limits_{|x|>r} |f'(x - \mathrm{i}\gamma)|\mathrm{d}x + \int\limits_{|x|<r} |f'(x - \mathrm{i}\gamma)|\mathrm{d}x + |f(-R_0 - \mathrm{i}\gamma)| + |f(R_0 - \mathrm{i}\gamma)|. \tag{5.15}$$

We now consider the integrals over the vertical sides $\mp R + \mathrm{i}[-\gamma, a]$. Let

$$I_2(\pm R, t) = \pm\int_{-\gamma}^{a} \mathrm{e}^{\mathrm{i}t(\pm R + \mathrm{i}y)} f(\pm R + \mathrm{i}y)\mathrm{d}y.$$

The Cauchy-Schwarz inequality gives

$$|I_2(\pm R, t)|^2 \leq \left(\int_{-\gamma}^{a} \mathrm{e}^{-2ty}\,\mathrm{d}y\right) g(\pm R),$$

where

$$g(R) = \int_{-\gamma}^{a} |f(\pm R + \mathrm{i}y)|^2\mathrm{d}y.$$

Assumption (4.2) for $j = 0$ yields that the function $R \mapsto g(-R) + g(R)$ is integrable on $[R_0, +\infty[$, and so there exists a sequence $(R_k)_{k\in\mathbb{N}^*}$ such that $\lim_{k\to\infty} R_k = \infty$ and $\lim_{k\to\infty} g(\pm R_k) = 0$. It follows that

$$\lim_{k\to\infty} I_2(\pm R_k, t) = 0. \tag{5.16}$$

We now consider the integral over the top side of Γ_R,

$$I_3(R,t) = -\mathrm{e}^{at}\int_{-R}^{R} \mathrm{e}^{-\mathrm{i}tx} f(x+\mathrm{i}a)\mathrm{d}x.$$

Since $a > m$ we have that, for $x \in \mathbb{R}$,

$$f(x+\mathrm{i}a) = \langle \phi, (x+\mathrm{i}a-L)^{-1}\psi\rangle = -\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}t(x+\mathrm{i}a)}\langle\phi, \mathrm{e}^{-\mathrm{i}tL}\psi\rangle\,\mathrm{d}t,$$

and so the left-hand side is the Fourier transform of the L^2-function

$$t \mapsto -\mathrm{i}\mathrm{e}^{-at}\langle\phi, \mathrm{e}^{-\mathrm{i}tL}\psi\rangle \mathbb{1}_{]0,\infty[}(t).$$

The Plancherel theorem gives that

$$\lim_{k\to\infty} I_3(R_k,t) = 2\pi\mathrm{i}\langle\phi, \mathrm{e}^{-\mathrm{i}tL}\psi\rangle \mathbb{1}_{]0,\infty[}(t)$$

where the limit is in $L^2(\mathbb{R}, \mathrm{d}t)$ and the R_k are as in (5.16). Hence, there exists a subsequence $(R_{k_n})_{n\in\mathbb{N}}$ such that for a.e. $t \geq 1$

$$\lim_{n\to\infty} I_3(R_{k_n},t) = 2\pi\mathrm{i}\langle\phi, \mathrm{e}^{-\mathrm{i}tL}\psi\rangle.$$

Combining (5.14) and (5.16) we derive that for a.e. $t \geq 1$, and with K given by (5.15),

$$|\langle\phi, \mathrm{e}^{-\mathrm{i}tL}\psi\rangle - \mathrm{e}^{-\mathrm{i}\mathfrak{r}t}p(t)| \leq K\mathrm{e}^{-\gamma t}. \tag{5.17}$$

Since both sides in (5.17) are continuous w.r.t. t, (5.17) holds for all $t \geq 1$.

5.9 Proof of Proposition 4.2

Since $\langle\phi, \mathrm{e}^{-\mathrm{i}tL}\psi\rangle = \mathrm{e}^{-\mathrm{i}t\mathfrak{r}}p(t) + R(t)$, with p a polynomial of degree $N-1$ and

$$|R(t)| \leq K\mathrm{e}^{-\gamma t}, \tag{5.18}$$

we have, for $z \in \mathbb{C}$ such that $\operatorname{Im} z > m$,

$$\langle\phi, (z-L)^{-1}\psi\rangle = -\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}tz}\langle\phi, \mathrm{e}^{-\mathrm{i}tL}\psi\rangle\mathrm{d}t = \frac{q(z-\mathfrak{r})}{(z-\mathfrak{r})^N} - \mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}zt}R(t)\mathrm{d}t, \tag{5.19}$$

where q is a polynomial of degree $N-1$. From the estimate (5.18), we deduce that the Laplace transform

$$z \mapsto \int_0^\infty \mathrm{e}^{\mathrm{i}zt}R(t)\mathrm{d}t$$

has an analytic continuation to the half-plane $\operatorname{Im} z > -\gamma$. This proves the first part of the proposition.

Turning to the resolvent estimates (4.3), we deal first with the case $j = 0$. Let $0 < \mu < \gamma$ and $r > |\mathfrak{r}| + 1$. It follows from (5.19) that for all $y > -\mu$ and $|x| > r$ we can write

$$\langle\phi, (x+\mathrm{i}y-L)^{-1}\psi\rangle = \frac{q(x-\mathfrak{r}+\mathrm{i}y)}{(x-\mathfrak{r}+\mathrm{i}y)^N} + \hat{g}(x), \tag{5.20}$$

where

$$\hat{g}(x) = -\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}tx}\mathrm{e}^{-ty}R(t)\mathrm{d}t.$$

There is a constant $C > 0$ such that, in the region $|x| > r$, the modulus of the first term on the right-hand side of Relation (5.20) is bounded, uniformly in y, by the function $x \mapsto \frac{C}{|x-\mathrm{Re}\,\mathfrak{r}|}$, which is square integrable on this region. The second term is the Fourier transform of the function

$$g(t) = -\mathrm{i}\mathrm{e}^{-ty}R(t)\mathbb{1}_{]0,\infty[}(t),$$

which is such that

$$\int_{\mathbb{R}} |g(t)|^2\mathrm{d}t \le \frac{K^2}{2(\gamma+y)} \le \frac{K^2}{2(\gamma-\mu)}.$$

The Plancherel theorem gives that the L^2-norm of $\hat{g}$ is uniformly bounded for $y > -\mu$.

In the case $j = 1$ we note that, for some polynomial $\tilde{q}$ of degree $N-1$,

$$\partial_x\langle\phi,(x+\mathrm{i}y-L)^{-1}\psi\rangle = \frac{\tilde{q}(x-\mathfrak{r}+\mathrm{i}y)}{(x-\mathfrak{r}+\mathrm{i}y)^{N+1}} + \int_0^\infty \mathrm{e}^{\mathrm{i}tx}\mathrm{e}^{-ty}tR(t)\mathrm{d}t,$$

where the first term on the right-hand side is bounded, uniformly in y, by a function $x \mapsto \frac{\tilde{C}}{|x-\mathrm{Re}\,\mathfrak{r}|^2}$ which is integrable on the region $|x| > r$. To deal with the second term we perform two integrations by parts, obtaining

$$\int_0^\infty \mathrm{e}^{\mathrm{i}tx}\mathrm{e}^{-ty}tR(t)\mathrm{d}t = -\frac{1}{(x+\mathrm{i}y)^2}\left(R(0) + \int_0^\infty \mathrm{e}^{\mathrm{i}tx}\mathrm{e}^{-ty}(2R^{(1)}(t)+tR^{(2)}(t))\mathrm{d}t\right),$$

which is clearly integrable on the region $|x| > r$, uniformly for $y > -\mu$.

5.10 Proof of Theorem 4.4

By Proposition 3.2(1+4), the formula

$$\mathfrak{F}^{2\mathrm{tm}}_{\omega,t}(\alpha) = \langle\Omega,\mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega\rangle$$

holds for α satisfying $\left|\mathrm{Re}\left(\alpha-\frac{1}{2}\right)\right| < \vartheta$. Assumption **(Deform2)**(d) and Proposition 4.1 applied to $\phi = \psi = \Omega$ and $L = -\mathscr{L}_{1/2-\alpha}$ give that

$$\langle\Omega,\mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega\rangle = \mathrm{e}^{-\mathrm{i}t\mathscr{E}(\frac{1}{2}-\alpha)}p_\alpha(t) + O(\mathrm{e}^{-t\gamma}),$$

where p_α is a non-zero polynomial by the first condition in **(Deform2)**(g)[24] and $\gamma > -\mathrm{Im}\,\mathscr{E}(\frac{1}{2}-\alpha)$. It follows that

$$\lim_{t\to\infty}\frac{1}{t}\log\langle\Omega,\mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega\rangle = -\mathrm{i}\mathscr{E}\left(\tfrac{1}{2}-\alpha\right) \tag{5.21}$$

for $-\vartheta+\frac{1}{2} < \mathrm{Re}\,\alpha < \vartheta+\frac{1}{2}$ and $|\mathrm{Im}\,\alpha| < \zeta$. This gives the existence of the first limit in (4.9). Part (1) follows from (5.21) and the observation that $\mathfrak{F}^{2\mathrm{tm}}_{\omega,t}(\alpha) > 0$ for $\alpha \in]-\vartheta,\vartheta[$.

[24] Recall from the proof of Proposition 4.1 that $p_\alpha(0) = \langle\Omega,\mathscr{R}_{\frac{1}{2}-\alpha}\Omega\rangle$.

Next, Definition (2.45), Theorem 4.3 and **(Deform2)**(a) give that

$$\mathfrak{F}^{\rm qpsc}_{\omega_+,t}(\alpha) = \omega_+([D\omega_{-t}:D\omega]_\alpha) = \langle \Omega, \mathscr{R}_{1/2} D^{-1}[D\omega_{-t}:D\omega]_\alpha \Omega\rangle,$$

for $\alpha \in B(\vartheta,\zeta)$. Invoking Relation (5.11) we further obtain

$$\mathfrak{F}^{\rm qpsc}_{\omega_+,t}(\alpha) = \langle D^{-1}\mathscr{R}^*_{1/2}\Omega, \mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega\rangle.$$

Assumption **(Deform2)**(c) allows us to write

$$\langle D^{-1}\mathscr{R}^*_{1/2}\Omega, (z+\mathscr{L}_{1/2-\alpha})^{-1}\Omega\rangle = \langle D^{-2}\mathscr{R}^*_{1/2}\Omega, D(z+\mathscr{L}_{1/2-\alpha})^{-1}D\Omega\rangle,$$

and so we can combine Assumption **(Deform2)**(d+e+g) and Proposition 4.1 with $\phi = D^{-1}\mathscr{R}^*_{1/2}\Omega$, $\psi = \Omega$, and $L = -\mathscr{L}_{1/2-\alpha}$. This yields the third limit in (4.9) for $-\vartheta + \frac{1}{2} < \mathrm{Re}\,\alpha < \vartheta$ and $|\mathrm{Im}\,\alpha| < \zeta$.

The second limit is handled by a very similar argument. Definition (2.15), Theorem 4.3 and **(Deform2)**(b) give that, for $\alpha \in B(\vartheta,\zeta)$,

$$\mathfrak{F}^{\rm ancilla}_{\omega_+,t}(\alpha) = \omega_+([D\omega_{-t}:D\omega]^*_{\frac{\overline{\alpha}}{2}}[D\omega_{-t}:D\omega]_{\frac{\alpha}{2}}) = \langle D^{-1}\mathscr{R}^*_{1/2}\Omega, [D\omega_{-t}:D\omega]^*_{\frac{\overline{\alpha}}{2}}[D\omega_{-t}:D\Omega]_{\frac{\alpha}{2}}\Omega\rangle.$$

Relation (5.12) further gives

$$\mathfrak{F}^{\rm ancilla}_{\omega_+,t}(\alpha) = \langle \Delta^{-\overline{\alpha}/2}D^{-1}\mathscr{R}^*_{1/2}\Omega, \mathrm{e}^{\mathrm{i}t\mathscr{L}_{1/2-\alpha}}\Omega\rangle.$$

Again **(Deform2)**(c) allows us to write

$$\langle \Delta^{-\overline{\alpha}/2}D^{-1}\mathscr{R}^*_{1/2}\Omega, (z+\mathscr{L}_{1/2-\alpha})^{-1}\Omega\rangle = \langle D^{-1}\Delta^{-\overline{\alpha}/2}D^{-1}\mathscr{R}^*_{1/2}\Omega, D(z+\mathscr{L}_{1/2-\alpha})^{-1}D\Omega\rangle,$$

and, invoking **(Deform2)**(f), we conclude the proof in the same way as in the case of the third limit.

5.11 Proof of Theorem 4.5

It suffices to show that, given $\alpha_0 \in B(\vartheta,\zeta)$ and a small enough open neighborhood $U \ni \alpha_0$ satisfying Conditions **(Deform3)**, the function $\alpha \mapsto \mathscr{E}(\alpha)$ is analytic in U. We set $R = \sup_{\alpha\in U}|\mathscr{E}(\alpha)| + r_U + 1$ and fix γ such that

$$\max(0, -\inf_{\alpha\in U}\mathrm{Im}\,\mathscr{E}(\alpha)) < \gamma < \mu.$$

By Assumptions **(Deform3)**(b+c) one has

$$K = \sup_{\alpha\in U}\left(2\int_{-\infty}^{+\infty}|\langle\Omega,(x-\mathrm{i}\gamma+\mathscr{L}_\alpha)^{-2}\Omega\rangle|\mathrm{d}x + |\langle\Omega,(-R-\mathrm{i}\gamma+\mathscr{L}_\alpha)^{-1}\Omega\rangle| + |\langle\Omega,(+R-\mathrm{i}\gamma+\mathscr{L}_\alpha)^{-1}\Omega\rangle|\right) < \infty.$$

We first note that $\alpha \mapsto f_t(\alpha) = \langle\Omega, \mathrm{e}^{\mathrm{i}t\mathscr{L}_\alpha}\Omega\rangle$ is analytic by (3.1) and **(AnC(ϑ))**. From the proof of Proposition 4.1 we further get that

$$f_t(\alpha) = \mathrm{e}^{-\mathrm{i}t\mathscr{E}(\alpha)}p_\alpha(t) + F_t(\alpha),$$

where p_α is a non-zero polynomial and $\sup_{\alpha\in U}|F_t(\alpha)| \le K\mathrm{e}^{-\gamma t}$. Thus, for large enough $T > 0$, one has $\{f_t(\alpha) \mid t \ge T, \alpha \in U\} \subset \mathbb{C}^*$. It follows that, for $t \ge T$, $\log f_t(\alpha)$ has an holomorphic branch satisfying

$$\frac{1}{t}\log f_t(\alpha) = -\mathrm{i}\mathscr{E}(\alpha) + O(t^{-1}),$$

so that Weierstrass convergence theorem allows us to conclude that $\mathscr{E}$ is holomorphic on U.

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