On the thermodynamic limit of two-times measurement entropy production

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Dedicated to the memory of Huzihiro Araki

Abstract. We provide a justification, via the thermodynamic limit, of the modular formula for entropy production in two-times measurement proposed in [BBJ+23]. We consider the cases of open quantum systems in which all thermal reservoirs are either (discrete) quantum spin systems or free Fermi gases.

Contents

1 Introduction

This work is a direct continuation of [BBJ+23], and we assume that the reader is familiar with the conceptual framework, notation, and results of this reference.

Let $(\mathcal{O}, \tau, \omega)$ be a modular C^*-dynamical system satisfying the regularity assumptions **(Reg1)** and **(Reg2)** of [BBJ+23]. Then, by Theorem 1.3 in this reference, for all $\nu \in \mathcal{N}$, $t \in \mathbb{R}$ and $\alpha \in \mathrm{i}\mathbb{R}$, the limit

$$\mathfrak{F}_{\nu,t}(\alpha) := \lim_{R\to\infty} \frac{1}{R}\int_0^R \nu\left(\varsigma_\omega^\theta\left([D\omega_{-t} : D\omega]_\alpha\right)\right) \mathrm{d}\theta \tag{1.1}$$

exists, and there exists unique Borel probability measure $Q_{\nu,t}$ on $\mathbb{R}$ such that

$$\mathfrak{F}_{\nu,t}(\alpha) = \int_{\mathbb{R}} \mathrm{e}^{-\alpha s}\mathrm{d}Q_{\nu,t}(s). \tag{1.2}$$

The family $(Q_{\nu,t})_{t\in\mathbb{R}}$ describes the statistics of the two-times measurement entropy production[1] of $(\mathcal{O}, \tau, \omega)$ with respect to ν. We recall that $\mathcal{N}$ denotes the set of all ω-normal states on $\mathcal{O}$, ς_ω is the modular group of ω, $\omega_t = \omega \circ \tau^t$, and $[D\omega_{-t} : D\omega]_\alpha$ is the Connes cocycle of the pair of states (ω_{-t}, ω).

In the special case of a finite quantum system, $Q_{\nu,t}$ is indeed the law of the entropy production of the system as defined by the two-times measurement protocol of the entropic observable $-\log\omega$, assuming that at the instant of the first measurement the system was in the state ν; see Section 1.3 in [BBJ+23]. The formulas (1.1)–(1.2) arise through the customary route of modular generalization. As repeatedly emphasized in [BBJ+23], this generalization requires the underlying thermodynamic limit[2] to be justified on solid physical grounds. In this work, we carry out this justification for two paradigmatic classes

[1] Abbreviated 2TMEP in the sequel.
[2] In the sequel abbreviated TDL.

of open quantum systems describing a finite quantum system coupled to several independent thermal reservoirs.

Our starting point is an abstract TDL scheme described in Section 2. This scheme is motivated by the specific models we will consider and has its roots in Araki's results on the continuity of the modular structure obtained in [AI74, Section 2], and in the specific form of the Araki–Wyss GNS-representation of CAR-algebras induced by a quasi-free state [AW64] [3]. One novel aspect of this scheme is the spectral assumption on the modular operators that, in our specific settings, forces the dynamical ergodicity assumption on thermal reservoirs. We remark that the same ergodicity assumption is required in the main stability result of [BBJ$^+$23].

The two specific settings to which we will apply our abstract TDL scheme are *Open Quantum Spin Systems*, abbreviated OQ2S, and *Electronic Black Box Models*, abbreviated EBBM. We comment on them separately.

OQ2S were introduced in [Rue01], in the study of entropy production in non-equilibrium quantum statistical mechanics. It is in this setting that we will make use of Araki's continuity results [AI74]. As emphasized in [Rue01], the fully interacting nature of the lattice spin thermal reservoirs brings to the forefront the role of the so-called *"boundary terms"* in the characterization of the KMS-states by the Araki–Gibbs Condition. Due to our current lack of understanding of the effect of these boundary terms, our results in the OQ2S setting are incomplete and a number of important questions touching on foundations of quantum statistical mechanics remain open. We will comment on some of them in Section 3.4.

EBBM describe open quantum systems consisting of an electronic gas in the tight binding approximation, interacting only in a finite subset of its countably infinite set of particle sites. Each thermal reservoir is a free Fermi gas and the local nature of the interaction makes the model amenable to rigorous analysis. Boundary terms play no role in free Fermi gas reservoirs and our results for the EBBM are complete. The above mentioned Araki–Wyss representation of a free Fermi gas plays a central role in our analysis and allows for a relatively effortless verification of the assumptions of our abstract TDL scheme. The literature on the EBMM and related models is vast, and an incomplete list of mathematically rigorous works on the subject is [Dav74, SL78, BM83, JP02a, JP02b, FMSU03, FMU03, AJPP06, JKP06, JOP06, JOP07, JP07, JOPP10, CMP14].

The paper is organized as follows. Our abstract TDL scheme is described in Section 2. In Section 3 this scheme is applied to OQ2S and in Section 4 to EBBM. The proofs follow the statements of the results. Sections 3.4 and 4.4, where we comment on the obtained results, are an important part of this work.

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[3] See [DG13, JOPP10] for pedagogical introductions to this topic.

2 An abstract TDL scheme

2.1 A general scheme

We denote by $(\mathcal{H}, \pi, \Omega)$ the GNS representation of the C^*-algebra $\mathcal{O}$ induced by the modular state ω and write A for $\pi(A)$ whenever the meaning is clear within the context. $\mathfrak{M} = \pi(\mathcal{O})''$ is the enveloping von Neumann algebra and we denote by $\mathcal{H}^+$ and J the natural cone and the modular conjugation of the pair $(\mathfrak{M}, \Omega)$. The set $\mathcal{N}$ of ω-normal states is identified with the set of density matrices on $\mathcal{H}$ and the state ω with the vector state $(\Omega, \cdot\, \Omega)$. Δ_ν denotes the modular operator of $\nu \in \mathcal{N}$, $\Delta_{\nu|\mu}$ the relative modular operator of a pair (ν, μ) of ω-normal states. Whenever both ν and μ are faithful on $\mathfrak{M}$, the associated Connes cocycle is given by $[D\nu : D\mu]_{\mathrm{i}s} = \Delta_{\nu|\mu}^{\mathrm{i}s}\Delta_{\mu}^{-\mathrm{i}s} \in \mathfrak{M}$.

Let $\mathcal{N}_0$ be the set of states of the form $\nu_B(\cdot) = \langle B\Omega, \cdot\, B\Omega\rangle$ where $B \in \mathfrak{M}'$ and $\|B\Omega\| = 1$. Since Ω is a cyclic vector for the von Neumann algebra $\mathfrak{M}'$, $\mathcal{N}_0$ is norm-dense in $\mathcal{N}$. Our abstract approximation scheme concerns the justification of the formula (1.1)–(1.2) for $\nu = \nu_B \in \mathcal{N}_0$ by a themodynamic limit. Under a mild regularity assumption, the time-evolved reference states $\omega_s = \omega \circ \tau^s$ are in $\mathcal{N}_0$.

Proposition 2.1 *Suppose that for some $s \in \mathbb{R}$ the map*

$$\mathrm{i}\mathbb{R} \ni z \mapsto [D\omega_s : D\omega]_z$$

has an analytic continuation to the vertical strip $0 < \mathrm{Re}\, z < \frac{1}{2}$ that is continuous and bounded on its closure. Then $\omega_s = \nu_{B_s} \in \mathcal{N}_0$, with

$$B_s = J[D\omega_s : D\omega]_{\frac{1}{2}} J \in \mathfrak{M}'.$$

Proof. Let Ω_s be the vector representative of ω_s in the natural cone $\mathcal{H}^+$. Then

$$\Delta_{\omega_s|\omega}^{\frac{1}{2}}\Omega = \Omega_s,$$

and so the function

$$\mathrm{i}\mathbb{R} \ni z \mapsto \Delta_{\omega_s|\omega}^{z}\Omega \in \mathcal{H}$$

has an analytic continuation to the strip $0 < \mathrm{Re}\, z < \frac{1}{2}$ that is continuous and bounded on its closure; see [Ara73, Lemma 3]. Note that for $z \in \mathrm{i}\mathbb{R}$,

$$[D\omega_s : D\omega]_z\Omega = \Delta_{\omega_s|\omega}^{z}\Omega, \tag{2.1}$$

and so by the Privalov theorem, see [Koo98, Section III.D], (2.1) holds for $0 \le \mathrm{Re}\, z \le \frac{1}{2}$. In particular,

$$[D\omega_s : D\omega]_{\frac{1}{2}}\Omega = \Delta_{\omega_s|\omega}^{\frac{1}{2}}\Omega = \Omega_s.$$

Since $J\Omega = \Omega$ and $J\Omega_s = \Omega_s$,

$$J[D\omega_s : D\omega]_{\frac{1}{2}} J\Omega = \Omega_s,$$

and the statement follows. □

Proposition 2.1 motivates our first assumption.

(TDL1) For all $s \in \mathbb{R}$ the map

$$\mathrm{i}\mathbb{R} \ni z \mapsto [D\omega_s : D\omega]_z$$

has an analytic continuation to the vertical strip $0 < \operatorname{Re} z < \frac{1}{2}$ that is continuous and bounded on its closure.

We consider a net of *finite-dimensional* quantum dynamical systems $(\mathcal{O}_\Lambda, \tau_\Lambda, \omega_\Lambda)_{\Lambda \in \mathcal{I}}$, where $\mathcal{I}$ is a directed set endowed with the order relation $\subseteq$. We denote by $(\mathcal{H}_\Lambda, \pi_\Lambda, \Omega_\Lambda)_{\Lambda \in \mathcal{I}}$ the associated net of GNS-representations. We further write $\omega_{\Lambda,t} = \omega_\Lambda \circ \tau_\Lambda^t$ and make the following assumptions:

(TDL2)

(1) For all $\Lambda \in \mathcal{I}$, $\mathcal{O}_\Lambda \subseteq \mathcal{O}$ and $\omega_\Lambda > 0$.

(2) If $\Lambda \subseteq \Lambda'$, then $\mathcal{O}_\Lambda \subseteq \mathcal{O}_{\Lambda'}$. Moreover,

$$\mathcal{O}_{\mathrm{loc}} := \bigcup_{\Lambda \in \mathcal{I}} \mathcal{O}_\Lambda$$

is dense in $\mathcal{O}$.

(3) For all $A \in \mathcal{O}_{\mathrm{loc}}$, $\lim_\Lambda \omega_\Lambda(A) = \omega(A)$.

(4) For all $A \in \mathcal{O}_{\mathrm{loc}}$, s–$\lim_\Lambda \tau_\Lambda^t(A) = \tau^t(A)$, locally uniformly for $t \in \mathbb{R}$.[4]

(5) For all $\Lambda \in \mathcal{I}$, $\mathcal{H}_\Lambda \subseteq \mathcal{H}$, and $\Omega_\Lambda = \Omega$.

(6) For all $t \in \mathbb{R}$ and $\alpha \in \mathrm{i}\mathbb{R}$,

$$\text{s–}\lim_{\Lambda} [D\omega_{\Lambda,t} : D\omega_\Lambda]_\alpha = [D\omega_t : D\omega]_\alpha.$$

We denote by Δ_{ω_Λ} and J_Λ the modular operator and the modular conjugation of the pair $(\pi_\Lambda(\mathcal{O}_\Lambda)'', \Omega)$, and extend them to $\mathcal{H}$ by setting Δ_{ω_Λ} to be the identity on $\mathcal{H}_\Lambda^\perp$ and J_Λ an arbitrary anti-unitary involution on $\mathcal{H}_\Lambda^\perp$. Our next assumption is

(TDL3)

(1) s–$\lim_\Lambda \Delta_{\omega_\Lambda}^{\mathrm{i}\theta} = \Delta_\omega^{\mathrm{i}\theta}$ for all $\theta \in \mathbb{R}$, and s–$\lim_\Lambda J_\Lambda = J$.

(2) For all $\Lambda \in \mathcal{I}$, $\ker \log \Delta_\omega \subseteq \ker \log \Delta_{\omega_\Lambda}$.

This assumption requires a comment. As we shall see, (TDL3)(2) holds automatically for open quantum systems in which each thermal reservoir is an ergodic quantum dynamical system. Regarding (TDL3)(1), the following set of results is established in [AI74, Section 2].[5]

[4] Here and in the following s–$\lim_\Lambda$ denotes a strong limit, *i.e.*, $\lim_\Lambda \tau_\Lambda^t(A)\Psi = \tau^t(A)\Psi$ for all $\Psi \in \mathcal{H}$.

[5] For an erratum, see [Ara76, Section 5, Remark 2].

Theorem 2.2 *Suppose that* (TDL2)(1–2) *hold, and that for all* $\Lambda \in \mathcal{I}$,

$$\omega_\Lambda = \omega\big|_{\mathcal{O}_\Lambda}, \qquad \pi_\Lambda = \pi\big|_{\mathcal{O}_\Lambda}, \qquad \Omega_\Lambda = \Omega,$$

where $(\,\cdot\,)\big|_{\mathcal{O}_\Lambda}$ *denotes the restriction from* $\mathcal{O}$ *to* $\mathcal{O}_\Lambda$*. Then the following hold:*

(1) $\omega(A) = \omega_\Lambda(A)$ *for* $A \in \mathcal{O}_\Lambda$.

(2) $\mathrm{s\text{–}lim}_\Lambda\, \Delta_{\omega_\Lambda}^{\mathrm{i}\theta} = \Delta_\omega^{\mathrm{i}\theta}$*, locally uniformly for* $\theta \in \mathbb{R}$.

(3) $\mathrm{s\text{–}lim}_\Lambda\, J_\Lambda = J$.

(4) *Let* $(Q_\Lambda)_{\Lambda\in\mathcal{I}}$ *with* $Q_\Lambda \in \mathcal{O}_\Lambda$ *be such that, for some* $Q \in \mathcal{O}$,

$$\operatorname*{s\text{–}lim}_\Lambda Q_\Lambda = Q, \qquad \operatorname*{s\text{–}lim}_\Lambda Q_\Lambda^* = Q^*.$$

Then,

$$\lim_\Lambda \Delta_{\omega_\Lambda}^z Q_\Lambda \Omega = \Delta_\omega^z Q\Omega,$$

locally uniformly for z *in the vertical strip* $0 \le \mathrm{Re}\, z \le \frac{1}{2}$.

We note in particular that (TDL2)(3) and (TDL3)(1) hold under the assumptions of Theorem 2.2.

The final assumption in this section is:

(TDL4) (TDL1) holds and for all $s \in \mathbb{R}$,

$$\operatorname*{s\text{–}lim}_\Lambda\, [D\omega_{\Lambda,s} : D\omega_\Lambda]_{\frac{1}{2}} = [D\omega_s : D\omega]_{\frac{1}{2}},$$
$$\operatorname*{s\text{–}lim}_\Lambda\, [D\omega_{\Lambda,s} : D\omega_\Lambda]_{\frac{1}{2}}^* = [D\omega_s : D\omega]_{\frac{1}{2}}^*.$$

For each $\nu_B \in \mathcal{N}_0$ we set $\nu_{B,\Lambda} := \nu_B\big|_{\mathcal{O}_\Lambda}$. The 2TMEP of $(\mathcal{O}_\Lambda, \tau_\Lambda, \omega_\Lambda)$ with respect to $\nu_{B,\Lambda}$ is defined by the formulas (1.1)–(1.2),

$$\mathfrak{F}_{\nu_{B,\Lambda},t}(\alpha) = \lim_{R\to\infty} \frac{1}{R}\int_0^R \nu_{B,\Lambda}\left(\varsigma_{\omega_\Lambda}^\theta\left([D\omega_{\Lambda,-t} : D\omega_\Lambda]_\alpha\right)\right)\mathrm{d}\theta,$$

$$\mathfrak{F}_{\nu_{B,\Lambda},t}(\alpha) = \int_{\mathbb{R}} \mathrm{e}^{-\alpha s}\mathrm{d}Q_{\nu_{B,\Lambda},t}(s).$$

Of course, in this case $Q_{\nu_{B,\Lambda},t}$ is just the law of the 2TMEP of ω_Λ, the system being in the state $\nu_{B,\Lambda}$ at the instant of the first measurement; see Section 1.3 in [BBJ+23]. If $\nu_{B,\Lambda}$ is replaced with $\omega_{\Lambda,s} = \omega_\Lambda \circ \tau_\Lambda^s$, then $\mathfrak{F}_{\omega_{\Lambda,s},t}$ and $Q_{\omega_{\Lambda,s},t}$ describe the 2TMEP of ω_Λ, the system being in the state $\omega_{\Lambda,s}$ at the instant of the first measurement. This second case is of importance for our study of a quantum Gallavotti–Cohen Fluctuation Theorem in [BBJ+]. Note that, except in trivial cases, $\omega_s\big|_{\mathcal{O}_\Lambda} \neq \omega_{\Lambda,s}$.

The main result in this section is :

Theorem 2.3 *Suppose that (TDL2) and (TDL3) hold. Then,*

(1) *Given any state $\nu_B \in \mathcal{N}_0$, for all $t \in \mathbb{R}$ and $\alpha \in \mathrm{i}\mathbb{R}$ one has*

$$\lim_{\Lambda} \mathfrak{F}_{\nu_{B,\Lambda},t}(\alpha) = \mathfrak{F}_{\nu_B,t}(\alpha), \tag{2.2}$$

and $\lim_\Lambda Q_{\nu_{B,\Lambda},t} = Q_{\nu_B,t}$ weakly.

(2) *Suppose in addition that (TDL4) holds. Then for all $s, t \in \mathbb{R}$ and $\alpha \in \mathrm{i}\mathbb{R}$ one has*

$$\lim_{\Lambda} \mathfrak{F}_{\omega_{\Lambda,s},t}(\alpha) = \mathfrak{F}_{\omega_s,t}(\alpha), \tag{2.3}$$

and $\lim_\Lambda Q_{\omega_{\Lambda,s},t} = Q_{\omega_s,t}$ weakly.

Proof. **(1)** Central to the argument are the formulas

$$\begin{aligned} \mathfrak{F}_{\nu_B,t}(\alpha) &= \langle B^*B\Omega, P[D\omega_{-t} : D\omega]_\alpha \Omega\rangle, \\ \mathfrak{F}_{\nu_{B,\Lambda},t}(\alpha) &= \langle B^*B\Omega, P_\Lambda[D\omega_{\Lambda,-t} : D\omega_\Lambda]_\alpha \Omega\rangle, \end{aligned} \tag{2.4}$$

where P and P_Λ denote the orthogonal projections onto $\ker \log \Delta_\omega$ and $\ker \log \Delta_{\omega_\Lambda}$ respectively. The formulas (2.4) hold for all $\nu_B \in \mathcal{N}_0$ and are easy consequences of the fact that $B \in \mathfrak{M}'$; see the proof of Theorem 1.3 in [BBJ+23]. By (TDL2)(6) and the polarization identity, to prove (2.2) it suffices to show that for all $\Psi \in \mathcal{H}$,

$$\lim_{\Lambda} \langle \Psi, P_\Lambda \Psi\rangle = \langle \Psi, P\Psi\rangle. \tag{2.5}$$

Assumption (TDL3)(2) gives that $\langle \Psi, P_\Lambda \Psi\rangle \geq \langle \Psi, P\Psi\rangle$ for all $\Lambda \in \mathcal{I}$, and so

$$\liminf_{\Lambda} \langle \Psi, P_\Lambda \Psi\rangle \geq \langle \Psi, P\Psi\rangle. \tag{2.6}$$

Theorem 2.2(2) gives that, for $t \in \mathbb{R}$,

$$\lim_{\Lambda} \langle \Psi, \Delta_{\omega_\Lambda}^{\mathrm{i}t} \Psi\rangle = \langle \Psi, \Delta_{\omega}^{\mathrm{i}t} \Psi\rangle.$$

By Lévy's continuity theorem, the spectral measure $\mu_{\Psi,\Lambda}$ of $\log \Delta_{\omega_\Lambda}$ for the vector Ψ converges weakly to the spectral measure μ_Ψ of $\log \Delta_\omega$ for the same vector Ψ. By the Portmanteau Theorem [Bog07, Theorem 8.2.3], this convergence gives

$$\limsup_{\Lambda} \mu_{\Psi,\Lambda}(\{0\}) \leq \mu_\Psi(\{0\}),$$

and so

$$\limsup_{\Lambda} \langle \Psi, P_\Lambda \Psi\rangle \leq \langle \Psi, P\Psi\rangle. \tag{2.7}$$

The inequalities (2.6) and (2.7) yield (2.5). The second claim follows from (2.2) and Lévy's continuity theorem.

(2) Writing $C_\Lambda = [D\omega_{\Lambda,s} : D\omega_\Lambda]_{\frac{1}{2}}$ and $C = [D\omega_s : D\omega]_{\frac{1}{2}}$, it follows from (2.4) and Proposition 2.1 that

$$\mathfrak{F}_{\omega_{\Lambda,s},t}(\alpha) = \langle J_\Lambda C_\Lambda^* C_\Lambda \Omega, P_\Lambda [D\omega_{\Lambda,-t} : D\omega_\Lambda]_\alpha \Omega\rangle,$$
$$\mathfrak{F}_{\omega_s,t}(\alpha) = \langle J C^* C \Omega, P [D\omega_{-t} : D\omega]_\alpha \Omega\rangle.$$

By (TDL3)(1), the isometric modular conjugations are strongly convergent, and by (TDL4) and the uniform boundedness principle, one has $\sup_\Lambda \|C_\Lambda\| < \infty$. Thus, it follows from the telescopic expansion

$$J_\Lambda C_\Lambda^* C_\Lambda \Omega - J C^* C \Omega = (J_\Lambda - J) C^* C \Omega + J_\Lambda (C_\Lambda^* - C^*) C \Omega + J_\Lambda C_\Lambda^* (C_\Lambda - C)\Omega,$$

that

$$\lim_\Lambda J_\Lambda C_\Lambda^* C_\Lambda \Omega = J C^* C \Omega,$$

while by (TDL2)(6),

$$\lim_\Lambda [D\omega_{\Lambda,-t} : D\omega_\Lambda]_\alpha \Omega = [D\omega_{-t} : D\omega]_\alpha \Omega.$$

Invoking (2.5) again gives (2.3) and Lévy's continuity theorem yields the second claim. □

2.2 Thermodynamic limit of Connes' cocycles

In this section we introduce more structure to the dynamical system $(\mathcal{O}, \tau, \omega)$. The purpose of these new elements is to allow us to exploit the structural stability of KMS states, as embodied in Araki's perturbation theory, see [BR81, Section 5.4]. Quite unexpectedly, this part of the algebraic theory of equilibrium quantum statistical mechanics developed in the 70', is also at the hearth of some more recent advances in nonequilibrium quantum statistical mechanics. The next two sets of assumptions will allow us to use the concept of entropy production in nonequilibrium processes to gain control on the Connes cocycles. This will lead us to check the corresponding Assumptions (TDL1), (TDL2)(6) and (TDL4).

We denote by δ_ω the $*$-derivation generating the modular group ς_ω: $\varsigma_\omega^\theta = \mathrm{e}^{\theta\delta_\omega}$. Our next assumption is:

(TDL5)

(1) ς_ω commutes with a "free" C^*-dynamics $\tau_{\rm fr}^t = \mathrm{e}^{t\delta_{\rm fr}}$ which leaves the state ω invariant:

$$\varsigma_\omega^\theta \circ \tau_{\rm fr}^t = \tau_{\rm fr}^t \circ \varsigma_\omega^\theta, \qquad \omega \circ \tau_{\rm fr}^t = \omega,$$

for all $\theta, t \in \mathbb{R}$.

(2) τ is a *local perturbation* of $\tau_{\rm fr}$:

$$\tau^t = \mathrm{e}^{t\delta}, \qquad \delta = \delta_{\rm fr} + \mathrm{i}[V, \cdot\,],$$

where V is a self-adjoint element of $\mathcal{O}$.

(3) $V \in \mathrm{dom}(\delta_\omega)$, and we define

$$\sigma = \delta_\omega(V).$$

(4) The map

$$\mathbb{R} \ni \theta \mapsto \varsigma_{\omega}^{\theta}(\sigma) \in \mathcal{O}$$

has an analytic continuation to the horizontal strip $|\mathrm{Im}\,\theta| < \frac{1}{2}$ that is bounded and continuous on its closure.

Starting with the works [JP01, Rue01], the operator σ has played an important role in many developments in non-equilibrium quantum statistical mechanics as the *entropy production observable* of the C^*-dynamical system $(\mathcal{O}, \tau, \omega)$. In the sequel, we denote by $\mathcal{L}$ (respectively $\mathcal{L}_{\mathrm{fr}}$) the standard Liouvillean of the dynamics τ (respectively τ_{fr}), *i.e.*, the unique self-adjoint operator on $\mathcal{H}$ such that

$$\pi(\tau^t(\,\cdot\,)) = \mathrm{e}^{\mathrm{i}t\mathcal{L}}\pi(\,\cdot\,)\mathrm{e}^{-\mathrm{i}t\mathcal{L}}, \qquad \mathrm{e}^{-\mathrm{i}t\mathcal{L}}\mathcal{H}^+ \subset \mathcal{H}^+,$$

for all $t \in \mathbb{R}$ (and similarly for the free dynamics).

Lemma 2.4 *Under the assumptions* (TDL5)(1–3)*, one has*

$$\log \Delta_{\omega_s|\omega} = \log \Delta_\omega + Q_s, \qquad Q_s = \int_0^s \tau^{-t}(\sigma)\mathrm{d}t,$$

for any $s \in \mathbb{R}$.

Proof. A proof of the statement is implicit in [JP03]. For the reader's convenience and later references, we sketch the argument. Let $(\Gamma_s)_{s\in\mathbb{R}}$ be the cocycle associated to the local perturbation V of the free dynamics τ_{fr}, *i.e.*, the solution of the Cauchy problem

$$\partial_s\Gamma_s = \mathrm{i}\Gamma_s\tau_{\mathrm{fr}}^s(V), \qquad \Gamma_0 = \mathbb{1}. \tag{2.8}$$

Γ_s is a unitary element of $\mathcal{O}$ with the norm convergent expansion

$$\Gamma_s = \mathbb{1} + \sum_{n\geq 1} \mathrm{i}^n \int\limits_{0\leq s_1\leq\cdots\leq s_n\leq s} \tau_{\mathrm{fr}}^{s_1}(V)\cdots\tau_{\mathrm{fr}}^{s_n}(V)\mathrm{d}s_1\cdots\mathrm{d}s_n, \tag{2.9}$$

and such that, as a consequence of (TDL5)(2),

$$\tau^s(\,\cdot\,) = \Gamma_s\tau_{\mathrm{fr}}^s(\,\cdot\,)\Gamma_s^*, \tag{2.10}$$

see, e.g., [BR81, Section 5.4.1]. From the perspective of the associated W^*-dynamics, one has

$$\mathcal{L} = \mathcal{L}_{\mathrm{fr}} + \pi(V) - J\pi(V)J,$$

and

$$\mathrm{e}^{\mathrm{i}s\mathcal{L}} = J\pi(\Gamma_s)J\pi(\Gamma_s)\mathrm{e}^{\mathrm{i}s\mathcal{L}_{\mathrm{fr}}}.$$

Taking (TDL5)(1) into account, we have $\mathrm{e}^{\mathrm{i}s\mathcal{L}_{\mathrm{fr}}}\Omega = \Omega$, so that the vector representative of the state ω_s in the natural cone $\mathcal{H}^+$ is

$$\Omega_s = \mathrm{e}^{-\mathrm{i}s\mathcal{L}}\Omega = J\pi(\Gamma_{-s})J\pi(\Gamma_{-s})\Omega. \tag{2.11}$$

For some arbitrary but fixed $s \in \mathbb{R}$ and any $\theta \in \mathbb{R}$, set

$$T_\theta = \Gamma_{-s} \varsigma_\omega^\theta(\Gamma_{-s}^*). \tag{2.12}$$

It follows from (TDL5)(3) and the expansion (2.9) that $\Gamma_{-s} \in \mathrm{dom}(\delta_\omega)$. Using (2.8), one easily checks that

$$\partial_\theta T_\theta = \mathrm{i} T_\theta \varsigma_\omega^\theta(Q_s), \qquad T_0 = \mathbb{1},$$

where $Q_s = \mathrm{i}\delta_\omega(\Gamma_{-s})\Gamma_{-s}^* = Q_s^* \in \mathcal{O}$. Thus, $(T_\theta)_{\theta\in\mathbb{R}}$ is the unitary cocycle associated to the local perturbation Q_s of the modular group

$$\alpha^\theta(\,\cdot\,) = \mathrm{e}^{\theta(\delta_\omega + \mathrm{i}[Q_s,\,\cdot\,])} = T_\theta \varsigma_\omega^\theta(\,\cdot\,) T_\theta^*.$$

Recalling that

$$\pi(\varsigma_\omega^\theta(\,\cdot\,)) = \mathrm{e}^{\mathrm{i}\theta \log \Delta_\omega} \pi(\,\cdot\,) \mathrm{e}^{-\mathrm{i}\theta \log \Delta_\omega},$$

we derive

$$\pi(\alpha^\theta(\,\cdot\,)) = \pi(T_\theta) \mathrm{e}^{\mathrm{i}\theta \log \Delta_\omega} \pi(\,\cdot\,) \mathrm{e}^{-\mathrm{i}\theta \log \Delta_\omega} \pi(T_\theta^*).$$

Thus, α extends to a W^*-dynamics on $\mathfrak{M}$ which we also denote by α,

$$\alpha^\theta(\,\cdot\,) = \mathrm{e}^{\mathrm{i}\theta(\log \Delta_\omega + Q_s)}(\,\cdot\,)\mathrm{e}^{-\mathrm{i}\theta(\log \Delta_\omega + Q_s)}, \qquad \mathrm{e}^{\mathrm{i}\theta(\log \Delta_\omega + Q_s)} = \pi(T_\theta)\mathrm{e}^{\mathrm{i}\theta \log \Delta_\omega}.$$

Using (2.12) we obtain, for any $A \in \mathcal{O}$ and $\theta \in \mathbb{R}$,

$$\mathrm{e}^{\mathrm{i}\theta(\log \Delta_\omega + Q_s)} \pi(A)\Omega = \pi(T_\theta)\mathrm{e}^{\mathrm{i}\theta \log \Delta_\omega} \pi(A)\Omega = \pi(\Gamma_{-s})\mathrm{e}^{\mathrm{i}\theta \log \Delta_\omega} \pi(\Gamma_{-s}^* A)\Omega.$$

Since the right-hand side has an analytic continuation to $\theta = -\mathrm{i}/2$, we further get

$$\mathrm{e}^{(\log \Delta_\omega + Q_s)/2} \pi(A)\Omega = \pi(\Gamma_{-s})\Delta_\omega^{\frac{1}{2}} \pi(\Gamma_{-s}^* A)\Omega = \pi(\Gamma_{-s}) J \pi(A^* \Gamma_{-s})\Omega,$$

and multiplication on the left by J yields, taking (2.11) into account,

$$J\mathrm{e}^{(\log \Delta_\omega + Q_s)/2} \pi(A)\Omega = J\pi(\Gamma_{-s}) J \pi(A^* \Gamma_{-s})\Omega = \pi(A^*) J\pi(\Gamma_{-s}) J \pi(\Gamma_{-s})\Omega = \pi(A^*)\Omega_s,$$

which shows that $\log \Delta_\omega + Q_s = \log \Delta_{\omega_s|\omega}$. To finish the proof, we note that a simple calculation using (2.8) yields

$$\partial_s Q_s = \tau^{-s}(\sigma),$$

so that

$$Q_s = \int_0^s \tau^{-t}(\sigma)\mathrm{d}t.$$

□

Proposition 2.5 *Assumption (TDL5) implies (TDL1).*

Proof. We shall use the notation and intermediate results from the previous proof. By Lemma 2.4, the cocycle

$$[D\omega_s : D\omega]_z = \mathrm{e}^{z\log\Delta_{\omega_s|\omega}}\mathrm{e}^{-z\log\Delta_\omega}$$

has, for $z \in \mathrm{i}\mathbb{R}$, the norm convergent expansion

$$[D\omega_s : D\omega]_z = \mathbb{1} + \sum_{n\geq 1} z^n \int\limits_{0\leq\theta_1\leq\cdots\leq\theta_n\leq 1} \varsigma_\omega^{-\mathrm{i}\theta_1 z}(Q_s)\cdots\varsigma_\omega^{-\mathrm{i}\theta_n z}(Q_s)\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n. \tag{2.13}$$

Using (2.10), we have for $\theta \in \mathbb{R}$,

$$\varsigma_\omega^\theta(Q_s) = \int_0^s \varsigma_\omega^\theta(\tau^{-t}(\sigma))\mathrm{d}t = \int_0^s \varsigma_\omega^\theta(\Gamma_{-t}\tau_{\mathrm{fr}}^{-t}(\sigma)\Gamma_{-t}^*)\mathrm{d}t,$$

and since the modular group commutes with the free dynamics, we can write

$$\varsigma_\omega^\theta(Q_s) = \int_0^s \varsigma_\omega^\theta(\Gamma_{-t})\tau_{\mathrm{fr}}^{-t}(\varsigma_\omega^\theta(\sigma))\varsigma_\omega^\theta(\Gamma_{-t}^*)\mathrm{d}t. \tag{2.14}$$

By Assumption (TDL5)(4), $\theta \mapsto \varsigma_\omega^\theta(\sigma)$ is analytic in the strip $|\mathrm{Im}\,\theta| < \frac{1}{2}$, and bounded and continuous on its closure. Since $\partial_\theta\varsigma_\omega^\theta(V) = \varsigma_\omega^\theta(\sigma)$, the same is true for

$$\theta \mapsto \varsigma_\omega^\theta(V) = V + \theta\int_0^1 \varsigma_\omega^{\theta t}(\sigma)\mathrm{d}t.$$

That the same is also true for $\theta \mapsto \varsigma_\omega^\theta(\Gamma_{-t})$ is a consequence of the expansion (2.9), since the latter implies

$$\varsigma_\omega^\theta(\Gamma_{-t}) = \mathbb{1} + \sum_{n\geq 1}(-\mathrm{i}t)^n \int\limits_{0\leq s_1\leq\cdots\leq s_n\leq 1} \tau_{\mathrm{fr}}^{-ts_1}(\varsigma_\omega^\theta(V))\cdots\tau_{\mathrm{fr}}^{-ts_n}(\varsigma_\omega^\theta(V))\mathrm{d}s_1\cdots\mathrm{d}s_n. \tag{2.15}$$

Invoking (2.14), we conclude that the function $\theta \mapsto \varsigma_\omega^\theta(Q_s)$ is analytic in the strip $|\mathrm{Im}\,\theta| < \frac{1}{2}$, and bounded and continuous on its closure. Finally, from the expansion (2.13) we conclude that the map

$$z \mapsto [D\omega_s : D\omega]_z$$

is analytic on the strip $|\mathrm{Re}\,z| < \frac{1}{2}$, and bounded and continuous on its closure. □

In a similar spirit, we will assume the following additional properties of the approximation scheme we have introduced in Section 2.

(TDL6) (TDL2)(5) and (TDL5) hold. Moreover, for all $\Lambda \in \mathcal{I}$:

(1) $\varsigma_{\omega_\Lambda}$ commutes with a "free" C^*-dynamics $\tau_{\Lambda,\mathrm{fr}}^t = \mathrm{e}^{t\delta_{\Lambda,\mathrm{fr}}}$ which leaves the state ω_Λ invariant:

$$\varsigma_{\omega_\Lambda}^\theta \circ \tau_{\Lambda,\mathrm{fr}}^t = \tau_{\Lambda,\mathrm{fr}}^t \circ \varsigma_{\omega_\Lambda}^\theta, \qquad \omega_\Lambda \circ \tau_{\Lambda,\mathrm{fr}}^t = \omega_\Lambda,$$

for all $\theta, t \in \mathbb{R}$.

(2) τ_Λ is a *local perturbation* of $\tau_{\Lambda,\mathrm{fr}}$:

$$\tau_\Lambda^t = \mathrm{e}^{t\delta_\Lambda}, \qquad \delta_\Lambda = \delta_{\Lambda,\mathrm{fr}} + \mathrm{i}[V_\Lambda, \cdot\,],$$

where V_Λ is a self-adjoint element of $\mathcal{O}_\Lambda$ such that

$$\lim_\Lambda V_\Lambda = V$$

holds in $\mathcal{O}$.

(3) Extending the standard Liouvillean $\mathcal{L}_{\Lambda,\mathrm{fr}}$ of the free dynamics $\tau_{\Lambda,\mathrm{fr}}$ to $\mathcal{H}$ by setting it to 0 on $\mathcal{H}_\Lambda^\perp$, one has

$$\mathrm{s}\text{–}\lim_\Lambda \mathrm{e}^{\mathrm{i}t\mathcal{L}_{\Lambda,\mathrm{fr}}} = \mathrm{e}^{\mathrm{i}t\mathcal{L}_{\mathrm{fr}}},$$

locally uniformly for $t \in \mathbb{R}$.

(4) The entropy production observable of the C^*-dynamical system $(\mathcal{O}_\Lambda, \tau_\Lambda, \omega_\Lambda)$,

$$\sigma_\Lambda = \delta_{\omega_\Lambda}(V_\Lambda),$$

satisfies

$$\mathrm{s}\text{–}\lim_\Lambda \varsigma_{\omega_\Lambda}^\theta(\sigma_\Lambda) = \varsigma_\omega^\theta(\sigma)$$

locally uniformly for $|\mathrm{Im}\,\theta| \le \frac{1}{2}$.

Proposition 2.6 *Assumption* (TDL6) *implies* (TDL2)(6) *and* (TDL4).

Proof. Since (TDL6)(1–2) imply that the C^*-dynamical system $(\mathcal{O}_\Lambda, \tau_\Lambda, \omega_\Lambda)$ satisfies the assumptions of Lemma 2.4, we can start, as in the proof of Proposition 2.5, with the expansions

$$\begin{aligned}
[D\omega_{\Lambda,s} : D\omega_\Lambda]_z &= \mathbb{1} + \sum_{n\ge1} z^n \int\limits_{0\le\theta_1\le\cdots\le\theta_n\le1} \varsigma_{\omega_\Lambda}^{-\mathrm{i}\theta_1 z}(Q_{\Lambda,s}) \cdots \varsigma_{\omega_\Lambda}^{-\mathrm{i}\theta_n z}(Q_{\Lambda,s})\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n, \\
[D\omega_{\Lambda,s} : D\omega_\Lambda]_z^* &= \mathbb{1} + \sum_{n\ge1} \overline{z}^n \int\limits_{0\le\theta_1\le\cdots\le\theta_n\le1} \varsigma_{\omega_\Lambda}^{\mathrm{i}\theta_n \overline{z}}(Q_{\Lambda,s}) \cdots \varsigma_{\omega_\Lambda}^{\mathrm{i}\theta_1 \overline{z}}(Q_{\Lambda,s})\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n,
\end{aligned} \tag{2.16}$$

where $Q_{\Lambda,s} = \int_0^s \tau_\Lambda^{-t}(\sigma_\Lambda)\mathrm{d}t$. $\mathcal{O}_\Lambda$ being finite dimensional, the relations (2.16) hold for all $z \in \mathbb{C}$. We again write

$$\varsigma_{\omega_\Lambda}^\theta(Q_{\Lambda,s}) = \int_0^s \varsigma_{\omega_\Lambda}^\theta(\Gamma_{\Lambda,-t})\tau_{\Lambda,\mathrm{fr}}^{-t}(\varsigma_{\omega_\Lambda}^\theta(\sigma_\Lambda))\varsigma_{\omega_\Lambda}^\theta(\Gamma_{\Lambda,-t}^*)\mathrm{d}t, \tag{2.17}$$

where

$$\varsigma_{\omega_\Lambda}^\theta(\Gamma_{\Lambda,-t}) = \mathbb{1} + \sum_{n\ge1}(-\mathrm{i}t)^n \int\limits_{0\le s_1\le\cdots\le s_n\le1} \tau_{\Lambda,\mathrm{fr}}^{-ts_1}(\varsigma_{\omega_\Lambda}^\theta(V_\Lambda))\cdots\tau_{\Lambda,\mathrm{fr}}^{-ts_n}(\varsigma_{\omega_\Lambda}^\theta(V_\Lambda))\mathrm{d}s_1\cdots\mathrm{d}s_n. \tag{2.18}$$

Since

$$\varsigma_\omega^\theta(V) - \varsigma_{\omega_\Lambda}^\theta(V_\Lambda) = V - V_\Lambda + \theta \int_0^1 (\varsigma_\omega^{s\theta}(\sigma) - \varsigma_{\omega_\Lambda}^{s\theta}(\sigma_\Lambda))\mathrm{d}s,$$

it follows from (TDL6)(2)+(4) that

$$\text{s–}\lim_\Lambda \varsigma_{\omega_\Lambda}^\theta(V_\Lambda) = \varsigma_\omega^\theta(V),$$

locally uniformly for $|\mathrm{Im}\,\theta| \leq \frac{1}{2}$. The group property and the isometric nature of the modular dynamics, together with the uniform boundedness principle yield

$$\sup_{\Lambda, |\mathrm{Im}\,\theta| \leq 1/2} \|\varsigma_{\omega_\Lambda}^\theta(V_\Lambda)\| < \infty, \qquad \sup_{\Lambda, |\mathrm{Im}\,\theta| \leq 1/2} \|\varsigma_{\omega_\Lambda}^\theta(\sigma_\Lambda)\| < \infty,$$

and since $\mathrm{e}^{\mathrm{i}t\mathcal{L}_{\Lambda,\mathrm{fr}}}$ is unitary, a telescopic expansion yields, as in the proof of Theorem 2.3(2),

$$\text{s–}\lim_\Lambda \varsigma_{\omega_\Lambda}^\theta(\Gamma_{\Lambda,-t}) = \varsigma_\omega^\theta(\Gamma_{-t}),$$

for $|\mathrm{Im}\,\theta| \leq \frac{1}{2}$. By the same argument, (2.17) yields

$$\text{s–}\lim_\Lambda \varsigma_{\omega,\Lambda}^\theta(Q_{\Lambda,s}) = \varsigma_\omega^\theta(Q_s),$$

and in particular

$$\sup_{\Lambda, |\mathrm{Im}\,\theta| \leq 1/2} \|\varsigma_{\omega_\Lambda}^\theta(Q_{\Lambda,s})\| < \infty.$$

Finally, we deduce from the expansion (2.16) that

$$\text{s–}\lim_\Lambda [D\omega_{\Lambda,s} : D\omega_\Lambda]_z = [D\omega_s : D\omega]_z, \qquad \text{s–}\lim_\Lambda [D\omega_{\Lambda,s} : D\omega_\Lambda]_z^* = [D\omega_s : D\omega]_z^*,$$

hold for all z in the closed strip $|\mathrm{Im}\, z| \leq \frac{1}{2}$. □

3 Open Quantum Spin Systems

3.1 Quantum spin systems

We follow [BR81]; see also [Isr79, Sim93, Rue69].

The C^*-algebra. Let G be a countably infinite set. At this point, no further structure on G is assumed. The collection of all finite subsets of G is denoted by $\mathfrak{G}_{\mathrm{fin}}$. Let $\mathfrak{h}$ be the finite dimensional Hilbert space of a single spin. To each $x \in G$ we associate a copy $\mathfrak{h}_x$ of $\mathfrak{h}$, and to each $\Lambda \in \mathfrak{G}_{\mathrm{fin}}$ the Hilbert space

$$\mathcal{K}_\Lambda = \bigotimes_{x\in\Lambda} \mathfrak{h}_x.$$

$\mathcal{O}_\Lambda$ denotes the C^*-algebra of all linear operators on $\mathcal{K}_\Lambda$. Its elements describe observables of the spins localized in the region Λ. For $\Lambda \subseteq \Lambda'$ one naturally identifies $\mathcal{O}_\Lambda$ with a C^*-subalgebra of $\mathcal{O}_{\Lambda'}$. The $*$-algebra of local observables is

$$\mathcal{O}_{\mathrm{loc}} = \bigcup_{\Lambda \in \mathfrak{G}_{\mathrm{fin}}} \mathcal{O}_\Lambda.$$

Finally, the C^*-algebra of the spin system over G is the norm closure $\mathcal{O}_G$ of $\mathcal{O}_{\mathrm{loc}}$. The algebra $\mathcal{O}_G$ is unital, simple, and separable. For any $G_0 \subset G$ one has a natural identification $\mathcal{O}_G = \mathcal{O}_{G_0} \otimes \mathcal{O}_{G_0^c}$.[6] Whenever G is understood, we write $\mathcal{O}$ for $\mathcal{O}_G$.

Dynamics. An *interaction* is a map

$$\Phi : \mathfrak{G}_{\mathrm{fin}} \to \mathcal{O}$$

such that $\Phi(X)$ is a self-adjoint element of $\mathcal{O}_X$. The *Hamiltonian* of a region $\Lambda \in \mathfrak{G}_{\mathrm{fin}}$ is the local observable defined by

$$H_\Lambda(\Phi) = \sum_{X \subseteq \Lambda} \Phi(X).$$

It generates a local C^*-dynamics $\tau_{\Phi,\Lambda}$ on $\mathcal{O}$, where

$$\tau^t_{\Phi,\Lambda}(A) = \mathrm{e}^{\mathrm{i}tH_\Lambda(\Phi)} A \mathrm{e}^{-\mathrm{i}tH_\Lambda(\Phi)}.$$

To control $\tau_{\Phi,\Lambda}$ in the thermodynamic limit $\Lambda \uparrow G$, one needs a suitable regularity assumption. We settle for

(SR) For some $\lambda > 0$,

$$\|\Phi\|_\lambda = \sup_{x \in G} \sum_{X \ni x} \|\Phi(X)\| \mathrm{e}^{\lambda(|X|-1)} < \infty,$$

where $|X|$ denotes the cardinality of the set X.

Theorem 6.2.4 in [BR81] and its proof give the following.

Theorem 3.1 *Suppose that (SR) holds. Then:*

(1) *For all $A \in \mathcal{O}$ the limit*

$$\tau^t_\Phi(A) := \lim_\Lambda \tau^t_{\Phi,\Lambda}(A)$$

exists in norm, locally uniformly for $t \in \mathbb{R}$.

(2) *$\tau_\Phi = \{\tau^t_\Phi \mid t \in \mathbb{R}\}$ is a C^*-dynamics on $\mathcal{O}$. We denote by δ_Φ its generator.*

(3) *$\mathcal{O}_{\mathrm{loc}}$ is a core of δ_Φ, and for $A \in \mathcal{O}_\Lambda$,*

$$\delta_\Phi(A) = \sum_{X \cap \Lambda \neq \emptyset} \mathrm{i}[\Phi(X), A].$$

[6] We will write A for $A \otimes \mathbb{1}$, $\mathbb{1} \otimes A$ whenever the meaning is clear within the context.

(4) *For $A \in \mathcal{O}_\Lambda$ and $n \geq 1$,*

$$\|\delta_\Phi^n(A)\| \leq \frac{2^n n!}{\lambda^n} \mathrm{e}^{\lambda|\Lambda|} \|\Phi\|_\lambda^n \|A\|.$$

In particular, for all $A \in \mathcal{O}_{\mathrm{loc}}$, the map

$$\mathbb{R} \ni t \mapsto \tau_\Phi^t(A) \in \mathcal{O}$$

has an analytic extension to the strip $|\mathrm{Im}\, z| < \frac{\lambda}{2\|\Phi\|_\lambda}$.

Until the end of this section we assume that (SR) holds. We will make frequent use of the fact that, whenever $A = \sum_n A_n$ is a norm convergent series with $A_n \in \mathcal{O}_{\mathrm{loc}}$ and $\sum_n \|\delta_\Phi(A_n)\| < \infty$, then Property (3) implies that $A \in \mathrm{dom}(\delta_\Phi)$ with $\delta_\Phi(A) = \sum_n \delta_\Phi(A_n)$.

KMS-states. For $\Lambda \in \mathfrak{G}_{\mathrm{fin}}$, the *local Gibbs state* on $\mathcal{O}_\Lambda$, at inverse temperature $\beta > 0$, is defined by the density matrix

$$\omega_{\beta,\Lambda} = \frac{\mathrm{e}^{-\beta H_\Lambda(\Phi)}}{\mathrm{tr}(\mathrm{e}^{-\beta H_\Lambda(\Phi)})}. \tag{3.1}$$

Using the identification $\mathcal{O} = \mathcal{O}_\Lambda \otimes \mathcal{O}_{\Lambda^c}$, one extends $\omega_{\beta,\Lambda}$ (in an arbitrary way) to a state on $\mathcal{O}$. Denoting this extension by $\overline{\omega}_{\beta,\Lambda}$, any weak*-limit point of the net $(\overline{\omega}_{\beta,\Lambda})_{\Lambda \in \mathfrak{G}_{\mathrm{fin}}}$ is a (τ_Φ, β)-KMS state on $\mathcal{O}$ [BR81, Theorem 6.2.15]. Any (τ_Φ, β)-KMS states that arises in this way is called a *thermodynamic limit* KMS state. This construction in particular gives that the set $\mathcal{S}_{\tau_\Phi,\beta}$ of all (τ_Φ, β)-KMS states is non-empty.

If it happens that $\mathcal{S}_{\tau_\Phi,\beta}$ is a singleton, then the net $(\overline{\omega}_{\beta,\Lambda})_{\Lambda \in \mathfrak{G}_{\mathrm{fin}}}$ converges to the unique (τ_Φ, β)-KMS state. This is known to be the case in the high-temperature regime, *i.e.*, for $\beta\|\Phi\|_\lambda$ small enough. For some concrete estimates, see [BR81, Proposition 6.2.45], or [FU15] for more recent results.

If $\mathcal{S}_{\tau_\Phi,\beta}$ is not a singleton, one needs to supply boundary conditions to the local Gibbs states in order to reach all (τ_Φ, β)-KMS states by the thermodynamic limit. That is our next topic.

The Araki–Gibbs Condition. For any $\Lambda \in \mathfrak{G}_{\mathrm{fin}}$, the so-called surface energies

$$W_\Lambda(\Phi) := \sum_{\substack{X \cap \Lambda \neq \emptyset \\ X \cap \Lambda^c \neq \emptyset}} \Phi(X)$$

are self-adjoint elements of $\mathcal{O}$. Let $\beta > 0$ and $V_\Lambda = \beta W_\Lambda(\Phi)$. Suppose that ω is a modular state on $\mathcal{O}$ and let δ_ω be the generator of its modular C^*-dynamics ς_ω. Consider the perturbed dynamics $\varsigma_{\omega,V_\Lambda}$ generated by $\delta_\omega + \mathrm{i}[V_\Lambda, \cdot\,]$, and let ω_{V_Λ} be the $(\varsigma_{\omega,V_\Lambda}, -1)$-KMS state associated to ω by Araki's perturbation theory [BR81, Theorem 5.4.4]. We say that ω satisfies the (β, Φ) Araki–Gibbs condition if, for all $\Lambda \in \mathfrak{G}_{\mathrm{fin}}$, the restriction of ω_{V_Λ} to $\mathcal{O}_\Lambda$ is given by (3.1). The Araki–Gibbs condition is the quantum extension of the DLR equation in equilibrium theory of classical spin systems. It has been introduced in [AI74], see also [Ara76, BR81, Sim93].

Theorem 3.2 *Suppose that ω is a modular state on $\mathcal{O}$ and $\beta > 0$. Then the following statements are equivalent.*

(1) *ω is a (τ_Φ, β)-KMS state.*

(2) *ω satisfies the (β, Φ) Araki–Gibbs condition.*

3.2 The OQ2S setting

Following [Rue01], we consider a quantum spin system over a set G with interaction Φ satisfying:

(DEC)

(1) G is the disjoint union

$$G = S \sqcup \left(\bigsqcup_{j=1}^{M} R_j \right)$$

of a finite set S and finitely many countably infinite sets $R_1, \ldots, R_M$.

(2) $\mathfrak{G}_{\text{fin}}$ is the set of finite subsets of G, and the indexing set is

$$\mathcal{I} = \{\Lambda \in \mathfrak{G}_{\text{fin}} \mid \Lambda \supset S\}.$$

(3) Besides satisfying Assumption (SR), the interaction $\Phi : \mathfrak{G}_{\text{fin}} \to \mathcal{O}_G$ is such that $\Phi(X) = 0$ whenever there exist $i \neq j$ with $X \cap R_i \neq \emptyset$ and $X \cap R_j \neq \emptyset$.

Assumption (DEC) implies

$$\mathcal{O}_G = \mathcal{O}_S \otimes \left(\bigotimes_{j=1}^{M} \mathcal{O}_{R_j} \right),$$

where $\mathcal{O}_S$ pertains to the *small systems* S, and $\mathcal{O}_{R_j}$ to the j^{th} *reservoir* R_j.

For each $j \in \{1, \ldots, M\}$, we define the interaction

$$\Phi_j(X) = \begin{cases} \Phi(X) & \text{if } X \subseteq R_j; \\ 0 & \text{otherwise}, \end{cases} \tag{3.2}$$

which clearly satisfies $\|\Phi_j\|_\lambda \leq \|\Phi\|_\lambda$. We denote by τ_{Φ_j} the associated C^*-dynamics on $\mathcal{O}_{R_j}$ and by δ_{Φ_j} its generator. We further define

$$V_j = \sum_{\substack{X \subseteq S \cup R_j \\ X \cap S \neq \emptyset, X \cap R_j \neq \emptyset}} \Phi(X), \qquad V = \sum_{j=1}^{M} V_j, \tag{3.3}$$

which are self-adjoint elements of $\mathcal{O}$. The "free" C^*-dynamics τ_{fr} on $\mathcal{O}_G$ is generated by

$$\delta_{\text{fr}} = \delta_S + \sum_{j=1}^{M} \delta_{\Phi_j},$$

where $\delta_S = \mathrm{i}[H_S(\Phi), \cdot\,]$. Obviously,

$$\delta_\Phi = \delta_{\text{fr}} + \mathrm{i}[V, \cdot\,].$$

The model is completed by the choice of a (τ_{Φ_j}, β_j)-KMS state ω_{β_j} on $\mathcal{O}_{R_j}$ for each $j \in \{1, \ldots, M\}$, and by taking

$$\omega = \omega_S \otimes \left(\bigotimes_{j=1}^{M} \omega_{\beta_j} \right) \tag{3.4}$$

for the reference state of $(\mathcal{O}, \tau_\Phi)$, where, for convenience, ω_S is taken to be the tracial state on $\mathcal{O}_S$[7],

$$\omega_S(A) = \frac{\mathrm{tr}(A)}{\dim \mathcal{K}_S}.$$

Obviously, $(\mathcal{O}, \tau_\Phi, \omega)$ is an example of open quantum system as discussed in [BBJ+23, Section 1.1] with reservoirs R_j described by the C^*-quantum dynamical systems $(\mathcal{O}_{R_j}, \tau_{\Phi_j}, \omega_{\beta_j})$. The modular group of the state ω is

$$\varsigma_\omega^\theta = \tau_{\Phi_1}^{-\beta_1\theta} \circ \cdots \circ \tau_{\Phi_M}^{-\beta_M\theta}, \tag{3.5}$$

and its generator is $\delta_\omega = -\sum_j \beta_j \delta_{\Phi_j}$. It follows from the definitions (3.3) that

$$\sigma = \delta_\omega(V) = -\sum_{j=1}^{M} \beta_j \delta_{\Phi_j}(V_j) = -\sum_{j=1}^{M} \beta_j \sum_{\substack{X \subseteq S \cup R_j \\ X \cap S \neq \emptyset, X \cap R_j \neq \emptyset}} \delta_{\Phi_j}(\Phi(X)). \tag{3.6}$$

From Theorem 3.1(4) and the definition (3.2), we further deduce that

$$\|\delta_{\Phi_j}(\Phi(X))\| \leq \frac{2}{\lambda} \|\Phi\|_\lambda \mathrm{e}^{\lambda|X|} \|\Phi(X)\|,$$

so that

$$\|\sigma\| \leq \sum_{j=1}^{M} \beta_j \sum_{x \in S} \sum_{X \ni x} \|\delta_{\Phi_j}(\Phi(X))\| \leq \frac{2|S|\mathrm{e}^\lambda}{\lambda} \|\Phi\|_\lambda^2 \sum_{j=1}^{M} \beta_j,$$

and in particular $V \in \mathrm{dom}(\delta_\omega)$.

We now describe our thermodynamic limit scheme. We write $\Lambda \in \mathcal{I}$ as the disjoint union

$$\Lambda = S \sqcup \left(\bigsqcup_{j=1}^{M} \Lambda_j \right), \qquad \Lambda_j \in \mathfrak{G}_{\mathrm{fin}} \cap R_j,$$

and use the identification

$$\mathcal{O}_\Lambda = \mathcal{O}_S \otimes \left(\bigotimes_{j=1}^{M} \mathcal{O}_{\Lambda_j} \right).$$

Let

$$\omega_{\Lambda_j} = \omega_{\beta_j}\big|_{\mathcal{O}_{\Lambda_j}}, \tag{3.7}$$

[7]None of our results depend on the choice of ω_S as long as $\omega_S > 0$.

and, identifying ω_{Λ_j} with a density matrix on $\mathcal{K}_{\Lambda_j}$,

$$\widehat{H}_{\mathrm{fr},\Lambda_j} = -\frac{1}{\beta_j}\log\omega_{\Lambda_j}. \tag{3.8}$$

The finite volume dynamics τ_Λ is generated by the Hamiltonian

$$\widehat{H}_\Lambda = \widehat{H}_{\mathrm{fr},\Lambda} + V_\Lambda, \tag{3.9}$$

where

$$\widehat{H}_{\mathrm{fr},\Lambda} = H_S(\Phi) + \sum_{j=1}^M \widehat{H}_{\mathrm{fr},\Lambda_j},$$

and

$$V_\Lambda = \sum_{j=1}^M V_{j,\Lambda}, \qquad V_{j,\Lambda} = \sum_{\substack{X\subseteq S\cup\Lambda_j \\ X\cap S\neq\emptyset, X\cap\Lambda_j\neq\emptyset}} \Phi(X).$$

Finally, we observe that

$$\omega_\Lambda = \omega_S \otimes \left(\bigotimes_{j=1}^M \omega_{\Lambda_j}\right) = \omega\big|_{\mathcal{O}_\Lambda}. \tag{3.10}$$

The net $(\mathcal{O}_\Lambda, \tau_\Lambda, \omega_\Lambda)_{\Lambda\in\mathcal{I}}$ defines our TDL scheme. The finite volume entropy production observable is

$$\sigma_\Lambda = \delta_{\omega_\Lambda}(V_\Lambda) = \mathrm{i}[\log\omega_\Lambda, V_\Lambda] = \sum_{j=1}^M \mathrm{i}[\log\omega_{\Lambda_j}, V_j]. \tag{3.11}$$

We denote by $(\mathcal{H}_S, \pi_S, \Omega_S)$ the GNS-representation of $\mathcal{O}_S$ induced by ω_S[8] and by $(\mathcal{H}_j, \pi_j, \Omega_j)$ the GNS-representation of $\mathcal{O}_{R_j}$ associated to ω_{β_j}. For the GNS-representation of $\mathcal{O}_G$ associated to ω we then take $(\mathcal{H}, \pi, \Omega)$ where

$$\Omega = \Omega_S \otimes \left(\bigotimes_{j=1}^M \Omega_j\right) \in \mathcal{H} = \mathcal{H}_S \otimes \left(\bigotimes_{j=1}^M \mathcal{H}_j\right),$$

and

$$\pi = \pi_S \otimes \left(\bigotimes_{j=1}^M \pi_j\right).$$

We have a similar product structure for the TDL scheme, where besides $(\mathcal{H}_S, \pi_S, \Omega_S)$ we take, for $j\in\{1,\ldots,M\}$, $(\mathcal{H}_{\Lambda_j}, \pi_{\Lambda_j}, \Omega_{\Lambda_j})$ to be

$$\pi_{\Lambda_j} = \pi_j\big|_{\mathcal{O}_{\Lambda_j}}, \qquad \Omega_{\Lambda_j} = \Omega_j, \qquad \mathcal{H}_{\Lambda_j} = \pi_j(\mathcal{O}_{\Lambda_j})\Omega_j \subset \mathcal{H}_j.$$

[8] As in [BBJ+23], we take $\mathcal{H}_S = \mathcal{K}_S\otimes\mathcal{K}_S$, $\pi_S(A) = A\otimes\mathbb{1}$, $\Omega_S = \frac{1}{\sqrt{N}}\sum_i \psi_i\otimes\psi_i$, where $\{\psi_i\}$ is an orthonormal basis of $\mathcal{K}_S$ and $N = \dim\mathcal{K}_S$.

Other choices for τ_Λ and ω_Λ are possible. An arguably simpler choice is to take in (3.10)

$$\omega_{\Lambda_j} = \frac{\mathrm{e}^{-\beta_j H_{\Lambda_j}(\Phi_j)}}{\mathrm{tr}(\mathrm{e}^{-\beta_j H_{\Lambda_j}(\Phi_j)})}, \tag{3.12}$$

and for τ_Λ the dynamics $\tau_{\Phi,\Lambda}$ generated by

$$H_\Lambda(\Phi) = H_S(\Phi) + \sum_{j=1}^{M} H_{\Lambda_j}(\Phi_j) + V_\Lambda. \tag{3.13}$$

However, our proofs do not work for this choice and we will comment more on this point in Sections 3.4 and 4.4.

3.3 TDL of 2TMEP

Besides Assumption (DEC), we will also need

(SE) Each reservoir system $(\mathcal{O}_{R_j}, \tau_{\Phi_j}, \omega_{\beta_j})$ is ergodic, *i.e.*, for any ω_{β_j}-normal state ν on $\mathcal{O}_{R_j}$ and any $A \in \mathcal{O}_{R_j}$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \nu \circ \tau^t_{\Phi_j}(A)\mathrm{d}t = \omega_{\beta_j}(A).$$

(SE) will be used only to verify (TDL3)(2). We set

$$\overline{\beta}_\lambda := \|\Phi\|_\lambda \max_j \beta_j.$$

Theorem 3.3 *Suppose that* (DEC) *and* (SE) *hold with* $\overline{\beta}_\lambda < \lambda$. *Then Assumptions* (TDL1)–(TDL3) *hold. In particular, Proposition 2.1 and Theorem 2.3(1) hold for* OQ2S.

The rest of this section is devoted to the proof of Theorem 3.3 through a sequence of Lemmata. The road map from (DEC)+(SE) to (TDL1)–(TDL3) is summarized in the following chart.

(DEC) with $\lambda > \overline{\beta}_\lambda$	Lemma 3.4 $\Longrightarrow$	(TDL5)	Proposition 2.5 $\Longrightarrow$	(TDL1)
(DEC)	Lemma 3.5 $\Longrightarrow$	(TDL2)(1–5)	(TDL6)(1–3)	
(DEC) with $\lambda > \overline{\beta}_\lambda$	Lemma 3.7 $\Longrightarrow$	(TDL2)(6)		
(DEC)(SE)	Lemma 3.8 $\Longrightarrow$	(TDL3)		

Lemma 3.4 *Suppose that (DEC) holds. Then,*

(1) *The map*

$$\mathbb{R} \ni \theta \mapsto \varsigma_\omega^\theta(V) \in \mathcal{O}$$

has an analytic continuation to the horizontal strip $|\mathrm{Im}\,\theta| < \frac{\lambda}{2\overline{\beta}_\lambda}$.

(2) *The map*

$$\mathbb{R} \ni \theta \mapsto \varsigma_\omega^\theta(\sigma) \in \mathcal{O}$$

has an analytic continuation to the horizontal strip $|\mathrm{Im}\,\theta| < \frac{\lambda}{2\overline{\beta}_\lambda}$.

(3) *For all* $s \in \mathbb{R}$ *the map*

$$\mathrm{i}\mathbb{R} \ni z \mapsto [D\omega_s : D\omega]_z$$

has an analytic continuation to the vertical strip $|\mathrm{Re}\,z| < \frac{\lambda}{2\overline{\beta}_\lambda}$.

Each of these maps is bounded on any closed substrip of the respective strip. In particular, if (SR) holds with $\lambda > \overline{\beta}_\lambda$*, then Assumptions (TDL1) and (TDL5) hold.*

Proof. **(1)** By Relations (3.3) and (3.5), one has

$$\varsigma_\omega^\theta(V) = \sum_{j=1}^M \tau_{\Phi_j}^{-\beta_j\theta}(V_j).$$

Hence, it suffices to show that for any $j \in \{1,\ldots,M\}$, the function

$$\mathbb{R} \ni t \mapsto \tau_{\Phi_j}^t(V_j) \tag{3.14}$$

has an analytic continuation to the strip $|\mathrm{Im}\,z| < \frac{\lambda}{2\|\Phi\|_\lambda}$. By Theorem 3.1(4) and the definition (3.2), for any positive integer n one has

$$\begin{aligned}\frac{1}{n!}\|\delta_{\Phi_j}^n(V_j)\| &\le \frac{1}{n!}\sum_{\substack{X\subseteq S\cup R_j\\ X\cap S\neq\emptyset, X\cap R_j\neq\emptyset}} \|\delta_{\Phi_j}^n(\Phi(X))\| \\ &\le \sum_{x\in S}\sum_{X\ni x}\left(\frac{2\|\Phi\|_\lambda}{\lambda}\right)^n \mathrm{e}^{\lambda|X|}\,\|\Phi(X)\| \le |S|\mathrm{e}^\lambda\|\Phi\|_\lambda\left(\frac{2\|\Phi\|_\lambda}{\lambda}\right)^n,\end{aligned}$$

so that the radius of convergence of the Taylor series of the function (3.14) is $\lambda/2\|\Phi\|_\lambda$. The result then follows from the group property of τ_{Φ_j}.

(2) Follows from (1) by differentiation.

(3) Follows from (2) and the proof of Proposition 2.5.

It is an immediate consequence of the group property and the isometric nature of ς_ω that the three maps are uniformly bounded on any closed substrip of their respective strip of analyticity. □

Lemma 3.5 *Assumption* (DEC) *implies* (TDL2)(1–5) *and* (TDL6)(1–3).

Proof. Parts (1–3) and (5) of (TDL2) as well as Part (1) of (TDL6) are immediate consequences of the definition of the TDL scheme and do not depend on (SR). The same is true for the first statement of (TDL6)(2), while (SR) implies the second statement, namely that

$$\lim_{\Lambda} V_\Lambda = V \tag{3.15}$$

holds in the norm of $\mathcal{O}$.

(TDL6)(3) is an immediate consequence of Theorem 2.2(2). Indeed,

$$\mathrm{e}^{\mathrm{i}t\mathcal{L}_{\mathrm{fr}}} = \prod_{j=1}^{M} \Delta_{\omega_{\beta_j}}^{-\mathrm{i}\beta_j t},$$

where $\Delta_{\omega_{\beta_j}}$ is the modular operator of ω_j, and similarly

$$\mathrm{e}^{\mathrm{i}t\mathcal{L}_{\Lambda,\mathrm{fr}}} = \prod_{j=1}^{M} \Delta_{\omega_{\Lambda_j}}^{-\mathrm{i}\beta_j t}.$$

These two observations and Theorem 2.2(2) give that

$$\text{s–}\lim_{\Lambda} \mathrm{e}^{\mathrm{i}t\mathcal{L}_{\Lambda,\mathrm{fr}}} = \mathrm{e}^{\mathrm{i}t\mathcal{L}_{\mathrm{fr}}}, \tag{3.16}$$

locally uniformly for $t \in \mathbb{R}$.

To prove Part (4) of (TDL2), consider the Dyson expansion

$$\begin{aligned}\mathrm{e}^{\mathrm{i}t(\mathcal{L}_{\Lambda,\mathrm{fr}}+V_\Lambda)} &= \mathrm{e}^{\mathrm{i}t\mathcal{L}_{\Lambda,\mathrm{fr}}}\\ &+ \sum_{n\geq 1} \mathrm{i}^n \int\limits_{0\leq t_1\leq\cdots\leq t_n\leq t} \mathrm{e}^{\mathrm{i}t_1\mathcal{L}_{\Lambda,\mathrm{fr}}} V_\Lambda \mathrm{e}^{\mathrm{i}(t_2-t_1)\mathcal{L}_{\Lambda,\mathrm{fr}}} V_\Lambda \cdots \mathrm{e}^{\mathrm{i}t(t_n-t_{n-1})\mathcal{L}_{\Lambda,\mathrm{fr}}} V_\Lambda \mathrm{e}^{\mathrm{i}(t-t_n)\mathcal{L}_{\Lambda,\mathrm{fr}}} \mathrm{d}t_1\cdots\mathrm{d}t_n.\end{aligned}$$

Invoking (3.15) and (3.16) gives that

$$\text{s–}\lim_{\Lambda} \mathrm{e}^{\mathrm{i}t(\mathcal{L}_{\Lambda,\mathrm{fr}}+V_\Lambda)} = \mathrm{e}^{\mathrm{i}t(\mathcal{L}_{\mathrm{fr}}+V)}, \tag{3.17}$$

locally uniformly for $t \in \mathbb{R}$. Finally, for $A \in \mathcal{O}_{\mathrm{loc}}$[9],

$$\begin{aligned}\text{s–}\lim_{\Lambda} \tau_\Lambda^t(A) &= \text{s–}\lim_{\Lambda} \mathrm{e}^{\mathrm{i}t(\mathcal{L}_{\Lambda,\mathrm{fr}}+V_\Lambda)} A \mathrm{e}^{-\mathrm{i}t(\mathcal{L}_{\Lambda,\mathrm{fr}}+V_\Lambda)}\\ &= \mathrm{e}^{\mathrm{i}t(\mathcal{L}_{\mathrm{fr}}+V)} A \mathrm{e}^{-\mathrm{i}t(\mathcal{L}_{\mathrm{fr}}+V)} = \tau^t(A)\end{aligned}$$

locally uniformly for $t \in \mathbb{R}$. Note that for this argument it is sufficient that s–$\lim_\Lambda V_\Lambda = V$. □

[9] One can take here any $A \in \mathcal{O}_G$.

Lemma 3.6 *Suppose that* (DEC) *holds with* $\lambda > \overline{\beta}_\lambda$. *Then,*

$$\operatorname*{s-lim}_{\Lambda} \varsigma^{\theta}_{\omega_\Lambda}(V_\Lambda) = \varsigma_\omega(V), \tag{3.18}$$

locally uniformly for $|\mathrm{Im}\,\theta| \le \frac{1}{2}$.

Proof. For $\Lambda \in \mathcal{I}$, let us define the interaction Φ_Λ by

$$\Phi_\Lambda(X) = \begin{cases} \Phi(X) & \text{if } X \subseteq \Lambda; \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$\lim_{\Lambda} \|\Phi - \Phi_\Lambda\|_\lambda = \lim_{\Lambda} \sup_{x\in G} \sum_{\substack{X \ni x \\ X\cap\Lambda^c \neq \emptyset}} \|\Phi(X)\| \mathrm{e}^{\lambda(|X|-1)} = 0.$$

Since, for $j \in \{1, \dots, M\}$,

$$D_{j,\Lambda} = V_j - V_{j,\Lambda} = \sum_{\substack{X\subseteq S\cup R_j \\ X\cap S\neq\emptyset\neq X\cap\Lambda_j^c}} \Phi(X),$$

the estimate of Theorem 3.1(4) gives

$$\frac{1}{n!}\|\delta^n_{\Phi_j}(D_\Lambda)\| \le |S|\mathrm{e}^\lambda \left(\frac{2\|\Phi\|_\lambda}{\lambda}\right)^n \sup_{x\in S} \sum_{\substack{X\ni x \\ X\cap\Lambda^c\neq\emptyset}} \|\Phi(X)\|\mathrm{e}^{\lambda(|X|-1)} \le |S|\mathrm{e}^\lambda \left(\frac{2\|\Phi\|_\lambda}{\lambda}\right)^n \|\Phi - \Phi_\Lambda\|_\lambda,$$

which, by the argument in the proof of Lemma 3.4, implies that

$$\lim_{\Lambda} \varsigma^\theta_\omega(V_\Lambda) = \varsigma^\theta_\omega(V) \tag{3.19}$$

in norm, locally uniformly for $|\mathrm{Im}\,\theta| < \frac{\lambda}{2\overline{\beta}_\lambda}$.

Since $\Omega_\Lambda = \Omega$ and $\pi_\Lambda = \pi$, for any $A \in \mathcal{O}_\Lambda$, the definition of the modular operator and conjugation gives

$$J_\Lambda \Delta^{\frac{1}{2}}_{\omega_\Lambda} \pi(A)\Omega = J\Delta^{\frac{1}{2}}_\omega \pi(A)\Omega.$$

Since $\pi(\mathcal{O}_\Lambda)\Omega = \mathcal{H}_\Lambda$, we have that, as operators on $\mathcal{H}_\Lambda$,

$$\Delta^{\frac{1}{2}}_{\omega_\Lambda} = J_\Lambda J \Delta^{\frac{1}{2}}_\omega, \qquad \Delta^{-\frac{1}{2}}_{\omega_\Lambda} = \Delta^{-\frac{1}{2}}_\omega J J_\Lambda,$$

and so

$$\begin{aligned} \varsigma^{\frac{1}{2}}_{\omega_\Lambda}(V_\Lambda) &= \Delta^{\frac{1}{2}}_{\omega_\Lambda} V_\Lambda \Delta^{-\frac{1}{2}}_{\omega_\Lambda} = J_\Lambda J \Delta^{\frac{1}{2}}_\omega V_\Lambda \Delta^{-\frac{1}{2}}_\omega J J_\Lambda, \\ \varsigma^{-\frac{1}{2}}_{\omega_\Lambda}(V_\Lambda) &= \Delta^{-\frac{1}{2}}_{\omega_\Lambda} V_\Lambda \Delta^{\frac{1}{2}}_{\omega_\Lambda} = J_\Lambda J \Delta^{-\frac{1}{2}}_\omega V_\Lambda \Delta^{\frac{1}{2}}_\omega J J_\Lambda, \end{aligned} \tag{3.20}$$

on $\mathcal{H}_\Lambda$. By Theorem 2.2(2–3), the relations (3.19)–(3.20) and the group property give that

$$\operatorname*{s-lim}_{\Lambda} \varsigma^{\theta}_{\omega_\Lambda}(V_\Lambda) = \operatorname*{s-lim}_{\Lambda} \Delta^{\mathrm{iRe}\,\theta}_{\omega_\Lambda} J_\Lambda J \varsigma^{\mathrm{iIm}\,\theta}_{\omega}(V_\Lambda) J J_\Lambda \Delta^{-\mathrm{iRe}\,\theta}_{\omega_\Lambda} = \varsigma^{\theta}_{\omega}(V),$$

locally uniformly for $|\mathrm{Im}\,\theta| = \frac{1}{2}$. Applying Hadamard's three lines theorem to the function

$$f(\theta) = (\varsigma^{\theta}_{\omega}(V) - \varsigma^{\theta}_{\omega_\Lambda}(V_\Lambda))\Psi$$

with $\Psi \in \mathcal{H}$ yields the convergence (3.18) with the required uniformity. □

Lemma 3.7 *Suppose that (DEC) holds with $\lambda > \overline{\beta}_\lambda$. Then (TDL2)(6) holds.*

Proof. By Relations (3.6) and (3.11) and Lemma 3.4(1), we have

$$\partial_\theta \varsigma^{\theta}_{\omega_\Lambda}(V_\Lambda) = \varsigma^{\theta}_{\omega_\Lambda}(\sigma_\Lambda), \qquad \partial_\theta \varsigma^{\theta}_{\omega}(V) = \varsigma^{\theta}_{\omega}(\sigma),$$

for $|\mathrm{Im}\,\theta| < \frac{1}{2}$. Thus, Lemma 3.6 and Weierstrass' convergence theorem allow us to conclude that

$$\operatorname*{s-lim}_{\Lambda} \varsigma^{\theta}_{\omega_\Lambda}(\sigma_\Lambda) = \varsigma^{\theta}_{\omega}(\sigma), \tag{3.21}$$

locally uniformly for $|\mathrm{Im}\,\theta| < \frac{1}{2}$. In particular,

$$\operatorname*{s-lim}_{\Lambda} \sigma_\Lambda = \sigma,$$

and, invoking the uniform boundedness principle, $\sup_\Lambda \|\sigma_\Lambda\| < \infty$. Hence, using (3.17), we get,

$$\operatorname*{s-lim}_{\Lambda} \int_0^s \tau_\Lambda^{-t}(\sigma_\Lambda)\mathrm{d}t = \int_0^s \tau^{-t}(\sigma)\mathrm{d}t. \tag{3.22}$$

By Lemma 2.4, we further have

$$\log \Delta_{\omega_{\Lambda,s}|\omega_\Lambda} = \log \Delta_{\omega_\Lambda} + \int_0^s \tau_\Lambda^{-t}(\sigma_\Lambda)\mathrm{d}t,$$
$$\log \Delta_{\omega_s|\omega} = \log \Delta_{\omega} + \int_0^s \tau^{-t}(\sigma)\mathrm{d}t,$$

for all $s \in \mathbb{R}$. By Theorem 2.2(2) and Trotter's theorem [RS80, Theorem VIII.21], these two relations and (3.22) allow us to conclude that, for all $s \in \mathbb{R}$,

$$\lim_{\Lambda} \log \Delta_{\omega_{\Lambda,s}|\omega_\Lambda} = \log \Delta_{\omega_s|\omega}$$

holds in the strong resolvent sense. By Trotter's theorem again

$$\operatorname*{s-lim}_{\Lambda} \Delta^{\mathrm{i}\theta}_{\omega_{\Lambda,s}|\omega_\Lambda} = \Delta^{\mathrm{i}\theta}_{\omega_s|\omega},$$

holds for all $\theta, s \in \mathbb{R}$, and together with Theorem 2.2(2),

$$\operatorname*{s-lim}_{\Lambda} \Delta^{\mathrm{i}\theta}_{\omega_{\Lambda,s}|\omega_\Lambda} \Delta^{-\mathrm{i}\theta}_{\omega_\Lambda} = \Delta^{\mathrm{i}\theta}_{\omega_s|\omega} \Delta^{-\mathrm{i}\theta}_{\omega},$$

which is (TDL2)(6). □

Finally, we complete the proof of Theorem 3.3 with

Lemma 3.8 *Assumption (SE) implies (TDL3).*

Proof. (TDL3)(1) follows from Theorem 2.2(2). Given the product structure (3.4) of the reference state ω, one has

$$\log \Delta_\omega = 0_S \oplus \left(\bigoplus_{j=1}^{M} \log \Delta_{\omega_{\beta_j}} \right) = 0_S \oplus \left(- \bigoplus_{j=1}^{M} \beta_j \mathcal{L}_j \right),$$

where 0_S denotes the zero operator on $\mathcal{H}_S$ and $\mathcal{L}_j$ is the standard Liouvillean of the C^*-dynamics τ_{Φ_j}. By the quantum Koopman lemma [Pil06, Section 4.7], the C^* dynamical system $(\mathcal{O}_{R_j}, \tau_{\Phi_j}, \omega_{\beta_j})$ is ergodic iff $\ker \mathcal{L}_j = \mathbb{C}\Omega_j$. Thus, Assumption (SE) gives that

$$\ker \log \Delta_\omega = \mathcal{H}_S \otimes \Omega_1 \otimes \cdots \otimes \Omega_M.$$

By the definition (3.10) of ω_Λ,

$$\log \Delta_{\omega_\Lambda} = 0_S \oplus \left(\bigoplus_{j=1}^{M} \log \Delta_{\omega_{\Lambda_j}} \right),$$

and since $\ker \log \Delta_{\omega_{\Lambda_j}} \ni \Omega_{\Lambda_j} = \Omega_j$, one has

$$\mathcal{H}_S \otimes \Omega_1 \otimes \cdots \otimes \Omega_M \subseteq \ker \log \Delta_{\omega_\Lambda}.$$

This gives that (TDL3)(2) follows. □

3.4 Remarks

1. On the choice of ω_Λ and τ_Λ. The proof of Theorem 3.3 combines fairly standard thermodynamic limit arguments with Araki's continuity Theorem 2.2. The definitions (3.7)–(3.9) of ω_Λ and τ_Λ take into the account the interaction "boundary terms" in a way that ensures the necessary continuity of the modular structure that is behind our TDL scheme. This continuity is broken with choice (3.12) and (3.13), and in this case it is likely that a completely new strategy of the proof is needed.

We remark that for the choice (3.12)–(3.13) it is not difficult to show that for all $t \in \mathbb{R}$ and $\alpha \in \mathrm{i}\mathbb{R}$,

$$\lim_{\Lambda} \varsigma^{\theta}_{\omega_\Lambda}([D\omega_{\Lambda,-t} : D\omega_\Lambda]_\alpha) = \varsigma^{\theta}_{\omega}([D\omega_{-t} : D\omega]_\alpha) \tag{3.23}$$

in norm, locally uniformly for $\theta \in \mathbb{R}$. Hence, if ω is a thermodynamic limit KMS-state and ω_{Λ_γ} is subnet such that $\lim_{\Lambda_\gamma} \overline{\omega}_{\Lambda_\gamma} = \omega$, we have

$$\lim_{\Lambda_\gamma} \varsigma^{\theta}_{\omega_{\Lambda_\gamma}}([D\omega_{\Lambda_\gamma,-t} : D\omega_{\Lambda_\gamma}]_\alpha) = \varsigma^{\theta}_{\omega}([D\omega_{-t} : D\omega]_\alpha),$$

providing the desired thermodynamic justification of (1.1)–(1.2) in the case $\nu = \omega$ that is much simpler than the proof of Theorem 3.3. The convergence (3.23) also gives that

$$\lim_{R\to\infty} \lim_{\Lambda} \frac{1}{R} \int_0^R \nu_\Lambda \left(\varsigma^{\theta}_{\omega_\Lambda} \left([D\omega_{\Lambda,-t} : D\omega_\Lambda]_\alpha \right) \right) \mathrm{d}\theta = \lim_{R\to\infty} \frac{1}{R} \int_0^R \nu \left(\varsigma^{\theta}_{\omega} \left([D\omega_{-t} : D\omega]_\alpha \right) \right) \mathrm{d}\theta$$

if $\nu_\Lambda \to \nu$ weakly. Thus, the TDL justification of (1.1)–(1.2) is equivalent to the justification of the exchange limits

$$\lim_{R\to\infty}\lim_{\Lambda}\frac{1}{R}\int_0^R \nu_\Lambda\left(\varsigma^\theta_{\omega_\Lambda}\left([D\omega_{\Lambda,-t}:D\omega_\Lambda]_\alpha\right)\right)\mathrm{d}\theta = \lim_{\Lambda}\lim_{R\to\infty}\frac{1}{R}\int_0^R \nu_\Lambda\left(\varsigma^\theta_{\omega_\Lambda}\left([D\omega_{\Lambda,-t}:D\omega_\Lambda]_\alpha\right)\right)\mathrm{d}\theta.$$

It remains an open question whether this holds for the choice (3.12)–(3.13). For the choice (3.7)–(3.8) and the special class of ν's this question is resolved in Theorem 3.3 by a strategy that invokes Araki's continuity results and the reservoir ergodicity assumptions.

2. On the ergodicity assumption. The reservoir ergodicity assumption plays an important role in the 2TMEP stability result established in [BBJ+23, Theorem 1.5]. However, its emergence in the study of TDL of 2TMEP perhaps comes as a surprise. Although the need for (SE) is clear given our strategy of the proof, its role in the TDL of 2TMEP remains to be understood better. Needless to say, although (SE) is believed to hold for generic spin interactions, there are very few examples for which it has been established. The case of EBBM is very different and there the respective reservoir ergodicity assumption follows from a simple natural criterion.

3. On the assumption $\lambda > \overline{\beta}_\lambda$. This high-temperature assumption also appeared in the recent work [JPT23, Theorem 2.11] and ensures that the reservoirs states ω_{β_j} are weak Gibbs, although we did not make use of this fact. The problem of TDL justification of the 2TMEP for OQ2S for arbitrary β_j's remains open.

4. TDL of the entropy balance equation. We recall that the relative entropy of two density matrices ρ and ν is

$$\mathrm{Ent}(\nu|\rho) = \mathrm{tr}(\nu(\log\rho - \log\nu)). \tag{3.24}$$

Its basic property is that $\mathrm{Ent}(\nu|\rho) \leq 0$ with equality iff $\rho = \nu$.

In the general setting of algebraic statistical mechanics, the relative entropy of a pair (ν, ρ) of normal states has been introduced by Araki in the seminal papers [Ara76, Ara77]. With the sign and ordering convention of [JP01], Araki's definition reads

$$\mathrm{Ent}(\nu|\rho) = \langle \Omega_\nu, \log\Delta_{\rho|\nu}\Omega_\nu\rangle.$$

It shares the above mentioned basic property with (3.24) and reduces to (3.24) in the finite dimensional setting.

For additional information about relative entropy we refer the reader to [OP93].

Returning to our OQ2S, Lemma 2.4 gives that

$$\mathrm{Ent}(\omega_s|\omega) = -\int_0^s \omega_t(\sigma)\mathrm{d}t, \tag{3.25}$$

and similarly, for either of the choices (3.7)–(3.9)/(3.12)–(3.13),

$$\mathrm{Ent}(\omega_{\Lambda,s}|\omega_\Lambda) = -\int_0^s \omega_{\Lambda,t}(\sigma_\Lambda)\mathrm{d}t, \tag{3.26}$$

with $\sigma_\Lambda = \delta_{\omega_\Lambda}(V_\Lambda)$. The identities (3.25)–(3.26) are the entropy balance equations of [Rue01, JP01]. Note that for the choice (3.12)–(3.13),

$$\sigma_\Lambda = \delta_{\omega_\Lambda}(V_\Lambda) = -\sum_{j=1}^{M} \beta_j \, \mathrm{i}[H_{\Lambda_j}(\Phi_j), V_\Lambda] = \sum_{j=1}^{M} \beta_j \, \mathrm{i}[H_\Lambda(\Phi), H_{\Lambda_j}(\Phi_j)], \tag{3.27}$$

which is Ruelle's original definition of the entropy production observable in [Rue01].

For the choice (3.12)–(3.13), assuming (SR), Theorem 3.1(1+4) ensures that for $A \in \mathcal{O}_{\mathrm{loc}}$ and locally uniformly for $t \in \mathbb{R}$,

$$\lim_\Lambda \sigma_\Lambda = \sigma, \qquad \lim_\Lambda \tau_\Lambda^t(A) = \tau^t(A),$$

hold in norm. Given $\epsilon > 0$, there is $\Lambda' \in \mathcal{I}$ such that $\|\sigma - \sigma_{\Lambda'}\| < \epsilon$. Let $\overline{\omega}_\Lambda$ be an arbitrary extension of ω_Λ to a state on $\mathcal{O}$ such that ω is the weak*-limit of the subnet $\overline{\omega}_{\Lambda_\gamma}$. Then, we have

$$\begin{aligned} |(\omega_t - \overline{\omega}_{\Lambda_\gamma,t})(\sigma)| &\leq |(\omega_t - \overline{\omega}_{\Lambda_\gamma,t})(\sigma_{\Lambda'})| + |(\omega_t - \overline{\omega}_{\Lambda_\gamma,t})(\sigma - \sigma_{\Lambda'})| \\ &\leq |(\omega_t - \overline{\omega}_{\Lambda_\gamma,t})(\sigma_{\Lambda'})| + 2\epsilon \\ &\leq |(\omega - \overline{\omega}_{\Lambda_\gamma}) \circ \tau^t(\sigma_{\Lambda'})| + |\overline{\omega}_{\Lambda_\gamma} \circ (\tau^t - \tau^t_{\Lambda_\gamma})(\sigma_{\Lambda'})| + 2\epsilon, \end{aligned}$$

so that

$$\limsup_{\Lambda_\gamma} |(\omega_t - \overline{\omega}_{\Lambda_\gamma,t})(\sigma)| \leq 2\epsilon,$$

from which we can conclude that $\lim_{\Lambda_\gamma} \overline{\omega}_{\Lambda_\gamma,t}(\sigma) = \omega_t(\sigma)$ and hence that

$$\lim_{\Lambda_\gamma} |\omega_t(\sigma) - \omega_{\Lambda_\gamma,t}(\sigma_{\Lambda_\gamma})| \leq \lim_{\Lambda_\gamma} |\omega_t(\sigma) - \overline{\omega}_{\Lambda_\gamma,t}(\sigma)| + \lim_{\Lambda_\gamma} |\overline{\omega}_{\Lambda_\gamma,t}(\sigma - \sigma_{\Lambda_\gamma})| = 0.$$

Thus, (3.25)-(3.26) give

$$\lim_{\Lambda_\gamma} \mathrm{Ent}(\omega_{\Lambda_\gamma,s}|\omega_{\Lambda_\gamma}) = \mathrm{Ent}(\omega_s|\omega),$$

for all $s \in \mathbb{R}$. Similarly, for the choice (3.7)–(3.9) and assuming that (SR) holds with $\lambda > \overline{\beta}_\lambda$,

$$\lim_{\Lambda} \mathrm{Ent}(\omega_{\Lambda,s}|\omega_\Lambda) = \mathrm{Ent}(\omega_s|\omega)$$

for all $s \in \mathbb{R}$.

To connect further with the work [Rue01], one can consider an intermediate scenario where the state ω_Λ is chosen by (3.7) and τ_Λ and σ_Λ by (3.13) and (3.27). The average entropy production over the time interval $[0, s]$ is then

$$-\int_0^s \omega_\Lambda(\tau_\Lambda^t(\sigma_\Lambda))\mathrm{d}t.$$

This entropy production cannot be directly linked to the relative entropy and does not need to be non-negative. However, since

$$-\lim_\Lambda \int_0^s \omega_\Lambda(\tau_\Lambda^t(\sigma_\Lambda))\mathrm{d}t = -\int_0^s \omega_t(\sigma)\mathrm{d}t = \mathrm{Ent}(\omega_s|\omega),$$

the basic properties of the averaged entropy production are restored in the thermodynamic limit. Note that for this choice our TDL proofs for 2TMEP do not work due to the lack of commutativity of the groups $\tau_{\Lambda_j,\mathrm{fr}}$ and $\varsigma_{\omega_{\Lambda_j}}$. Attempts to control this non-commutativity fail for reasons we will discuss in the next and final remark.

In summary, the TDL of the averaged entropy production of OQ2S is robust and does not exhibit the same subtleties as the TDL of 2TMEP.

5. On the Araki–Gibbs Condition. The reader may have noticed that our failure to establish (TDL6)(4) stems from the restriction $|\mathrm{Im}\,\theta| \leq \frac{1}{2}$ in Lemma 3.6 that is forced by the identities (3.20). This restriction allows us to establish (3.21) only for $|\mathrm{Im}\,\theta| < \frac{1}{2}$ and we cannot reach (TDL6)(4) due to the lack of control on the line $\mathrm{Im}\,\theta = \frac{1}{2}$. One can attempt a different approach by using directly (3.20) with σ_Λ instead of V_Λ or by using Theorem 2.2(4), and indeed for the choice (3.12)–(3.13) where σ_Λ is given by (3.27) this works relatively effortlessly. However, for the choice (3.7)–(3.8) one faces perennial difficulties in controlling $\log \omega_\Lambda$ on the basis of the Araki–Gibbs Condition; see [JPT23, Section 2.2] for related discussion and references. Comparing with Remark 1, one arrives at the following dual problems that must be understood better before further progress is made:

(a) The lack of continuity of modular structure, in the spirit of Theorem 2.2, for the choice (3.12)–(3.13).

(b) The lack of control of $\log \omega_\Lambda$ and σ_Λ, on the basis of the Araki–Gibbs condition, for the choice (3.7)–(3.8).

These problems are absent in the EBBM and related models that involve non-interacting ideal reservoirs.

4 The Electronic Black Box Model

4.1 The free Fermi gas

Let $\mathfrak{h}$ and h be the single fermion Hilbert space and Hamiltonian. We denote by $\mathrm{CAR}(\mathfrak{h})$ the CAR algebra over $\mathfrak{h}$, and by $a^*(f)/a(f)/\varphi(f)$ the creation/annihilation/field operator associated to $f \in \mathfrak{h}$. $a^\#$ stands for either a or a^*, and we recall that $\|a^\#(f)\| = \|f\|$. The group τ of Bogoliubov $*$-automorphisms of $\mathrm{CAR}(\mathfrak{h})$ generated by h is uniquely specified by $\tau^t(a^\#(f)) = a^\#(\mathrm{e}^{ith} f)$. The C^*-dynamical system $(\mathrm{CAR}(\mathfrak{h}), \tau)$ describes the free Fermi gas over $(\mathfrak{h}, h)$. We denote by $\mathrm{CAR}_g(\mathfrak{h})$ the gauge-invariant C^*-subalgebra of $\mathrm{CAR}(\mathfrak{h})$ generated by $\{a^*(f)a(g) \mid f, g \in \mathfrak{h}\} \cup \{\mathbb{1}\}$. The dynamics τ obviously preserves $\mathrm{CAR}_g(\mathfrak{h})$. For a given distribution operator $0 \leq T < \mathbb{1}$ on $\mathfrak{h}$, ω_T denotes the quasi-free state on $\mathrm{CAR}(\mathfrak{h})$ generated by T. The same letter will be used for the restriction of ω_T to $\mathrm{CAR}_g(\mathfrak{h})$. ω_T is faithful iff $\ker T$ is trivial, which we assume to hold in what follows. Of particular importance is the Fermi-Dirac distribution

$$T_{\beta\mu} = \frac{1}{1 + \mathrm{e}^{\beta(h - \mu\mathbb{1})}},$$

where $\beta > 0$ and $\mu \in \mathbb{R}$. We write $\omega_{\beta\mu}$ for $\omega_{T_{\beta\mu}}$ and remark that $\omega_{\beta\mu}$ is a (τ, β)-KMS state on $\mathrm{CAR}_g(\mathfrak{h})$. The C^*-quantum dynamical system $(\mathrm{CAR}(\mathfrak{h}), \tau, \omega_{\beta\mu})$ describes a free Fermi gas in thermal equilibrium at inverse temperature β and chemical potential μ. The subsystem $(\mathrm{CAR}_g(\mathfrak{h}), \tau, \omega_{\beta\mu})$ describes its gauge-invariant part, and will be the focus of this section. The full system $(\mathrm{CAR}(\mathfrak{h}), \tau, \omega_{\beta\mu})$ will be discussed in Section 4.4.

We recall the following well-known fact, see [Pil06] for a pedagogical discussion.

Theorem 4.1 *Suppose that h commutes with T and has purely absolutely continuous spectrum. Then the quantum dynamical systems $(\mathrm{CAR}(\mathfrak{h}), \tau, \omega_T)$ and $(\mathrm{CAR}_g(\mathfrak{h}), \tau, \omega_T)$ are mixing, and in particular ergodic.*

We will make use of the Araki–Wyss GNS-representation $(\mathcal{H}_{\mathrm{AW}}, \pi_{\mathrm{AW}}, \Omega_{\mathrm{AW}})$ of $\mathrm{CAR}(\mathfrak{h})$ associated to ω_T. This representation was constructed in [AW64]; see [DG13, JOPP10] for pedagogical introductions to the topic. We assume that the reader is familiar with the fermionic second quantization, and for definiteness we will use the same notation as in [JP02b]. We fix a complex conjugation $\bar{\cdot}$ on $\mathfrak{h}$ and assume that it commutes with T[10] and h. The Araki–Wyss representation is given by

$$
\begin{aligned}
\mathcal{H}_{\mathrm{AW}} &= \Gamma_-(\mathfrak{h}) \otimes \Gamma_-(\mathfrak{h}), \\
\pi_{\mathrm{AW}}(a(f)) &= a((\mathbb{1} - T)^{1/2} f) \otimes \mathbb{1} + \vartheta \otimes a^*(T^{1/2}\overline{f}), \\
\pi_{\mathrm{AW}}(a^*(f)) &= a^*((\mathbb{1} - T)^{1/2} f) \otimes \mathbb{1} + \vartheta \otimes a(T^{1/2}\overline{f}), \\
\Omega_{\mathrm{AW}} &= \Omega_{\mathrm{f}} \otimes \Omega_{\mathrm{f}},
\end{aligned}
$$

where $\Gamma_-(\mathfrak{h})$ is the fermionic Fock space over $\mathfrak{h}$, $\vartheta = \Gamma(-\mathbb{1}) = \mathrm{e}^{\mathrm{i}\pi N}$ where N is the number operator, and Ω_{f} is the Fock vacuum on $\Gamma_-(\mathfrak{h})$. The standard Liouvillean of τ is

$$
\mathcal{L} = \mathrm{d}\Gamma(h) \otimes \mathbb{1} - \mathbb{1} \otimes \mathrm{d}\Gamma(h),
$$

and

$$
\log \Delta_{\omega_T} = \mathrm{d}\Gamma(k_T) \otimes \mathbb{1} - \mathbb{1} \otimes \mathrm{d}\Gamma(k_T),
$$

where $k_T = \log(T(\mathbb{1} - T)^{-1})$. Note that if $T = T_{\beta\mu}$, then $k_T = -\beta(h - \mu\mathbb{1})$ and

$$
\Delta_{\omega_{\beta\mu}} = \mathrm{e}^{-\beta(\mathcal{L} - \mu\mathcal{N})},
$$

where $\mathcal{N} = N \otimes \mathbb{1} - \mathbb{1} \otimes N$. The chosen complex conjugation on $\mathfrak{h}$ naturally extends to $\Gamma_-(h)$ and we denote it by the same symbol $\Psi \mapsto \overline{\Psi}$. The modular conjugation acts as

$$
J(\Phi \otimes \Psi) = u\overline{\Psi} \otimes u\overline{\Phi}, \tag{4.1}
$$

where $u = \mathrm{e}^{\mathrm{i}\pi N(N-\mathbb{1})/2}$.

The Araki–Wyss representation of $\mathrm{CAR}_g(\mathfrak{h})$ associated to ω_T is obtained by the obvious restriction.

4.2 The EBBM setting

Let G be a countably infinite set satisfying Assumption (DEC)(1) of Section 3.2. Let h be a bounded self-adjoint operator on the Hilbert space $\ell^2(G)$. We denote by $(\delta_x)_{x \in G}$ the standard basis of $\ell^2(G)$ and

[10] In the case of $T_{\beta\mu}$ this is the same as assuming that it commutes with h

set $h(x,y) = \langle \delta_x, h\delta_y \rangle$.[11] Obviously,

$$\ell^2(G) = \ell^2(S) \oplus \left(\bigoplus_{j=1}^{M} \ell^2(R_j) \right).$$

Let h_S and h_j be the compressions of h to $\ell^2(S)$ and $\ell^2(R_j)$[12], and

$$h_{\mathrm{fr}} = h_S + \left(\bigoplus_{j=1}^{M} h_j \right).$$

The C^*-algebra of the EBBM is $\mathcal{O} = \mathrm{CAR}_g(\ell^2(G))$ and its free C^*-dynamics τ_{fr} is the group of Bogoliubov $*$-automorphisms generated by h_{fr}. The C^*-algebra of the small system S and the j-th reservoir R_j are $\mathcal{O}_S = \mathrm{CAR}_g(\ell^2(S))$ and $\mathcal{O}_{R_j} = \mathrm{CAR}_g(\ell^2(R_j))$. Their dynamics τ_S and τ_j are the groups of Bogoliubov $*$-automorphisms generated by h_S and h_j.

We write $h = h_{\mathrm{fr}} + v$ and, besides (DEC)(1), we assume

(DEC)

(4) No direct coupling between reservoirs

$$(x,y) \in R_i \times R_j \text{ for } i \neq j \quad \Longrightarrow \quad v(x,y) = 0.$$

(5) v is finite rank, *i.e.*, for some finite $G_0 \subset G$ containing S,

$$(x,y) \notin G_0 \times G_0 \quad \Longrightarrow \quad v(x,y) = 0.$$

(6) $\mathfrak{G}_{\mathrm{fin}}$ is the set of finite subsets of G and the indexing set is

$$\mathcal{I} = \{\Lambda \in \mathfrak{G}_{\mathrm{fin}} \mid \Lambda \supset G_0\}.$$

Note that $v = \sum_{j=1}^{M} v_j$, where v_j denotes the compression of v to $\ell^2(S \cup R_j)$. The coupling between S and R_j is given by the "hopping" term

$$V_j = \mathrm{d}\Gamma(v_j) = \sum_{x,y \in S \cup R_j} v_j(x,y) a_x^* a_y,$$

where $a_x^{\#} = a^{\#}(\delta_x)$. Note that since v_j is finite rank, $V_j \in \mathrm{CAR}(\ell^2(S \cup R_j))$.

Finally, the self-interaction, restricted to the small system S, is described by a self-adjoint element W of $\mathrm{CAR}_g(\ell^2(S))$. The interacting C^*-dynamics τ on $\mathcal{O}$ is generated by $\delta = \delta_{\mathrm{fr}} + \mathrm{i}[V, \cdot\,]$ where

$$V = W + \sum_{j=1}^{M} V_j$$

[11] We use the corresponding notation for matrix elements of bounded operators on $\ell^2(G)$.

[12] Whenever the meaning is clear within the context, we denote with the same letters the extensions of these operators to $\ell^2(G)$ by setting them to zero on $\ell^2(S)^\perp$ and $\ell^2(R_j)^\perp$. Analogous extensions will be used without further saying.

is a self-adjoint element of $\mathcal{O}$.

The reference state of the coupled system is the quasi-free state ω generated by

$$T = \mathbb{1}_S \oplus \left(\bigoplus_{j=1}^{M} T_{\beta_j \mu_j} \right),$$

where $\beta_j > 0$ and $\mu_j \in \mathbb{R}$ are the inverse temperature and chemical potential of the R_j and[13]

$$T_{\beta_j \mu_j} = \frac{1}{\mathbb{1} + \mathrm{e}^{\beta_j (h_j - \mu_j \mathbb{1}_{R_j})}}$$

is the corresponding density operator on $\ell^2(R_j)$. The EBBM is described by the quantum dynamical system $(\mathcal{O}, \tau, \omega)$. Note that $\omega\big|_{\mathcal{O}_{R_j}} = \omega_{\beta_j \mu_j}$ is the (τ_j, β_j)-KMS state on $\mathcal{O}_{R_j}$.

Due to the fermionic statistics, the open quantum system $S + R_1 + \cdots + R_M$ does not have the tensor product structure with respect to its subsystems, as postulated in [BBJ+23, Section 1.1]. This however does not affect any of the results of [BBJ+23], the proof of Theorem 1.5 in [BBJ+23] requiring only notational changes. We note in particular that disjointedly supported elements of $\mathcal{O}$ commute, *i.e.*, if $A \in \mathcal{O}_T$, $B \in \mathcal{O}_{T'}$ with $T, T' \subset G$ such that $T \cap T' = \emptyset$, then $[A, B] = 0$.

We now describe the TDL approximation scheme. We start with $\Lambda \in \mathcal{I}$ and write it as

$$\Lambda = S \sqcup \left(\bigsqcup_{j=1}^{M} \Lambda_j \right),$$

where $\Lambda_j \subset R_j$. Let

$$\mathcal{O}_{\Lambda_j} = \mathrm{CAR}_g(\ell^2(\Lambda_j)), \qquad \mathcal{O}_\Lambda = \mathrm{CAR}_g(\ell^2(\Lambda)).$$

The free dynamics $\tau_{\mathrm{fr},\Lambda}$ is the group of Bogoliubov $*$-automorphisms generated by

$$h_{\mathrm{fr},\Lambda} = h_S \oplus \left(\bigoplus_{j=1}^{M} h_{\Lambda_j} \right),$$

where h_{Λ_j} denotes the compression of h_j to $\ell^2(\Lambda_j)$. By Assumption (DEC)(5–6), $V \in \mathcal{O}_\Lambda$, and the dynamics τ_Λ is generated by $\delta_\Lambda = \delta_{\mathrm{fr},\Lambda} + \mathrm{i}[V, \cdot\,]$. The reference state ω_Λ is the quasi-free state on $\mathcal{O}_\Lambda$ with density

$$T_\Lambda = \mathbb{1}_S \oplus \left(\bigoplus_{j=1}^{M} T_{\Lambda_j, \beta_j \mu_j} \right), \qquad T_{\Lambda_j, \beta_j \mu_j} = \frac{1}{1 + \mathrm{e}^{\beta_j (h_{\Lambda_j} - \mu_j \mathbb{1}_{\Lambda_j})}}. \tag{4.2}$$

Our TDL scheme is defined by the net $(\mathcal{O}_\Lambda, \tau_\Lambda, \omega_\Lambda)$.

Other approximation schemes are of course possible, and we will comment on them in Section 4.4.

4.3 TDL of 2TMEP

We start with assumption

[13] We denote by $\mathbb{1}_S$, $\mathbb{1}_{R_j}$, etc, the orthogonal projection onto $\ell^2(S)$, $\ell^2(R_j)$, etc.

(FE) The reservoir's one-fermion Hamiltonians h_j, $1 \leq j \leq M$, have purely absolutely continuous spectrum.

This assumption ensures that all reservoir subsystems $(\mathcal{O}_j, \tau_j, \omega_{\beta_j\mu_j})$ are mixing, and hence ergodic, and is used only for the verification of Assumption (TDL3)(2).

Theorem 4.2 *Suppose that* (FE) *holds. Then Assumptions* (TDL1)–(TDL6) *hold. In particular, Proposition* 2.1 *and Theorem* 2.3 *hold for the* EBBM.

In the process of the proof we will establish regularity properties of the EBBM that are much stronger than needed for the verification of (TDL1)–(TDL6).

Lemma 4.3 **(1)** $\mathrm{s}\text{–}\lim_{\Lambda_j} h_{\Lambda_j} = h_j$.

(2) $\mathrm{s}\text{–}\lim_{\Lambda} h_{\Lambda,\mathrm{fr}} = h_{\mathrm{fr}}$.

(3) $\mathrm{s}\text{–}\lim_{\Lambda} \Delta_{\omega_\Lambda}^{\mathrm{i}\theta} = \Delta_{\omega}^{\mathrm{i}\theta}$ *for all* $\theta \in \mathbb{R}$.

(4) *For any* $A \in \mathcal{O}_{\mathrm{loc}}$,

$$\lim_{\Lambda} \tau_{\Lambda}^{z}(A) = \tau^{z}(A)$$

in norm, locally uniformly for $z \in \mathbb{C}$.

(5) *For any* $A \in \mathcal{O}$,

$$\lim_{\Lambda} \tau_{\Lambda}^{t}(A) = \tau^{t}(A)$$

in norm, locally uniformly for $t \in \mathbb{R}$.

(6) *For all* $\Lambda \in \mathcal{I}$ *one has* $\sigma_\Lambda = \delta_{\omega_\Lambda}(V) = \sigma$. *Moreover,*

$$\lim_{\Lambda} \varsigma_{\omega_\Lambda}^{z}(\sigma) = \varsigma_{\omega}^{z}(\sigma)$$

in norm, locally uniformly for $z \in \mathbb{C}$. *In particular, the function* $z \mapsto \varsigma_{\omega}^{z}(\sigma)$ *is entire.*

(7) *For all* $s \in \mathbb{R}$ *the map*

$$\mathrm{i}\mathbb{R} \ni z \mapsto [D\omega_s : D\omega]_z \in \mathcal{O}$$

extends to an entire function. Moreover, for $s \in \mathbb{R}$,

$$\lim_{\Lambda} [D\omega_{\Lambda,s} : D\omega_\Lambda]_z = [D\omega_s : D\omega]_z, \qquad \lim_{\Lambda} [D\omega_{\Lambda,s} : D\omega_\Lambda]_z^* = [D\omega_s : D\omega]_z^*,$$

in norm, locally uniformly for $z \in \mathbb{C}$.

Proof. **(1)** Since $h_{\Lambda_j} = \mathbb{1}_{\Lambda_j} h \mathbb{1}_{\Lambda_j}$, with $\mathrm{s}\text{–}\lim_{\Lambda} \mathbb{1}_{\Lambda_j} = \mathbb{1}_{R_j}$, one has the estimate

$$\|(h_j - h_{\Lambda_j})f\| \leq \|(\mathbb{1}_{R_j} - \mathbb{1}_{\Lambda_j})hf\| + \|h(\mathbb{1}_{R_j} - \mathbb{1}_{\Lambda_j})f\|,$$

which yields the statement. We observe that this implies, in particular, that $\text{s–lim}_\Lambda \, \mathrm{e}^{zh_{\Lambda_j}} = \mathrm{e}^{zh_j}$, locally uniformly for $z \in \mathbb{C}$.

(2) Writing

$$h_{\Lambda,\mathrm{fr}} - h_{\mathrm{fr}} = h_S \oplus \left(\bigoplus_{j=1}^{M} (h_{\Lambda_j} - h_j) \right),$$

the result follows directly from (1). Here again, we note that this implies $\text{s–lim}_\Lambda \, \mathrm{e}^{zh_{\Lambda,\mathrm{fr}}} = \mathrm{e}^{zh_{\mathrm{fr}}}$, locally uniformly for $z \in \mathbb{C}$.

(3) Let $\ell = -\sum_{j=1}^{M} \beta_j (h_j - \mu_j)$ and $\ell_\Lambda = -\sum_{j=1}^{M} \beta_j (h_{\Lambda_j} - \mu_j)$. By the final remark in the proof of Part (1), we have $\text{s–lim}_\Lambda \, \mathrm{e}^{z\ell_\Lambda} = \mathrm{e}^{z\ell}$, locally uniformly for $z \in \mathbb{C}$. We identify $\Gamma_-(\ell^2(\Lambda))$ with a subspace of $\Gamma_-(\ell^2(G))$ and denote by $\Gamma_-^{(N)}(\ell^2(G))$ the N-particle sector of the latter. Then, the respective Araki–Wyss representations give that, for any $N, N' \in \mathbb{N}$, $\theta \in \mathbb{R}$ and $\Psi \in \Gamma_-^{(N)}(\ell^2(G)) \otimes \Gamma_-^{(N')}(\ell^2(G))$,

$$\Delta_{\omega_\Lambda}^{\mathrm{i}\theta} \Psi = \left(\mathrm{e}^{\mathrm{i}\theta \ell_\Lambda} \right)^{\otimes N} \otimes \left(\mathrm{e}^{-\mathrm{i}\theta \ell_\Lambda} \right)^{\otimes N'} \Psi,$$

$$\Delta_{\omega}^{\mathrm{i}\theta} \Psi = \left(\mathrm{e}^{\mathrm{i}\theta \ell} \right)^{\otimes N} \otimes \left(\mathrm{e}^{-\mathrm{i}\theta \ell} \right)^{\otimes N'} \Psi,$$

and the result follows from a simple telescopic expansion of the difference $\Delta_{\omega_\Lambda}^{\mathrm{i}\theta} \Psi - \Delta_{\omega}^{\mathrm{i}\theta} \Psi$.

(4) Consider first $A = a_x^* a_y$ for some $x, y \in G$. Since

$$\tau_{\Lambda,\mathrm{fr}}^z(A) = a^*(\mathrm{e}^{zh_{\Lambda,\mathrm{fr}}} \delta_x) a(\mathrm{e}^{zh_{\Lambda,\mathrm{fr}}} \delta_y),$$

$$\tau_{\mathrm{fr}}^z(A) = a^*(\mathrm{e}^{zh_{\mathrm{fr}}} \delta_x) a(\mathrm{e}^{zh_{\mathrm{fr}}} \delta_y),$$

one has

$$\begin{aligned}\|\tau_{\Lambda,\mathrm{fr}}^z(A) - \tau_{\mathrm{fr}}^z(A)\| &\le \|a^*((\mathrm{e}^{zh_{\Lambda,\mathrm{fr}}} - \mathrm{e}^{zh_{\mathrm{fr}}})\delta_x)\| \|a(\mathrm{e}^{zh_{\Lambda,\mathrm{fr}}} \delta_y)\| + \|a^*(\mathrm{e}^{zh_{\mathrm{fr}}} \delta_x)\| \|a((\mathrm{e}^{zh_{\Lambda,\mathrm{fr}}} - \mathrm{e}^{zh_{\mathrm{fr}}})\delta_y)\| \\ &\le \|(\mathrm{e}^{zh_{\Lambda,\mathrm{fr}}} - \mathrm{e}^{zh_{\mathrm{fr}}})\delta_x\| \|\mathrm{e}^{zh_{\Lambda,\mathrm{fr}}} \delta_y\| + \|(\mathrm{e}^{zh_{\Lambda,\mathrm{fr}}} - \mathrm{e}^{zh_{\mathrm{fr}}})\delta_y\| \|\mathrm{e}^{zh_{\mathrm{fr}}} \delta_x\|,\end{aligned}$$

and it follows from (2) that

$$\lim_\Lambda \tau_{\Lambda,\mathrm{fr}}^z(A) = \tau_{\mathrm{fr}}^z(A) \tag{4.3}$$

in norm, locally uniformly for $z \in \mathbb{C}$. Since any $A \in \mathcal{O}_{\mathrm{loc}}$ is a finite linear combination of finite products of factors of the form $a_x^* a_y$, (4.3) holds for all $A \in \mathcal{O}_{\mathrm{loc}}$ with the required uniformity. Since $V \in \mathcal{O}_{\mathrm{loc}}$, (4) follows from the expansions

$$\tau_\Lambda^z(A) = \tau_{\Lambda,\mathrm{fr}}^z(A) + \sum_{n \ge 1} z^n \int_{0 \le t_1 \le \cdots \le t_n \le 1} \mathrm{i}[\tau_{\Lambda,\mathrm{fr}}^{zt_n}(V), \cdots, \mathrm{i}[\tau_{\Lambda,\mathrm{fr}}^{zt_1}(V), \tau_{\Lambda,\mathrm{fr}}^z(A)] \cdots] \mathrm{d}t_1 \cdots \mathrm{d}t_n,$$

$$\tau^z(A) = \tau_{\mathrm{fr}}^z(A) + \sum_{n \ge 1} z^n \int_{0 \le t_1 \le \cdots \le t_n \le 1} \mathrm{i}[\tau_{\mathrm{fr}}^{zt_n}(V), \cdots, \mathrm{i}[\tau_{\mathrm{fr}}^{zt_1}(V), \tau_{\mathrm{fr}}^z(A)] \cdots] \mathrm{d}t_1 \cdots \mathrm{d}t_n,$$

and a telescopic expansion of their difference.

(5) Since τ_Λ^t and τ^t are isometric for real t, the result follows from Part (4) and the dense inclusion $\mathcal{O}_{\mathrm{loc}} \subset \mathcal{O}$ by an elementary $\epsilon/3$ argument.

(6) Let c be a finite rank operator on $\ell^2(G)$, with singular value decomposition $c = \sum_{k=1}^m s_k f_k (g_k, \,\cdot\,)$. Then, $\mathrm{d}\Gamma(c) = \sum_{k=1}^m s_k a^*(f_k) a(g_k) \in \mathcal{O}$, and one easily checks that

$$\|\mathrm{d}\Gamma(c)\| \le \|c\|_1 = \sum_{k=1}^m s_k \|f_k\| \|g_k\|,$$

where $\|\,\cdot\,\|_1$ denotes the trace norm. Since finite rank operators are dense in the Banach space of trace class operators, one has $\mathrm{d}\Gamma(c) \in \mathcal{O}$ for trace class c, with the same inequality.

By assumption, v is finite rank and $V = W + \mathrm{d}\Gamma(v)$ with $W \in \mathcal{O}_S$. It follows that

$$[\mathrm{d}\Gamma(\ell), W] = \sum_{x,y \in G \setminus S} \ell(x,y) [a_x^* a_y, W] = 0,$$

which gives

$$\varsigma_\omega^\theta(V) = \mathrm{e}^{\mathrm{i}\theta \mathrm{d}\Gamma(\ell)} V \mathrm{e}^{-\mathrm{i}\theta \mathrm{d}\Gamma(\ell)} = \mathrm{e}^{\mathrm{i}\theta \mathrm{d}\Gamma(\ell)} W \mathrm{e}^{-\mathrm{i}\theta \mathrm{d}\Gamma(\ell)} + \mathrm{d}\Gamma\left(\mathrm{e}^{\mathrm{i}\theta\ell} v \mathrm{e}^{-\mathrm{i}\theta\ell}\right) = W + \mathrm{d}\Gamma\left(\mathrm{e}^{\mathrm{i}\theta\ell} v \mathrm{e}^{-\mathrm{i}\theta\ell}\right), \quad (4.4)$$

and so

$$\sigma = \delta_\omega(V) = \frac{\mathrm{d}}{\mathrm{d}\theta} \varsigma_\omega^\theta(V)\big|_{\theta=0} = \mathrm{d}\Gamma(\mathrm{i}[\ell, v]) = \sum_{j=1}^M \beta_j \mathrm{d}\Gamma(\phi_j), \qquad \phi_j = \mathrm{i}[v_j, h_j] = \mathrm{i}[h, h_j].$$

Similarly, recalling that $V \in \mathcal{O}_\Lambda$ for all $\Lambda \in \mathcal{I}$, we get

$$\sigma_\Lambda = \delta_{\omega_\Lambda}(V) = \mathrm{d}\Gamma(\mathrm{i}[\ell_\Lambda, v]) = \sum_{j=1}^M \beta_j \mathrm{d}\Gamma(\phi_{j,\Lambda}), \qquad \phi_{j,\Lambda} = \mathrm{i}[v_j, h_{j,\Lambda}] = \mathrm{i}[v_j, h_j] = \phi_j,$$

which shows that $\sigma_\Lambda = \sigma$.

For $\Lambda \in \mathcal{I}$ and $z \in \mathbb{C}$, we have

$$\varsigma_{\omega_\Lambda}^z(\sigma) = \mathrm{d}\Gamma\left(\phi_\Lambda(z)\right), \qquad \varsigma_\omega^z(\sigma) = \mathrm{d}\Gamma\left(\phi(z)\right),$$

where

$$\phi_\Lambda(z) = \mathrm{e}^{\mathrm{i}z\ell_\Lambda} \phi \mathrm{e}^{-\mathrm{i}z\ell_\Lambda}, \qquad \phi(z) = \mathrm{e}^{\mathrm{i}z\ell} \phi \mathrm{e}^{-\mathrm{i}z\ell},$$

and $\phi = \sum_{j=1}^M \beta_j \phi_j$ is finite rank. It follows from the initial remark in the proof of Part (3) that

$$\lim_\Lambda \phi_\Lambda(z) = \phi(z)$$

holds in trace norm, locally uniformly for $z \in \mathbb{C}$, and the result follows from the estimate

$$\|\varsigma_{\omega_\Lambda}^z(\sigma) - \varsigma_\omega^z(\sigma)\| \le \|\mathrm{d}\Gamma(\phi_\Lambda(z) - \phi(z))\| \le \|\phi_\Lambda(z) - \phi(z)\|_1.$$

(7) We follow the proofs of Propositions 2.5 and 2.6. Relation (4.4) gives that $\theta \mapsto \varsigma_\omega^\theta(V)$ is entire, from which (2.15) shows that the same is true of $\theta \mapsto \varsigma_\omega^\theta(\Gamma_{-t})$. It then follows from (2.14) that (2.13) holds for all $z \in \mathbb{C}$ and gives the first part of the statement.

With (2.16), (2.17), and (2.18), to get the second assertion it is sufficient to prove that

$$\lim_\Lambda \tau_{\Lambda,\mathrm{fr}}^t(\varsigma_{\omega_\Lambda}^\theta(V)) = \tau_{\mathrm{fr}}^t(\varsigma_\omega^\theta(V)),$$

$$\lim_\Lambda \tau_{\Lambda,\mathrm{fr}}^t(\varsigma_{\omega_\Lambda}^\theta(\sigma)) = \tau_{\mathrm{fr}}^t(\varsigma_\omega^\theta(\sigma)),$$

locally uniformly for $t \in \mathbb{R}$ and $\theta \in \mathbb{C}$. This follows from (5) and (6). □

Proof of Theorem 4.2.

(TDL1)+(TDL4) Follow from Lemma 4.3(7).

(TDL2) Parts (1–2) are obvious consequences of our setup. Part (3) follows from the definition (4.2) of T_Λ, Lemma 4.3(1) and $\lim_\Lambda \mathbb{1}_{\Lambda_j} = \mathbb{1}_j$ which ensure that s–$\lim_\Lambda T_\Lambda = T$. The general formula for quasi-free states

$$\omega_T(a^*(f_n)\cdots a^*(f_1)a(g_1)\cdots a(g_m)) = \delta_{nm} \det(\{(g_i, Tf_j)\})$$

then implies the weak* convergence $\omega_{T_\Lambda} \to \omega_T$. Part (4) follows from Lemma 4.3(4). As in the proof of Lemma 4.3(3), we identify $\Gamma_-(\ell^2(\Lambda))$ with a subspace of $\Gamma_-(\ell^2(G))$. Then the respective Araki–Wyss representations give that Part (5) holds with $\Omega_\Lambda = \Omega_{\mathrm{AW}} = \Omega_\mathrm{f} \otimes \Omega_\mathrm{f}$.[14] Part (6) follows from Lemma 4.3(7).

(TDL3) Part (1) follows from Lemma 4.3(3) and Formula (4.1) that allow us to take $J_\Lambda = J$. Regarding Part (2), setting $R = \cup_{j=1}^M R_j$ and invoking the fermionic exponential law [BSZ92, Theorem 3.2], we may identify

$$\Gamma_-(\ell^2(G)) = \Gamma_-(\ell^2(S)) \otimes \Gamma_-(\ell^2(R)),$$

and the corresponding Fock vacua

$$\Omega_\mathrm{f} = \Omega_{\mathrm{f},S} \otimes \Omega_{\mathrm{f},R}.$$

Considering the operators ℓ and ℓ_Λ defined in the proof of Lemma 4.3(3) as acting on $\ell^2(R)$, we have

$$\Delta_\omega^\theta = (I \otimes \mathrm{e}^{\mathrm{i}\theta \mathrm{d}\Gamma(\ell)}) \otimes (I \otimes \mathrm{e}^{-\mathrm{i}\theta \mathrm{d}\Gamma(\ell)}),$$

and

$$\Delta_{\omega_\Lambda}^\theta = (I \otimes \mathrm{e}^{\mathrm{i}\theta \mathrm{d}\Gamma(\ell_\Lambda)}) \otimes (I \otimes \mathrm{e}^{-\mathrm{i}\theta \mathrm{d}\Gamma(\ell_\Lambda)}).$$

Since $\mathrm{d}\Gamma(\ell_\Lambda)\Omega_{\mathrm{f},R} = 0$, we have

$$(\Gamma_-(\ell^2(S)) \otimes \Omega_{\mathrm{f},R}) \otimes (\Gamma_-(\ell^2(S)) \otimes \Omega_{\mathrm{f},R}) \subseteq \ker \log \Delta_{\omega_\Lambda}.$$

Assumption (FE) implies that

$$\ker \log \Delta_\omega = (\Gamma_-(\ell^2(S)) \otimes \Omega_{\mathrm{f},R}) \otimes (\Gamma_-(\ell^2(S)) \otimes \Omega_{\mathrm{f},R}),$$

[14] Note, however, that $\pi_\Lambda \neq \pi$ on $\mathcal{O}_\Lambda$ since $T_\Lambda \neq T|_{\ell^2(\Lambda)}$.

which gives Part (2).

(TDL5) Parts (1–3) follow from our setup, Part (4) from Lemma 4.3(6).

(TDL6) Parts (1–2) follow from the setup and Parts (3–4) from Lemma 4.3(2+6). □

4.4 Remarks

1.The Spin–Fermion Model. This open quantum system was studied in the early seminal works [Dav74, SL78] and has remained one of the paradigmatic models of quantum statistical mechanics. The transfer operator techniques of [JP02b, JOPP10] are applicable to this model, and we will return to it in the continuation of this paper [BBJ+]. Just like OQ2S, the Spin–Fermion Model (in the sequel abbreviated SFM) has the tensor product structure of open quantum systems discussed in Section 1.2 of [BBJ+23]. The reservoirs are free Fermi gasses over $(\mathfrak{h}_j, h_j)$ described by $(\mathrm{CAR}(\mathfrak{h}_j), \tau_j, \omega_{\beta_j})$, where ω_{β_j} is a quasi-free state generated by $T_{\beta_j 0}$. The interaction of S with R_j is described by a self-adjoint V_j which is a finite sum of terms

$$Q \otimes \varphi_j(f_1) \cdots \varphi_j(f_n), \tag{4.5}$$

where Q is a self-adjoint element of $\mathcal{O}_S$ and φ_j denotes the fermionic field operator in $\mathrm{CAR}(\mathfrak{h}_j)$. If $\mathfrak{h}_j = \ell^2(R_j)$ for some countably infinite R_j, the TDL of the 2TMEP is carried out in the same way as for the EBMM. The continuous reservoir case $\mathfrak{h}_j = L^2(\mathbb{R}^d, \mathrm{d}x)$ and $h_j = -\frac{1}{2m}\Delta$ is of particular importance in connection to the transfer operator techniques of [JP02b, JOPP10]. The TDL scheme is carried out by considering subspaces $L^2([-L, L]^d, \mathrm{d}x)$ and by restricting $-\frac{1}{2m}\Delta$ to $[-L, L]^d$ with periodic (or Dirichlet, or Neumann...) boundary condition. To define a finite dimensional TDL scheme, one also introduces a high energy cut-off $E > 0$, and so the scheme is indexed by $(E, L)_{E>0, L>0}$. The details are the same as in [JOPP10, Exercise 6.4], and we leave them to an interested reader. Assumptions (TDL1)–(TDL6) hold if the test functions f_k in (4.5) are in $\mathrm{dom}(\mathrm{e}^{\lambda_j h_j})$ for some $\lambda_j > \beta_j$.

2. The Spin–Boson Model. This model has the same general structure as the SFM upon replacing CAR with CCR. The model cannot be defined on the C^*-level and the interacting dynamics is introduced only in the GNS-representation. For technical reasons due to the unboundedness of the bosonic fields operators, in (4.5) one always takes $n \leq 2$. In spite of the relatively large literature on the Spin–Boson Model, its TDL has rarely been discussed. Although it is likely that the TDL justification of the 2TMEP for the Spin–Boson Model can be carried out along similar lines as for the SFM model, the technical details remain to be worked out.

3. Other TDL schemes. Just like in the OQ2S case (3.7)–(3.8), one can also consider the TDL scheme where

$$\omega_{\Lambda_j} = \omega_{\beta_j}\big|_{\mathcal{O}_{\Lambda_j}}. \tag{4.6}$$

Note that ω_{Λ_j} is quasi-free state on $\mathcal{O}_{\Lambda_j}$ generated by the compression of $T_{\beta_j\mu_j}$ on the subspace $\ell^2(\Lambda_j)$, which we denote by $[T_{\beta_j\mu_j}]_{\Lambda_j}$. Obviously, $[T_{\beta_j\mu_j}]_{\Lambda_j}$ differs from $T_{\Lambda_j, \beta_j\mu_j}$ given by (4.2). One then takes

$$\overline{h}_{\Lambda_j} = -\frac{1}{\beta_j} \log([T_{\beta_j\mu_j}]_{\Lambda_j}(\mathbb{1}_{\Lambda_j} - [T_{\beta_j\mu_j}]_{\Lambda_j})^{-1}) + \mu_j,$$

so that

$$[T_{\beta_j\mu_j}]_{\Lambda_j} = \frac{1}{1 + \mathrm{e}^{\beta_j(\overline{h}_{\Lambda_j} - \mu_j \mathbb{1}_{\Lambda_j})}}.$$

$\overline{h}_{\Lambda_j}$ is also obviously not equal to the compression h_{Λ_j} of h_j to $\ell^2(\Lambda_j)$, but we still have that

$$\operatorname*{s-lim}_{\Lambda_j} \overline{h}_{\Lambda_j} = h_j.$$

This gives that Lemma 4.3 holds as formulated with the same proof, and that the same holds for Theorem 4.2.

Finally, one can consider a combined TDL scheme where ω_{Λ_j} is given by (4.6) and h_{Λ_j} is the restriction of h_j to Λ_j. The proof now requires a slight modification since the groups $\tau_{\Lambda_j,\mathrm{fr}}$ and $\varsigma_{\omega_{\Lambda_j}}$ do not commute anymore. However, since $\mathrm{s\text{-}lim}_{\Lambda_j}[h_{\Lambda_j}, \overline{h}_{\Lambda_j}] = 0$, the required changes of the argument are minor.

4. On the role of free reservoirs. Free reservoirs play a distinguished role in both experimental and theoretical studies of open quantum systems. The free reservoir structure brings to the focus interaction processes involving the small system. In OQ2S, the study of these interaction processes is affected by the reservoir complexity, and the respective open quantum system remains poorly understood simply because its reservoirs are poorly understood. From a mathematical perspective, free Fermi (and Bose) gas reservoirs allow for a simple implementation of the TDL with respect to its explicitly identifiable modular structure given by the Araki–Wyss (or Araki–Woods [AW63]) GNS-representation. They also allow for simple criteria ensuring the reservoirs' ergodicity, a question which remains poorly understood for quantum spin systems.

This being said, the study of OQ2S brings to the forefront some foundational problems of quantum statistical mechanics that stem from the pioneering works of Araki, Haag, Ruelle, and many others in 1960's and 70's.[15] Although relatively little progress has been made in the last forty years, these problems remain at the core of quantum statistical mechanics and are a challenge for future generations.

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