A note on two-times measurement entropy production and modular theory

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Abstract. Recent theoretical investigations of the two-times measurement entropy production (2TMEP) in quantum statistical mechanics have shed a new light on the mathematics and physics of the quantum-mechanical probabilistic rules. Among notable developments are the extensions of entropic fluctuation relations to the quantum domain and the discovery of a deep link between 2TMEP and modular theory of operator algebras. All these developments concerned the setting where the state of the system at the instant of the first measurement is the same as the state whose entropy production is measured. In this work we consider the case where these two states are different and link this more general 2TMEP to modular theory. The established connection allows us to show that under general ergodicity assumptions the 2TMEP is essentially independent of the choice of the system state at the instant of the first measurement due to a decoherence effect induced by the first measurement. This stability sheds a new light on the concept of quantum entropy production, and, in particular, on possible quantum formulations of the celebrated classical Gallavotti–Cohen Fluctuation Theorem which will be studied in a continuation of this work.

Contents

1 Introduction

Starting with the seminal work [HHW67], the mathematical theory of equilibrium quantum statistical mechanics based on the KMS-condition has developed rapidly in 1970's, resulting in a structure of rare unity and beauty summarized in the classical monographs [BR87, BR81]. A large part of these developments was centered around the link between the KMS-condition and the modular theory of operator algebras.

Modular theory has also played a central role in the more recent developments in non-equilibrium quantum statistical mechanics initiated in [JP01, Rue01], where the definition of an entropy production observable and the related entropy balance equation are given in terms of basic objects of modular theory;[1] a non-exhaustive list of related works is [AS07, ASF07, HA00, AP03, AJPP06, AJPP07, FMSU03, FMU03, JKP06, JOP06a, JLP13, JOP06c, JOP06b, JOP07, JOPP10, JOPS12, JP02a, JP02b, JP07, MMS07b, MMS07a, MO03, Oga04, Oji89, Oji91, OHI88, Pil01, Rue02, Tas06, TM03, TM05].

Perhaps more surprising were parallel developments related to the search for a quantum extension of the celebrated fluctuation relations of classical non-equilibrium statistical mechanics [ES94, GC95a, GC95b], see also the review [JPRB11]. The first of them introduced the two-times measurement entropy production [Kur00, Tas00]. The spectral measure of a relative modular operator was the central object of the second one [TM03].[2] These two proposals turned out to be equivalent, shedding an unexpected light on both quantum mechanical probabilistic rules and modular theory. A pedagogical discussion of this topic can be found in the lecture notes [JOPP10].

In this work we continue to study the link between the two-times measurement entropy production and modular theory. The more general setting we consider concerns the choice of the system state at the

[1] See [PW78] for a pioneering work on the subject.

[2] Another early work on the subject is [DR09],

instant of the first measurement, which is here assumed to be arbitrary. Somewhat surprisingly, under mild ergodicity assumptions, the modular link we establish gives that the two-times measurement entropy production in open quantum systems is essentially independent of the state of the system at the instant of the first measurement. This stability result will be the starting point of our follow-up work [BBJ+a], in which we propose an extension of the celebrated Gallavotti–Cohen Fluctuation Theorem [GC95a, GC95b] to the quantum domain; see also Remark 4 in Section 1.5.

This note is organized as follows. For notational purposes, the elements of algebraic quantum statistical mechanics and modular theory that we will need are briefly reviewed in Sections 1.1 and 1.2. The two-times measurement entropy production of finite quantum system is discussed in Section 1.2. Our results are stated in Section 1.4 and are briefly discussed in Section 1.5. The proofs are given in Section 2.

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1.1 Algebraic quantum statistical mechanics

We start with the setting of a quantum system with finite dimensional Hilbert space $\mathcal{K}$. We will refer to such quantum systems as *finite*. $\mathcal{O}_\mathcal{K}$ denotes the C^*-algebra of all linear maps $A : \mathcal{K} \to \mathcal{K}$ and $\mathcal{S}_\mathcal{K} \subset \mathcal{O}_\mathcal{K}$ the set of all density matrices on $\mathcal{K}$. Observables of the system are identified with elements of $\mathcal{O}_\mathcal{K}$ and states with elements of $\mathcal{S}_\mathcal{K}$, with the usual duality $\nu(A) = \mathrm{tr}(\nu A)$, $\nu \in \mathcal{S}_\mathcal{K}$, $A \in \mathcal{O}_\mathcal{K}$. The number $\nu(A)$ is interpreted as the expectation value of the observable A when the system is in the state ν. A state ν is called faithful if $\nu > 0$. The dynamics is described by the system Hamiltonian $H = H^* \in \mathcal{O}_\mathcal{K}$ and the induced group $\tau = \{\tau^t \mid t \in \mathbb{R}\}$ of $*$-automorphisms of $\mathcal{O}_\mathcal{K}$ defined by

$$\tau^t(A) = \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH}.$$

We will sometimes write A_t for $\tau^t(A)$ and call the map $A \mapsto A_t$ the Heisenberg picture dynamics. In the dual Schrödinger picture the states evolve in time as $\nu \mapsto \nu_t$ where

$$\nu_t = \mathrm{e}^{-\mathrm{i}tH} \nu \mathrm{e}^{\mathrm{i}tH}.$$

Obviously, $\nu_t(A) = \nu(A_t)$. The time-correlations are quantified by the function

$$F_{\nu,A,B}(t) = \nu(AB_t). \tag{1.1}$$

A triple $(\mathcal{O}_\mathcal{K}, \tau, \omega)$, where ω is the reference state of the system, is called a finite quantum dynamical system. This system is said to be in thermal equilibrium at inverse temperature $\beta \in \mathbb{R}$ if its reference state is the Gibbs canonical ensemble

$$\omega = \frac{\mathrm{e}^{-\beta H}}{\mathrm{tr}(\mathrm{e}^{-\beta H})}. \tag{1.2}$$

The Gibbs ensemble (1.2) is the unique state $\nu \in \mathcal{S}_\mathcal{K}$ satisfying the KMS relation

$$\nu(AB_{t+\mathrm{i}\beta}) = \nu(B_t A)$$

for all $A, B \in \mathcal{O}_\mathcal{K}$ and $t \in \mathbb{R}$.

More generally, in algebraic quantum statistical mechanics observables are described by elements of a C^*-algebra $\mathcal{O}$ with identity $\mathbb{1}$. For a large part of the general theory, no other structure is imposed on $\mathcal{O}$. States are elements of $\mathcal{S}_\mathcal{O}$, the set of positive normalized[3] elements ν of the dual $\mathcal{O}^*$ of $\mathcal{O}$. The number $\nu(A)$ is interpreted as the expectation value of the observable A when the system is in the state ν.

The Heisenberg picture dynamics is described by a strongly continuous[4] group $\tau = \{\tau^t \mid t \in \mathbb{R}\}$ of $*$-automorphisms of $\mathcal{O}$. The group τ is called C^*-dynamics and the pair $(\mathcal{O}, \tau)$ a C^*-dynamical system. The dual group τ^* preserves $\mathcal{S}_\mathcal{O}$ and describes the Schrödinger picture dynamics. We write A_t for $\tau^t(A)$, ν_t for $\tau^{t*}(\nu) = \nu \circ \tau^t$, and use the same shorthand (1.1) for time-correlation functions. A state ν is called τ-invariant (or stationary) if $\nu_t = \nu$ for all $t \in \mathbb{R}$. The set of all τ-invariant states is denoted by $\mathcal{S}_\tau$ and is always non-empty. A triple $(\mathcal{O}, \tau, \omega)$, where ω is the reference state of the system, is called C^*-quantum dynamical system. A state $\omega \in \mathcal{S}_\tau$ is called ergodic if

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \omega(B^* A_t B)\mathrm{d}t = \omega(A)\omega(B^* B)$$

holds for all $A, B \in \mathcal{O}$.

Time-reversal plays an important role in statistical mechanics. An anti-linear involutive $*$-automorphism Θ of $\mathcal{O}$ is called time-reversal of $(\mathcal{O}, \tau)$ if

$$\Theta \circ \tau^t = \tau^{-t} \circ \Theta$$

for all $t \in \mathbb{R}$. A state ω is called time-reversal invariant if there exists a time-reversal Θ such that $\omega \circ \Theta(A) = \omega(A^*)$ for all $A \in \mathcal{O}$.

For $\beta \in \mathbb{R}^*$, $\nu \in \mathcal{S}_\mathcal{O}$ is called (τ, β)-KMS state if, for all $A, B \in \mathcal{O}$, the function $\mathbb{R} \ni t \mapsto F_{\nu,A,B}(t)$ has an analytic extension to the strip $0 < \mathrm{sgn}(\beta)\mathrm{Im}z < |\beta|$ that is bounded and continuous on its closure and satisfies the KMS-boundary condition

$$F_{\nu,A,B}(t + \mathrm{i}\beta) = \nu(B_t A)$$

for all $t \in \mathbb{R}$. We denote by $\mathcal{S}_{(\tau,\beta)}$ the set of all (τ, β)-KMS states. At the current level of generality this set might be empty. One always has $\mathcal{S}_{(\tau,\beta)} \subset \mathcal{S}_\tau$. A C^*-quantum dynamical system $(\mathcal{O}, \tau, \omega)$ is said to be in thermal equilibrium at inverse temperature $\beta \in \mathbb{R}^*$ (or just thermal) if $\omega \in \mathcal{S}_{(\tau,\beta)}$.

A state ν is called modular if there exists a C^*-dynamics ς_ν on $\mathcal{O}$ such that $\nu \in \mathcal{S}_{(\varsigma_\nu,-1)}$. ς_ν is called modular group of ν and is unique when it exists. We denote by δ_ν the generator of ς_ν with the convention $\varsigma_\nu^t = \mathrm{e}^{t\delta_\nu}$. If $\nu \in \mathcal{S}_{(\tau,\beta)}$, then it is modular and its modular group is $\varsigma_\nu^t = \tau^{-\beta t}$ (or equivalently, $\delta_\nu = -\beta\delta$, where δ is the generator of τ).

[3] $\nu(A^*A) \geq 0$ for all $A \in \mathcal{O}$ and $\nu(\mathbb{1}) = 1$.
[4] $\lim_{t\to 0} \|\tau^t(A) - A\| = 0$ for all $A \in \mathcal{O}$.

A special class of quantum dynamical systems, the so-called open quantum systems, play a privileged role in the study of non-equilibrium quantum statistical mechanics, and we proceed to describe them.

Consider M thermal reservoirs $\mathcal{R}_j$ described by C^*-quantum dynamical systems $(\mathcal{O}_j, \tau_j, \omega_j)$. We denote by δ_j the generator of τ_j. The reservoir $\mathcal{R}_j$ is assumed to be in thermal equilibrium at inverse temperature $\beta_j > 0$, that is, we assume that ω_j is a (τ_j, β_j)-KMS state on $\mathcal{O}_j$. In the absence of interaction, the combined reservoir system $\mathcal{R} = \mathcal{R}_1 + \cdots + \mathcal{R}_M$ is described by the quantum dynamical system $(\mathcal{O}_\mathcal{R}, \tau_\mathcal{R}, \omega_\mathcal{R})$, where[5]

$$\begin{aligned}\mathcal{O}_\mathcal{R} &= \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_M,\\ \tau_\mathcal{R} &= \tau_1 \otimes \cdots \otimes \tau_M,\\ \omega_\mathcal{R} &= \omega_1 \otimes \cdots \otimes \omega_M.\end{aligned}$$

We will consider two kinds of systems: directly coupled reservoirs and reservoirs coupled through a small system $\mathcal{S}$, with Hilbert space $\mathcal{K}_\mathcal{S}$. With a slight abuse of terminology, we will refer to both of them as *open quantum systems*.

In the first case, the interaction is described by a self-adjoint $V \in \mathcal{O}_\mathcal{R}$ and the interacting dynamics τ is generated by $\delta = \delta_\mathcal{R} + \mathrm{i}[V, \cdot\,]$, where $\delta_\mathcal{R} = \sum_j \delta_j$ is the generator of $\tau_\mathcal{R}$.

In the second case, let $(\mathcal{O}_\mathcal{S}, \tau_\mathcal{S}, \omega_\mathcal{S})$ be the finite dimensional C^*-quantum dynamical system describing $\mathcal{S}$[6], where we assume that $\omega_\mathcal{S} > 0$. The generator of $\tau_\mathcal{S}$ is $\delta_\mathcal{S} = \mathrm{i}[H_\mathcal{S}, \cdot\,]$, where $H_\mathcal{S}$ is the Hamiltonian of $\mathcal{S}$. In the absence of interaction, the joint system $\mathcal{S} + \mathcal{R}$ is described by the C^*-quantum dynamical system $(\mathcal{O}, \tau_{\mathrm{fr}}, \omega)$ where

$$\mathcal{O} = \mathcal{O}_\mathcal{S} \otimes \mathcal{O}_\mathcal{R}, \qquad \tau_{\mathrm{fr}} = \tau_\mathcal{S} \otimes \tau_\mathcal{R}, \qquad \omega = \omega_\mathcal{S} \otimes \omega_\mathcal{R}.$$

The state ω is obviously modular. The interaction of $\mathcal{S}$ with $\mathcal{R}_j$ is described by a self-adjoint element $V_j \in \mathcal{O}_\mathcal{S} \otimes \mathcal{O}_j$, and the full interaction by $V = \sum_j V_j$. The interacting dynamics τ is generated by $\delta = \delta_\mathcal{S} + \delta_\mathcal{R} + \mathrm{i}[V, \cdot\,]$. In what follows, we will always take

$$\omega_\mathcal{S} = \frac{\mathbb{1}}{\dim \mathcal{K}_\mathcal{S}} \tag{1.3}$$

for the reference state of $\mathcal{S}$. This choice is made for convenience. It is easy to show that none of our results depend on a specific choice of $\omega_\mathcal{S}$ as long as $\omega_\mathcal{S} > 0$; see Remark 6 in Section 1.5.

The above description of open quantum system is sometimes modified in the case of fermionic systems. The modifications are straightforward, and they do not affect any of our results; see [AJPP06, JOP07].

Modular theory and the closely related Araki's perturbation theory of KMS-structure play a central role in algebraic quantum statistical mechanics. A basic introduction to this subject can be found in [BR87, BR81]; see also [DJP03] and references therein for modern expositions. A pedagogical introduction to modular theory in the context of finite quantum systems can be found in [JOPP10]. We will not give a detailed review of modular theory in this paper and only a short introduction to basic notions will be

[5] Whenever the meaning is clear within the context, we write A for $A \otimes \mathbb{1}$ and $\mathbb{1} \otimes A$, δ_j for $\delta_j \otimes \mathrm{Id}$, $\mathrm{Id} \otimes \delta_j$, etc.

[6] We abbreviated by $\mathcal{O}_\mathcal{S}$ the C^*-algebra $\mathcal{O}_{\mathcal{K}_\mathcal{S}}$ of all linear operators on $\mathcal{K}_\mathcal{S}$.

presented in Section 1.2. However, as we proceed with the proofs, we will give references to the results we will use.

W^*-dynamical systems play a distinguished role in modular theory. Consider a pair $(\mathfrak{M}, \tau)$ where $\mathfrak{M}$ is W^*-algebra and $\tau = \{\tau^t \mid t \in \mathbb{R}\}$ is a pointwise σ-weakly continuous group of $*$-automorphisms on $\mathfrak{M}$. We shall refer to such τ as W^*-dynamics. A triple $(\mathfrak{M}, \tau, \omega)$, where ω is a normal state on $\mathfrak{M}$, is called a W^*-dynamical system. In the general development of non-equilibrium quantum statistical mechanics, the C^*-quantum dynamical systems are preferred starting point since the central notion of non-equilibrium steady states cannot be naturally defined in the W^*-setting.

1.2 GNS-representation and modular structure

Let ω be a modular state on $\mathcal{O}$. We denote by $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ the GNS-representation of $\mathcal{O}$ associated to ω, and by $\mathfrak{M}_\omega = \pi_\omega(\mathcal{O})''$ the enveloping von Neumann algebra of bounded operators on $\mathcal{H}_\omega$. In what follows, we drop the subscript ω whenever the meaning is clear within the context. Since the state ω is assumed to be modular, the cyclic vector Ω is separating for $\mathfrak{M}$[7], and in particular $\|\pi(A)\| = \|A\|$ for all $A \in \mathcal{O}$. Whenever the meaning is clear within the context, we will denote $\pi(A)$ by A.

$\mathcal{N}$ denotes the set of all normal states on $\mathfrak{M}$, *i.e.*, the states described by density matrices on $\mathcal{H}$. Obviously, an element of $\mathcal{N}$ also defines a state on $\mathcal{O}$ and any state on $\mathcal{O}$ that arises in this way is called ω-normal. Again, whenever the meaning is clear within the context, we will denote such states by the same letter. In particular, the vector state $\mathfrak{M} \ni A \mapsto \langle \Omega, A\Omega \rangle$ is denoted by ω.

We will assume that the reader is familiar with the basic notions of Tomita-Takesaki's modular theory; see any of the references [BR87, BR81, DJP03, Haa96, OP93, Pil06, Str81]. For definiteness, we will use the same notation and terminology as in [JOPS12, Section 5]. $\mathcal{H}^+$ and J denote the natural cone and modular conjugation associated to the pair $(\mathfrak{M}, \Omega)$. The unique vector representative of $\nu \in \mathcal{N}$ in the natural cone is denoted by Ω_ν. The modular operator of $\nu \in \mathcal{N}$ is denoted by Δ_ν. The relative modular operator of a pair (ν, ρ) of ω-normal states is denoted by $\Delta_{\nu|\rho}$.

The relative entropy of a pair (ν, ρ) of ω-normal faithful states is

$$\mathrm{Ent}(\nu|\rho) = \langle \Omega_\nu, \log \Delta_{\rho|\nu} \Omega_\nu \rangle.$$

This is the original definition of Araki [Ara76, Ara77], with the sign and ordering convention of [JP01]. In particular, $\mathrm{Ent}(\nu|\rho) \le 0$ with equality iff $\rho = \nu$. For additional information about relative entropy we refer the reader to [OP93].

Since ω is ς_ω invariant, the family $\{\pi \circ \varsigma_\omega^t \mid t \in \mathbb{R}\}$ extends to a W^*-dynamics on $\mathfrak{M}$ which we again denote by ς_ω. For $A \in \mathfrak{M}$, one has

$$\varsigma_\omega^t(A) = \Delta_\omega^{\mathrm{i}t} A \Delta_\omega^{-\mathrm{i}t}.$$

More generally, to any faithful $\nu \in \mathcal{N}$ one associates the W^*-dynamics ς_ν given by

$$\varsigma_\nu^t(A) = \Delta_\nu^{\mathrm{i}t} A \Delta_\nu^{-\mathrm{i}t}.$$

ς_ν is called the modular dynamics of ν, and ν is a $(\varsigma_\nu, -1)$-KMS state on $\mathfrak{M}$.

Throughout the paper we assume that the following holds:

[7] See [BR81, Corollary 5.3.9]

(Reg1) The family $\{\pi \circ \tau^t \mid t \in \mathbb{R}\}$ extends the interacting dynamics τ to a W^*-dynamics on $\mathfrak{M}$ which we again denote by τ.

This assumption is automatically satisfied by the interacting dynamics of the two kinds of open quantum systems introduced in the previous section. It ensures that ω_t is a ω-normal faithful state on $\mathfrak{M}$ and that there exists a unique self-adjoint operator $\mathcal{L}$ on $\mathcal{H}$, called the standard Liouvillean of τ, such that

$$\tau^t(A) = \mathrm{e}^{\mathrm{i}t\mathcal{L}} A \mathrm{e}^{-\mathrm{i}t\mathcal{L}}, \qquad \mathrm{e}^{-\mathrm{i}t\mathcal{L}}\mathcal{H}^+ = \mathcal{H}^+,$$

for all $A \in \mathfrak{M}$ and $t \in \mathbb{R}$. The vector representative of ω_t in $\mathcal{H}^+$ is $\mathrm{e}^{-\mathrm{i}t\mathcal{L}}\Omega$. We will make use of the following well-known result.

Theorem 1.1 (1) $\omega \in \mathcal{S}_\tau \Leftrightarrow \mathcal{L}\Omega = 0$.

(2) *Suppose that $\omega \in \mathcal{S}_\tau$. Then the quantum dynamical system $(\mathcal{O}, \tau, \omega)$ is ergodic iff 0 is a simple eigenvalue of $\mathcal{L}$.*

For latter reference we also recall the following well-known result that identifies ergodicity with the so-called property of return to equilibrium [Rob73].

Theorem 1.2 *Suppose that $\omega \in \mathcal{S}_\tau$. Then the quantum dynamical system $(\mathcal{O}, \tau, \omega)$ is ergodic iff for any ω-normal state ν and all $A \in \mathcal{O}$ one has*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \nu(\tau^t(A))\mathrm{d}t = \omega(A).$$

To any pair of faithful ω-normal states ν and ρ one associates the Connes cocycle

$$[D\nu : D\rho]_{\mathrm{i}t} = \Delta_{\nu|\rho}^{\mathrm{i}t} \Delta_{\rho}^{-\mathrm{i}t}, \qquad (t \in \mathbb{R}), \tag{1.4}$$

which is obviously a family of unitary operators. Its basic property is that $[D\nu : D\rho]_{\mathrm{i}t} \in \mathfrak{M}$; for additional properties of Connes' cocycle, see [AM82, Appendix C].

The Connes cocycle

$$[D\omega_t : D\omega]_\alpha, \qquad \alpha \in \mathrm{i}\mathbb{R},$$

will play a central role in our work. In what follows, we assume

(Reg2) For all $t \in \mathbb{R}$ and $\alpha \in \mathrm{i}\mathbb{R}$,

$$[D\omega_t : D\omega]_\alpha \in \pi(\mathcal{O}).$$

$\pi^{-1}([D\omega_t : D\omega]_\alpha)$ will be also denoted by $[D\omega_t : D\omega]_\alpha$. For the two kinds of open quantum systems introduced in the previous section, (Reg2) holds if $V \in \mathrm{Dom}\,(\delta_\omega)$. This follows from the definition (1.4), the fact that

$$\Delta_{\omega_t|\omega}^{\alpha} = \mathrm{e}^{\alpha(\log \Delta_\omega + \pi_\omega(Q_t))}, \qquad Q_t = \int_0^t \tau^{-s}(\delta_\omega(V))\mathrm{d}s,$$

as established in the proof of [JP03, Theorem 1.1], and time-dependent perturbation theory.

1.3 Two-times measurement entropy production in finite quantum systems

This notion of entropy production goes back to [Kur00, Tas00] and has been studied in detail in [JOPP10]. Consider a finite quantum dynamical system on a Hilbert space $\mathcal{K}$ with Hamiltonian H. The measurement protocol is defined with respect to two faithful states ν and ω. The first one, ν, is the state of system at the instant of the first measurement, and will be a variable in our work. The second one, ω, is assumed to be fixed and defines the observable to be measured. More precisely, we will consider two consecutive measurements of the observable

$$S = -\log \omega$$

interpreting the increase ΔS in the outcomes of these two measurement as the entropy produced by the system during the time interval between the two measurements. To motivate this interpretation, let the small system $\mathcal{S}$, with $\omega_{\mathcal{S}}$ given by (1.3), be coupled to thermal reservoirs $\mathcal{R}_1, \ldots, \mathcal{R}_M$ at inverse temperatures $\beta_1, \ldots, \beta_M$. Setting

$$\omega = \omega_{\mathcal{S}} \otimes \left(\bigotimes_{j=1}^{M} \frac{\mathrm{e}^{-\beta_j H_j}}{\operatorname{tr} \mathrm{e}^{-\beta_j H_j}} \right)$$

where H_j denotes the Hamiltonian of the j^{th} reservoir, we get

$$\Delta S = \sum_{j=1}^{M} \beta_j \Delta E_j$$

where ΔE_j is the change in the energy of the j^{th} reservoir. Thus, ΔS can be identified with the entropy dumped by the system $\mathcal{S}$ in the reservoirs.

Let $\mathcal{A}$ be a finite alphabet indexing the distinct eigenvalues $(\lambda_a)_{a\in\mathcal{A}}$ of ω, and let P_a be the eigenprojection corresponding to λ_a. The observable to be measured is described by the partition of unity $(P_a)_{a\in\mathcal{A}}$ on $\mathcal{K}$, with outcomes of the measurement labeled by the letters of $\mathcal{A}$. The two-times measurement protocol goes as follows. At the instant of the first measurement, when the system is in the state ν, the outcome $a \in \mathcal{A}$ is observed with probability

$$p_\nu(a) = \nu(P_a).$$

After the measurement the system is in the reduced state

$$\frac{1}{p_\nu(a)} P_a \nu P_a,$$

which evolves under the system dynamics over the time interval of length t to

$$\frac{1}{p_\nu(a)} \mathrm{e}^{-\mathrm{i}tH} P_a \nu P_a \mathrm{e}^{\mathrm{i}tH}.$$

The second measurement, performed at the end of this time interval, yields the outcome $b \in \mathcal{A}$ with probability

$$p_{\nu,t}(b|a) = \frac{1}{p_\nu(a)} \operatorname{tr}(\mathrm{e}^{-\mathrm{i}tH} P_a \nu P_a \mathrm{e}^{\mathrm{i}tH} P_b).$$

Finally, the probability of observing the pair (b, a) in this two-times measurement protocol is

$$p_{\nu,t}(b, a) = p_{\nu,t}(b|a)p_\nu(a) = \mathrm{tr}(\mathrm{e}^{-\mathrm{i}tH} P_a \nu P_a \mathrm{e}^{\mathrm{i}tH} P_b). \tag{1.5}$$

The formula (1.5) defines a probability measure on $\mathcal{A} \times \mathcal{A}$ even when ν is not faithful, and from now on we drop this restriction.[8]

The entropy production random variable $\mathcal{E} : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ is defined by

$$\mathcal{E}(b, a) = -\log \lambda_b + \log \lambda_a,$$

and its probability distribution with respect to $p_{\nu,t}$ is denoted by $Q_{\nu,t}$,

$$Q_{\nu,t}(s) = \sum_{\mathcal{E}(b,a)=s} p_{\nu,t}(b, a).$$

The statistics of two-times measurement entropy production is described by the family $(Q_{\nu,t})_{t>0}$. In the case $\nu = \omega$, this family of probability measures was studied in detail in [JOPP10]. To the best of our knowledge, the case $\nu \neq \omega$ was not considered before in the mathematical physics literature.

We set

$$\mathfrak{F}_{\nu,t}(\alpha) = \int_{\mathbb{R}} \mathrm{e}^{-\alpha s} \mathrm{d}Q_{\nu,t}(s), \qquad (\alpha \in \mathbb{C}).$$

The definition of $Q_{\nu,t}$ gives

$$\mathfrak{F}_{\nu,t}(\alpha) = \mathrm{tr}(\omega_{-t}^{\alpha} \omega^{-\alpha} \overline{\nu})$$

where $\overline{\nu} = \sum_{a\in\mathcal{A}} P_a \nu P_a$. Taking $\nu = \omega$ leads to the formulas

$$\mathfrak{F}_{\omega,t}(\alpha) = \omega([D\omega_{-t} : D\omega]_\alpha) = \langle \Omega_\omega, \Delta^{\alpha}_{\omega_{-t}|\omega} \Omega_\omega \rangle$$

and to the identification of $Q_{\omega,t}$ with the spectral measure of the operator $-\log \Delta_{\omega_{-t}|\omega}$ for the vector Ω_ω. This deep link between the statistics of the two-times measurement entropy production and modular theory has a somewhat unusual history and was discussed in detail in [JOPP10].

The starting point of this work is the observation that for general ν one can also link $Q_{\nu,t}$ to the modular structure via the formula

$$\mathfrak{F}_{\nu,t}(\alpha) = \lim_{R\to\infty} \frac{1}{R} \int_0^R \nu\left(\varsigma_\omega^\theta\left([D\omega_{-t} : D\omega]_\alpha\right)\right) \mathrm{d}\theta \tag{1.6}$$

that follows by an elementary computation. This modular representation of $\mathfrak{F}_{\nu,t}$ for general ν, to the best of our knowledge, has not appeared previously in the literature.

We are now ready to state our main results.

1.4 Main results

Throughout this section $(\mathcal{O}, \tau, \omega)$ is a fixed C^*-quantum dynamical system with modular reference state ω. Recall that Assumptions (Reg1) and (Reg2) are in force throughout the paper.

[8] For non-faithful ν the protocol can be implemented in a limiting sense by considering a sequence of faithful ν_n's such that $\lim_n \nu_n = \nu$.

Theorem 1.3 *For all $\nu \in \mathcal{N}$, $t \in \mathbb{R}$, and $\alpha \in \mathrm{i}\mathbb{R}$, the limit*

$$\mathfrak{F}_{\nu,t}(\alpha) = \lim_{R\to\infty} \frac{1}{R}\int_0^R \nu\left(\varsigma_\omega^\theta\left([D\omega_{-t} : D\omega]_\alpha\right)\right) \mathrm{d}\theta \tag{1.7}$$

exists, and there exists unique Borel probability measure $Q_{\nu,t}$ on $\mathbb{R}$ such that

$$\mathfrak{F}_{\nu,t}(\alpha) = \int_{\mathbb{R}} \mathrm{e}^{-\alpha s}\mathrm{d}Q_{\nu,t}(s). \tag{1.8}$$

The family $(Q_{\nu,t})_{t>0}$ describes the statistics of two-times measurement entropy production of $(\mathcal{O}, \tau, \omega)$ with respect to ν in the above general setting. This definition, that arises by modular extension of the finite quantum system physical notion discussed in the previous section, requires a general comment in relation to the thermodynamic limit procedure; see Remark 1 in Section 1.5 and [JOPP10, Chapter 5].

In the case $\nu = \omega$,

$$\omega\left(\varsigma_\omega^\theta\left([D\omega_{-t} : D\omega]_\alpha\right)\right) = \langle \Omega_\omega, \Delta^\alpha_{\omega_{-t}|\omega}\Omega_\omega\rangle$$

for all θ, and so $Q_{\omega,t}$ is the spectral measure of $-\log\Delta_{\omega_{-t}|\omega}$ for the vector Ω_ω. Thus, $Q_{\omega,t}$ coincides with the proposal of [TM03], where the authors, unaware of the works [Kur00, Tas00], were searching for a quantum version of the fluctuation relation of classical statistical mechanics; see Proposition 1.9 below. That, in the finite quantum system setting, this spectral measure coincides with the two-times measurement entropy production statistics discussed in the previous section did not appear in print until [JOPP10]. The basic properties of $Q_{\omega,t}$ are summarized in

Theorem 1.4 (1) $\int_{\mathbb{R}} s\,\mathrm{d}Q_{\omega,t}(s) = -\mathrm{Ent}(\omega_t|\omega)$. *In particular,* $\int_{\mathbb{R}} s\,\mathrm{d}Q_{\omega,t}(s) \geq 0$ *with the equality iff* $\omega = \omega_t$.

(2) *The map* $\mathrm{i}\mathbb{R} \ni \alpha \mapsto \mathfrak{F}_{\omega,t}(\alpha)$ *has an analytic extension to the vertical strip* $0 < \mathrm{Re}\,\alpha < 1$ *that is bounded and continuous on its closure and satisfies*

$$\mathfrak{F}_{\omega,t}(\alpha) = \overline{\mathfrak{F}_{\omega,-t}(1-\overline{\alpha})}$$

for $0 \leq \mathrm{Re}\,\alpha \leq 1$ *and* $t \in \mathbb{R}$.

In the remaining statements we assume that ω is time-reversal invariant.

(3) *For any α satisfying $0 \leq \mathrm{Re}\,\alpha \leq 1$,*

$$\mathfrak{F}_{\omega,t}(\alpha) = \overline{\mathfrak{F}_{\omega,t}(1-\overline{\alpha})}. \tag{1.9}$$

(4) *Let $\mathfrak{r} : \mathbb{R} \to \mathbb{R}$ be the reflection $\mathfrak{r}(s) = -s$ and $\overline{Q}_{\nu,t} = Q_{\nu,t} \circ \mathfrak{r}$. Then the measures $Q_{\omega,t}$ and $\overline{Q}_{\omega,t}$ are equivalent and*

$$\frac{\mathrm{d}\overline{Q}_{\omega,t}}{\mathrm{d}Q_{\omega,t}}(s) = \mathrm{e}^{-s}. \tag{1.10}$$

The relations (1.9) and (1.10) are known as the finite time quantum fluctuation relations.[9] Theorem 1.4 was essentially proven in [TM03, Theorem 7]. For the reader convenience and future reference, we provide its proof in Section 2.2.

We now return to general $\nu \in \mathcal{N}$. Our first result is an immediate consequence of Theorems 1.2 and 1.3:

[9] They are also sometimes called the Evans–Searles quantum fluctuation relations.

Theorem 1.5 *Suppose that the system $(\mathcal{O}, \varsigma_\omega, \omega)$ is ergodic. Then for all $t \in \mathbb{R}$ and $\nu \in \mathcal{N}$,*

$$Q_{\nu,t} = Q_{\omega,t}.$$

The above theorem applies to open quantum systems with directly coupled reservoirs and its assumption holds iff each reservoir system $(\mathcal{O}_j, \tau_j, \omega_j)$ is ergodic. In what follows we consider open quantum systems featuring a small system $\mathcal{S}$. For any $\nu \in \mathcal{S}_\mathcal{O}$ we denote by $\nu_\mathcal{S}$ the restriction of ν to $\mathcal{O}_\mathcal{S}$.[10]

Theorem 1.6 *Consider an open quantum system where the reservoirs $\mathcal{R}_1, \ldots, \mathcal{R}_M$ are coupled through the small system $\mathcal{S}$, each reservoir subsystem $(\mathcal{O}_j, \tau_j, \omega_j)$ being ergodic. Let $\nu \in \mathcal{N}$.*

(1) *For all $\alpha \in \mathrm{i}\mathbb{R}$,*

$$\begin{aligned}\mathfrak{F}_{\nu,t}(\alpha) &= \nu_\mathcal{S} \otimes \omega_\mathcal{R}\left([D\omega_{-t} : D\omega]_\alpha\right) \\ &= \langle \Omega_{\nu_\mathcal{S}\otimes\omega_\mathcal{R}}, \Delta^\alpha_{\omega_{-t}|\omega} \Omega_{\nu_\mathcal{S}\otimes\omega_\mathcal{R}} \rangle.\end{aligned}$$

In particular, $Q_{\nu,t}$ is the spectral measure of $-\log \Delta_{\omega_{-t}|\omega}$ for the vector $\Omega_{\nu_\mathcal{S}\otimes\omega_\mathcal{R}}$.

(2) *The measure $Q_{\nu,t}$ is absolutely continuous with respect to $Q_{\omega,t}$ and*

$$\frac{\mathrm{d}Q_{\nu,t}}{\mathrm{d}Q_{\omega,t}} \leq \dim \mathcal{K}_\mathcal{S}. \tag{1.11}$$

If $\nu_\mathcal{S}$ is faithful and γ is its smallest eigenvalue, then also

$$\gamma \dim \mathcal{K}_\mathcal{S} \leq \frac{\mathrm{d}Q_{\nu,t}}{\mathrm{d}Q_{\omega,t}}. \tag{1.12}$$

We equip $\mathcal{S}_\mathcal{O}$ with the weak*-topology and the set $\mathcal{P}(\mathbb{R})$ of all Borel probability measures on $\mathbb{R}$ with the weak topology. By [Tak55, Lemma 2.1] and [Fel60, Theorem 1.1], the set of ω-normal states $\mathcal{N}$ is dense in $\mathcal{S}_\mathcal{O}$. This gives that under the assumptions of either Theorem 1.5 or 1.6, the map

$$\mathcal{N} \ni \nu \mapsto Q_{\nu,t} \in \mathcal{P}(\mathbb{R}) \tag{1.13}$$

uniquely extends to a continuous map

$$\mathcal{S}_\mathcal{O} \ni \nu \mapsto Q_{\nu,t} \in \mathcal{P}(\mathbb{R}).$$

This continuous extension defines $Q_{\nu,t}$ for all $\nu \in \mathcal{S}_\mathcal{O}$. In the case of Theorem 1.5, obviously $Q_{\nu,t} = Q_{\omega,t}$ for all ν. In the case of Theorem 1.6, $Q_{\nu,t}$ is again the spectral measure of $-\log \Delta_{\omega_{-t}|\omega}$ for the vector $\Omega_{\nu_\mathcal{S}\otimes\omega_\mathcal{R}}$ and Part (2) holds. We summarize:

[10] For $A \in \mathcal{O}_\mathcal{S}$, $\nu_\mathcal{S}(A) = \nu(A \otimes \mathbb{1})$.

Theorem 1.7 *Consider a non-ω-normal state $\nu \in \mathcal{S}_{\mathcal{O}}$. Then, Theorems 1.5 and 1.6 hold, with $Q_{\nu,t}$ being the above mentioned continuous extension.*

1.5 Remarks

1. Thermodynamic limit. Thermodynamic limit (abbreviated TDL) plays a distinguished role in statistical mechanics. It realizes infinitely extended systems through a limiting procedure involving only finite quantum systems and is central for the identification of physically relevant objects in the infinite setting. The precise way the TDL is taken depends on the structure of the specific physical model under consideration, and often different approximation routes are possible. This topic is well-understood and discussed in many places in the literature; see, for example, [BR87, BR81, Rue69] and [JOPP10, Chapter 5]. Since early days, it is known that the modular structure is stable under TDL [AI74, Ara76, Ara76, Ara77], and this fact plays an important role in the foundations of quantum statistical mechanics. The customary route in discussions of the structural theory is the following:

Step 1. A physical notion, introduced in the context of finite quantum systems, is expressed in a modular form, and through this form is directly extended, by definition, to a general C^*/W^*-dynamical system. One basic example of such procedure is the introduction of the KMS-condition as characterization of thermal equilibrium states. This is the approach we have taken in this work in the introduction of $Q_{\nu,t}$.

Step 2. In concrete physical models the definitions of Step 1 are justified by the TDL limit.

Step 2 has been extensively studied in early days of quantum statistical mechanics and the wealth of obtained results make its implementation in modern literature most often a routine exercise. For this reason this step is often skipped. There is rarely a need for making an exception to this rule, but one is, we believe, in the context of our work. The mixture of quantum measurements and thermodynamics, which is central to the definition of two-times quantum measurement entropy production, has a number of unexpected features that, we believe, require the TDL justification to be put on solid physical grounds. More precisely, the formula (1.6), which gives the modular characterization of the initial state decoherence induced by the first measurement, also allows to postulate this decoherence effect for infinitely extended systems. The physical and mathematical implications of this Step 1 definition make its TDL justification necessary.

In the forthcoming article [BBJ$^+$b] we will carry out Step 2 for the 2TMEP of two paradigmatic models in non-equilibrium quantum statistical mechanics: open quantum spin system on a lattice[11] and open Spin-Fermion Model[12].

2. The effect of the first measurement. The somewhat striking rigidity of the two-times measurement entropy production statistics that follows from Theorems 1.5 and 1.6 can be understood in terms of the dominating effect of the first decoherence inducing measurement. In the TDL, this effect is dramatic as the measurement induced decoherence forces the (ergodic) reservoirs into their unique invariant

[11] See [Rue01].
[12] See the seminal works [Dav74, SL78].

state, while the state of the small system $\mathcal{S}$ has a marginal effect, being finite dimensional. In mathematical terms, the first measurement decoherence corresponds to a projection on $\mathrm{Ker}\,\log\Delta_\omega$ (see the formula (2.1) below), which, for infinitely extended systems and due to the ergodicity assumptions of Theorems 1.5 and 1.6, is finite dimensional.[13] Along the sequence of TDL approximations the size of this kernel grows to infinity together with the dimension of the reservoir Hilbert spaces to suddenly drastically shrink in the limiting state. This dimension reduction is at the core of the need for the TDL justification of the Step 1 definition of the 2TMEP.

A proposal for a less invasive measurement protocol through an auxiliary two-dimensional quantum system (ancilla), that we call Entropic Ancilla State Tomography, avoids the above decoherence effect. This alternative protocol and its stability are the topics of [BBJ$^+$a], see also Remark 4 below.

3. On the extension to $\nu \in \mathcal{S}_\mathcal{O}$. In Theorem 1.7, the 2TMEP $Q_{\nu,t}$ of $\nu \notin \mathcal{N}$ is defined by the continuous extension of the map (1.13) that builds on the stability results of Theorems 1.5 and 1.6. For $\nu \notin \mathcal{N}$ the representation (1.7) fails (with $\mathfrak{F}_{\nu,t}(\alpha)$ defined by (1.8)). In the same vein, the direct TDL limit justification of $Q_{\nu,t}$ is not possible for $\nu \notin \mathcal{N}$. It is replaced by the TDL limit justification of the approximating sequence $Q_{\nu_n,t}$, $\nu_n \in \mathcal{N}$, $\nu_n \to \nu$, with ν_n's chosen to reflect the physics of the limiting ν. We emphasize that when $\nu \notin \mathcal{N}$, $Q_{\nu,t}$ is defined by first taking the limit $R \to \infty$ in (1.7) for a fixed normal approximation ν_n of ν, and then by taking the limit $\nu_n \to \nu$. Taken in reverse order, this double limit does not necessarily exist, and will not produce the same limiting measure in general.

4. NESS and quantum Gallavotti–Cohen Fluctuation Theorem. This is the topic of the continuation of this work [BBJ$^+$a] whose starting points are Theorems 1.3–1.7, and we limit ourselves here to a brief comment. The quantum Evans–Searles and Gallavotti–Cohen Fluctuation Theorems deal with quantum extensions of the celebrated works [ES94, GC95a, GC95b] in classical statistical mechanics, see also the review [JPRB11]. The quantum Evans–Searles Fluctuation Theorem concerns the Large Deviation Principle (LDP) for the family of probability measures $(Q_{\omega,t}(t\,\cdot\,))_{t>0}$ in the limit $t \uparrow \infty$. In parallel with the classical theory, any putative quantum Gallavotti–Cohen Fluctuation Theorem should concern the entropic fluctuations with respect to the Non-Equilibrium Steady State (NESS) that the system $(\mathcal{O}, \tau, \omega)$ reaches in the large time limit $t \uparrow \infty$. This NESS is defined as the weak*-limit $\omega_+ = \lim_{t\to\infty}\omega_t$.[14] In typical non-equilibrium setting $\omega_+ \notin \mathcal{N}$, and for a long time it was unclear how to define the statistics $Q_{\omega_+,t}$. Theorems 1.5–1.7 provide a route to this definition that, together with the TDL justification of $Q_{\omega_T,t}$ for all $T, t \geq 0$, is both physically and mathematically natural. However, due to a quantum decoherence effect, this route comes with a degree of stability that has no classical counterpart and that identifies the two Fluctuation Theorems under very general ergodic assumptions that are satisfied in paradigmatic models of open quantum systems. This triviality aspect is addressed in [BBJ$^+$a] by the introduction of Entropic Ancilla State Tomography that we have already mentioned in Remark 2. Entropic Ancilla State Tomography provides a novel physical and mathematical perspective on the entropic fluctuations in quantum statistical mechanics and links the two quantum Fluctuation Theorems in a non-trivial way.

[13] In the case of Theorem 1.5 this kernel has dimension 1. In the case of Theorem 1.6 its dimension is $(\dim \mathcal{K}_\mathcal{S})^2$.

[14] The existence of this limit is typically a deep dynamical problem.

5. Repeated two-times measurement protocol. Besides the dynamical system approach to classical entropy production in which the reference state plays a central role (see the review [JPRB11] for a discussion of this point), an altogether different random path approach has been developed in [Kur98, LS99, Mae99] that does not make use of the reference state and is applicable to stochastic processes. Its quantum formulation in the setting of repeated quantum measurement processes goes back to [Cro08], and was elaborated in [BJPP18, BCJP21]. The advent of experimental methods in cavity and circuit QED, and in particular the experimental breakthroughs of the Haroche–Raimond and Wineland groups [Har13, HR06, Win13] make this complementary approach particularly relevant. We postpone the comparative discussion of the two approaches to the forthcoming review.

6. On the choice of $\omega_{\mathcal{S}}$. The proof of Theorem 1.6 makes explicit use of the special form (1.3) of $\omega_{\mathcal{S}}$, and that has the effect on the values of the constants in the estimates (1.11) and (1.12). However, if ν_1 and ν_2 are any two states in $\mathcal{N}_\omega$ such that the restrictions $\nu_{1\mathcal{S}}$ and $\nu_{2\mathcal{S}}$ are faithful with the smallest eigenvalues γ_1 and γ_2, then the chain rule

$$\frac{\mathrm{d}Q_{\nu_1,t}}{\mathrm{d}Q_{\nu_2,t}} = \frac{\mathrm{d}Q_{\nu_1,t}}{\mathrm{d}Q_{\omega,t}}\frac{\mathrm{d}Q_{\omega,t}}{\mathrm{d}Q_{\nu_2,t}}$$

and (1.11), (1.12), give that

$$\gamma_1 \leq \frac{\mathrm{d}Q_{\nu_1,t}}{\mathrm{d}Q_{\nu_2,t}} \leq \frac{1}{\gamma_2}.$$

7. On the definition of $\mathfrak{F}_{\nu,t}$. In the context of Theorem 1.3, one also has that

$$\mathfrak{F}_{\nu,t}(\alpha) = \lim_{R\to\infty}\frac{1}{R}\int_0^R \nu\left(\varsigma_\omega^\theta\left([D\omega_{-t} : D\omega]^*_{\frac{\alpha}{2}}[D\omega_{-t} : D\omega]_{\frac{\alpha}{2}}\right)\right)\mathrm{d}\theta. \tag{1.14}$$

For finite quantum systems (1.14) is an immediate consequence of the relation $[\omega,\overline{\nu}] = 0$. In the general case, (1.14) is established by a simple modification of the proof of Theorem 1.3. In the continuation of this work [BBJ$^+$a] we will make use of (1.14) in the context of the Entropic Ancilla State Tomography.

2 Proofs

We shall need the following results.

Lemma 2.1 *For any $t \in \mathbb{R}$, one has*

$$\mathrm{e}^{\mathrm{i}t\mathcal{L}}\Delta_{\omega|\omega_t}\mathrm{e}^{-\mathrm{i}t\mathcal{L}} = \Delta_{\omega_{-t}|\omega}.$$

Proof. For $A \in \mathfrak{M}$ and $t \in \mathbb{R}$, we have, taking into account the fact that $J\mathcal{L} + \mathcal{L}J = 0$,

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}t\mathcal{L}}\Delta_{\omega|\omega_t}^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}t\mathcal{L}}A\Omega &= \mathrm{e}^{\mathrm{i}t\mathcal{L}}\Delta_{\omega|\omega_t}^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}t\mathcal{L}}A\mathrm{e}^{\mathrm{i}t\mathcal{L}}\mathrm{e}^{-\mathrm{i}t\mathcal{L}}\Omega \\
&= \mathrm{e}^{\mathrm{i}t\mathcal{L}}JJ\Delta_{\omega|\omega_t}^{\frac{1}{2}}\tau^{-t}(A)\Omega_t \\
&= \mathrm{e}^{\mathrm{i}t\mathcal{L}}J\tau^{-t}(A^*)\Omega \\
&= \mathrm{e}^{\mathrm{i}t\mathcal{L}}J\mathrm{e}^{-\mathrm{i}t\mathcal{L}}A^*\mathrm{e}^{\mathrm{i}t\mathcal{L}}\Omega \\
&= JA^*\Omega_{-t} \\
&= \Delta_{\omega_{-t}|\omega}^{\frac{1}{2}}A\Omega,
\end{aligned}$$

where we used that $\mathrm{e}^{\mathrm{i}t\mathcal{L}}J = J\mathrm{e}^{\mathrm{i}t\mathcal{L}}$. □

Proposition 2.2 *Suppose that ω is time-reversal invariant with respect to the time-reversal Θ. Then, there exists an anti-unitary involution U on $\mathcal{H}$ such that:*

(1) *For any $A \in \mathfrak{M}$, $\Theta(A) = UAU^*$.*

(2) *$U\Omega = \Omega$ and $U\mathcal{H}^+ = \mathcal{H}^+$.*

(3) $[U, J] = 0$.

(4) $[U, \mathcal{L}] = 0$.

(5) *$U^*\Delta_{\omega_{-t}|\omega}U = \Delta_{\omega_t|\omega}$ for any $t \in \mathbb{R}$.*

Proof. The existence of U as well as Parts (1)–(3) follow from a simple adaptation of the proof of the corresponding statements of [BR87, Corrolary 2.5.32].

To prove Part (4), we start with the relation $\tau^t \circ \Theta = \Theta \circ \tau^{-t}$, which yields that

$$\mathrm{e}^{\mathrm{i}t\mathcal{L}}UAU^*\mathrm{e}^{-\mathrm{i}t\mathcal{L}} = U\mathrm{e}^{-\mathrm{i}t\mathcal{L}}A\mathrm{e}^{\mathrm{i}t\mathcal{L}}U^*$$

for any $t \in \mathbb{R}$ and $A \in \mathfrak{M}$. It follows that

$$(U^*\mathrm{e}^{\mathrm{i}t\mathcal{L}}U)A(U^*\mathrm{e}^{-\mathrm{i}t\mathcal{L}}U) = \mathrm{e}^{-\mathrm{i}t\mathcal{L}}A\mathrm{e}^{\mathrm{i}t\mathcal{L}},$$

and since Part (2) gives $U^*\mathrm{e}^{\mathrm{i}t\mathcal{L}}U\mathcal{H}^+ \subset \mathcal{H}^+$, the unicity of the standard Liouvillean yields that

$$\mathrm{e}^{-\mathrm{i}tU^*\mathcal{L}U} = U^*\mathrm{e}^{\mathrm{i}t\mathcal{L}}U = \mathrm{e}^{-\mathrm{i}t\mathcal{L}},$$

from which (4) follows.

By Parts (1–2), for any $t \in \mathbb{R}$ and $A \in \mathfrak{M}$, one has

$$\begin{aligned}
U^*\Delta_{\omega_{-t}|\omega}^{\frac{1}{2}}UA\Omega &= U^*\Delta_{\omega_{-t}|\omega}^{\frac{1}{2}}\Theta(A)\Omega \\
&= U^*J\Theta(A)^*\mathrm{e}^{\mathrm{i}t\mathcal{L}}\Omega \\
&= (U^*JU)A^*(U^*\mathrm{e}^{\mathrm{i}t\mathcal{L}}U)U^*\Omega.
\end{aligned}$$

Invoking Parts (3–4) further gives

$$U^*\Delta^{\frac{1}{2}}_{\omega_{-t}|\omega}UA\Omega = JA^*\mathrm{e}^{-\mathrm{i}t\mathcal{L}}\Omega = JA^*\Omega_t = \Delta^{\frac{1}{2}}_{\omega_t|\omega}A\Omega,$$

which yields Part (5). □

2.1 Proof of Theorem 1.3.

In parts, the arguments follow closely the proof of Theorem 3.3, Part (3) in [AJPP06]. We give details for the reader's convenience.

We consider first a state of the form $\nu_B(\cdot) = (B\Omega, \cdot\, B\Omega)$ where $B \in \pi(\mathcal{O})'$ and $\|B\Omega\| = 1$. Then, for any $A \in \mathcal{O}$,

$$\begin{aligned}\frac{1}{R}\int_0^R \nu_B(\varsigma^\theta_\omega(A))\mathrm{d}\theta &= \frac{1}{R}\int_0^R \langle B\Omega, \mathrm{e}^{\mathrm{i}\theta\log\Delta_\omega}A\mathrm{e}^{-\mathrm{i}\theta\log\Delta_\omega}B\Omega\rangle\mathrm{d}\theta \\ &= \frac{1}{R}\int_0^R \langle B^*B\Omega, \mathrm{e}^{\mathrm{i}\theta\log\Delta_\omega}A\Omega\rangle\mathrm{d}\theta.\end{aligned}$$

The von Neumann ergodic theorem gives that, for all $A \in \mathcal{O}$,

$$\nu_{B+}(A) = \lim_{R\to\infty}\frac{1}{R}\int_0^R \nu_B(\varsigma^\theta_\omega(A))\mathrm{d}\theta = \langle B^*B\Omega, PA\Omega\rangle,$$

where P is the orthogonal projection on $\mathrm{Ker}\,\log\Delta_\omega$. In particular,

$$\nu_{B+}([D\omega_{-t} : D\omega]_\alpha) = \langle B^*B\Omega, P\Delta^\alpha_{\omega_{-t}|\omega}\Omega\rangle.$$

This proves that for $\alpha \in \mathrm{i}\mathbb{R}$,

$$\mathfrak{F}_{\nu_B,t}(\alpha) = \lim_{R\to\infty}\frac{1}{R}\int_0^R \nu_B(\varsigma^\theta_\omega([D\omega_{-t} : D\omega]_\alpha))\mathrm{d}\theta = \langle B^*B\Omega, P\Delta^\alpha_{\omega_{-t}|\omega}\Omega\rangle. \tag{2.1}$$

The function $\mathrm{i}\mathbb{R} \ni \alpha \mapsto \mathfrak{F}_{\nu_B,t}(\alpha)$ is continuous. Moreover, this function is also positive-definite since, for $z_1, \cdots, z_N \in \mathbb{C}$ and $\alpha_1, \cdots, \alpha_N \in \mathrm{i}\mathbb{R}$,

$$\begin{aligned}\sum_{k,l=1}^N \mathfrak{F}_{\nu_B,t}(\alpha_k - \alpha_l)z_k\bar{z}_l &= \left\langle B^*B\Omega, P\left[\sum_{k=1}^M z_k\Delta^{\alpha_k}_{\omega_{-t}|\omega}\right]^*\left[\sum_{l=1}^N z_l\Delta^{\alpha_l}_{\omega_{-t}|\omega}\right]\Omega\right\rangle \\ &= \left\langle B^*B\Omega, P\left[\sum_{k=1}^N z_k\Delta^{\alpha_k}_{\omega_{-t}|\omega}\Delta^{-\alpha_k}_\omega\right]^*\left[\sum_{l=1}^N z_l\Delta^{\alpha_l}_{\omega_{-t}|\omega}\Delta^{-\alpha_l}_\omega\right]\Omega\right\rangle \\ &= \lim_{R\to\infty}\frac{1}{R}\int_0^R\left\langle B^*B\Omega, \varsigma^\theta_\omega\left(\left[\sum_{k=1}^N z_k[D\omega_{-t} : D\omega]_{\alpha_k}\right]^*\left[\sum_{l=1}^N z_l[D\omega_{-t} : D\omega]_{\alpha_l}\right]\right)\Omega\right\rangle\mathrm{d}\theta \\ &= \lim_{R\to\infty}\frac{1}{R}\int_0^R \nu_B\circ\varsigma^\theta_\omega\left(\left[\sum_{k=1}^N z_k[D\omega_{-t} : D\omega]_{\alpha_k}\right]^*\left[\sum_{l=1}^N z_l[D\omega_{-t} : D\omega]_{\alpha_l}\right]\right)\mathrm{d}\theta \geq 0.\end{aligned}$$

Hence, by the Bochner-Khinchine theorem, there exists unique Borel probability measure $Q_{\nu_B,t}$ on $\mathbb{R}$ such that, for all $\alpha \in \mathrm{i}\mathbb{R}$,

$$\mathfrak{F}_{\nu_B,t}(\alpha) = \int_{\mathbb{R}} \mathrm{e}^{-\alpha s}\mathrm{d}Q_{\nu_B,t}(s).$$

Let now ν be an arbitrary ω-normal state on $\mathcal{O}$. Since Ω is a cyclic vector for $\pi(\mathcal{O})'$, for any $n \in \mathbb{N}$ there exists $B_n \in \pi(\mathcal{O})'$ such that

$$\|\nu - \nu_{B_n}\| \leq \frac{1}{n}.$$

This gives that the sequence ν_{B_n} is Cauchy in norm. If ν_+ is any limit point of the net

$$\frac{1}{R}\int_0^R \nu \circ \varsigma_\omega^\theta \mathrm{d}\theta \tag{2.2}$$

as $R \uparrow \infty$, we have that

$$\|\nu_+ - \nu_{B_n+}\| \leq \|\nu - \nu_{B_n}\| \leq \frac{1}{n}.$$

It follows that ν_+ is the norm limit of ν_{B_n+} and in particular that the net (2.2) has the unique limit ν_+. This gives that for all $\alpha \in \mathrm{i}\mathbb{R}$,

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R \nu(\varsigma_\omega^\theta([D\omega_{-t} : D\omega]_\alpha))\mathrm{d}\theta = \lim_{n\to\infty}\nu_{B_n+}([D\omega_{-t} : D\omega]_\alpha) = \nu_+([D\omega_{-t} : D\omega]_\alpha),$$

establishing the existence of $\mathfrak{F}_{\nu,t}$. In addition, we have that for $\alpha \in \mathrm{i}\mathbb{R}$,

$$\mathfrak{F}_{\nu,t}(\alpha) = \lim_{n\to\infty}\int_{\mathbb{R}} \mathrm{e}^{-\alpha s}\mathrm{d}Q_{\nu_{B_n},t}(s),$$

and so, by the Lévy continuity theorem, there exists unique Borel probability measure $Q_{\nu,t}$ on $\mathbb{R}$ such that

$$\mathfrak{F}_{\nu,t}(\alpha) = \int_{\mathbb{R}} \mathrm{e}^{-\alpha s}\mathrm{d}Q_{\nu,t}(s).$$

□

2.2 Proof of Theorem 1.4.

(1) By definition of the relative entropy we have

$$\mathrm{Ent}(\omega_t|\omega) = \langle \Omega_t, \log \Delta_{\omega|\omega_t}\Omega_t\rangle.$$

Lemma 2.1 and the functional calculus allow us to write

$$\mathrm{Ent}(\omega_t|\omega) = \langle \Omega, \mathrm{e}^{\mathrm{i}t\mathcal{L}} \log \Delta_{\omega|\omega_t}\mathrm{e}^{-\mathrm{i}t\mathcal{L}}\Omega\rangle = \langle \Omega, \log \Delta_{\omega_{-t}|\omega}\Omega\rangle = -\int_{\mathbb{R}} s\,\mathrm{d}Q_{\omega,t}(s).$$

(2) Since $\Omega \in \mathrm{Dom}\,(\Delta_{\omega_{-t}|\omega}^{1/2})$,

$$\int_{\mathbb{R}} \mathrm{e}^{-s}\mathrm{d}Q_{\omega,t}(s)\mathrm{d}s < \infty,$$

and this implies the stated regularity of $\mathfrak{F}_{\omega,t}$. Invoking again Lemma 2.1, and using the fact that $J^*\Delta_{\nu|\mu}J = \Delta_{\mu|\nu}^{-1}$, we write

$$\begin{aligned}
\mathfrak{F}_{\omega,t}(\alpha) &= \langle\Omega, \Delta_{\omega_{-t}|\omega}^{\alpha}\Omega\rangle \\
&= \langle\Omega, \mathrm{e}^{\mathrm{i}t\mathcal{L}}\Delta_{\omega|\omega_t}^{\alpha}\mathrm{e}^{-\mathrm{i}t\mathcal{L}}\Omega\rangle \\
&= \langle J\Delta_{\omega_t|\omega}^{1/2}\Omega, \Delta_{\omega|\omega_t}^{\alpha}J\Delta_{\omega_t|\omega}^{1/2}\Omega\rangle \\
&= \overline{\langle\Omega, \Delta_{\omega_t|\omega}^{1/2}J^*\Delta_{\omega|\omega_t}^{\alpha}J\Delta_{\omega_t|\omega}^{1/2}\Omega\rangle} \\
&= \overline{\langle\Omega, \Delta_{\omega_t|\omega}^{1/2}\Delta_{\omega_t|\omega}^{-\overline{\alpha}}\Delta_{\omega_t|\omega}^{1/2}\Omega\rangle} \\
&= \overline{\langle\Omega, \Delta_{\omega_t|\omega}^{1-\overline{\alpha}}\Omega\rangle},
\end{aligned}$$

which yields the required identity.

(3) By Proposition 2.2, the time-reversal map Θ has a standard anti-unitary implementation U on $\mathcal{H}$, such that $U\Omega = \Omega$. It follows from Part (5) of the above mentioned proposition that

$$\mathfrak{F}_{\omega,t}(\alpha) = \langle\Omega, \Delta_{\omega_{-t}|\omega}^{\alpha}\Omega\rangle = \langle U\Omega, \Delta_{\omega_{-t}|\omega}^{\alpha}U\Omega\rangle = \overline{\langle\Omega, U^*\Delta_{\omega_{-t}|\omega}^{\alpha}U\Omega\rangle} = \overline{\langle\Omega, \Delta_{\omega_t|\omega}^{\overline{\alpha}}\Omega\rangle} = \langle\Omega, \Delta_{\omega_t|\omega}^{\alpha}\Omega\rangle.$$

Thus, time-reversal invariance implies that

$$\mathfrak{F}_{\omega,t}(\alpha) = \mathfrak{F}_{\omega,-t}(\alpha)$$

for $0 \le \operatorname{Re}\alpha \le 1$ and $t \in \mathbb{R}$. Combined with the identity obtained in Part (2), this yields the result.

(4) It follows from Part (3) that for $\alpha \in \mathrm{i}\mathbb{R}$,

$$\int_{\mathbb{R}} \mathrm{e}^{-\alpha s}\mathrm{d}Q_{\omega,t}(s) = \int_{\mathbb{R}} \mathrm{e}^{-\alpha s}\mathrm{e}^{s}\mathrm{d}\overline{Q}_{\omega,t}(s),$$

which ends the proof. □

2.3 Proof of Theorem 1.6.

Let $(\psi_1, \dots, \psi_N)$ be an orthonormal basis of $\mathcal{K}_{\mathcal{S}}$ consisting of eigenvectors of $\nu_{\mathcal{S}}$, and set $P_{ij} = |\psi_i\rangle\langle\psi_j|$. For the GNS-representation $(\mathcal{H}_{\mathcal{S}}, \pi_{\mathcal{S}}, \Omega_{\mathcal{S}})$ of $\mathcal{O}_{\mathcal{S}}$ induced by the state $\omega_{\mathcal{S}}$ given by (1.3) we take

$$\mathcal{H}_{\mathcal{S}} = \mathcal{K}_{\mathcal{S}} \otimes \mathcal{K}_{\mathcal{S}}, \qquad \pi_{\mathcal{S}}(A) = A \otimes \mathbb{1}, \qquad \Omega_{\mathcal{S}} = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\psi_i \otimes \psi_i.$$

If $(\mathcal{H}_{\mathcal{R}}, \pi_{\mathcal{R}}, \Omega_{\mathcal{R}})$ is the GNS-representation of $\mathcal{O}_{\mathcal{R}}$ induced by $\omega_{\mathcal{R}}$, then

$$\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}, \qquad \pi = \pi_{\mathcal{S}} \otimes \pi_{\mathcal{R}}, \qquad \Omega = \Omega_{\mathcal{S}} \otimes \Omega_{\mathcal{R}},$$

and

$$\log \Delta_\omega = \log \Delta_{\omega_{\mathcal{R}}} = \sum_{j=1}^{M} \log \Delta_{\omega_j}.$$

By our ergodicity assumption, it follows from Theorem 1.1(2) and the fact that the Liouvillean of the j-th reservoir is $-\beta_j^{-1} \log \Delta_{\omega_j}$), that $\mathrm{Ker}\ \log \Delta_\omega$ is spanned by the family $(\psi_i \otimes \psi_j \otimes \Omega_{\mathcal{R}})_{1\le i,j\le N}$.

We follow up on the proof of Theorem 1.3. With ν_{B_n} and P as in that proof, we have that

$$\begin{aligned}\mathfrak{F}_{\nu_{B_n},t}(\alpha) &= \langle B_n^* B_n \Omega, P\Delta^\alpha_{\omega_{-t}|\omega}\Omega\rangle \\ &= \sum_{i,j=1}^{N} \langle B_n^* B_n \Omega, \psi_i \otimes \psi_j \otimes \Omega_{\mathcal{R}}\rangle \langle \psi_i \otimes \psi_j \otimes \Omega_{\mathcal{R}}, \Delta^\alpha_{\omega_{-t}|\omega}\Omega\rangle.\end{aligned} \tag{2.3}$$

Note that

$$\psi_i \otimes \psi_j \otimes \Omega_{\mathcal{R}} = [P_{ij}\psi_j] \otimes \psi_j \otimes \Omega_{\mathcal{R}} = \sqrt{N}\pi(P_{ij})\Omega,$$

and so (2.3) gives that

$$\mathfrak{F}_{\nu_{B_n},t}(\alpha) = \sqrt{N}\sum_{i,j=1}^{N} \nu_{B_n}(\pi(P_{ij}))\langle \psi_i \otimes \psi_j \otimes \Omega_{\mathcal{R}}, \Delta^\alpha_{\omega_{-t}|\omega}\Omega\rangle.$$

Since $\lim_{n\to\infty} \nu_{B_n}(\pi(P_{ij})) = \nu_{\mathcal{S}}(P_{ij})$, and since by the choice of the ψ_i's, $\nu_{\mathcal{S}}(P_{ij}) = \lambda_i\delta_{ij}$, where λ_i denotes the eigenvalue of $\nu_{\mathcal{S}}$ for ψ_i, one has

$$\mathfrak{F}_{\nu,t}(\alpha) = \lim_{n\to\infty} \mathfrak{F}_{\nu_{B_n},t}(\alpha) = \sqrt{N}\sum_{i=1}^{N} \lambda_i \langle \psi_i \otimes \psi_i \otimes \Omega_{\mathcal{R}}, \Delta^\alpha_{\omega_{-t}|\omega}\Delta_\omega^{-\alpha}\Omega\rangle.$$

Expressing Ω in terms of the ψ_i's and invoking (1.4) further leads to

$$\begin{aligned}\mathfrak{F}_{\nu,t}(\alpha) &= \sum_{i,k=1}^{N} \lambda_i \langle \psi_i \otimes \psi_i \otimes \Omega_{\mathcal{R}}, [D\omega_{-t} : D\omega]_\alpha \psi_k \otimes \psi_k \otimes \Omega_{\mathcal{R}}\rangle \\ &= \sum_{i=1}^{N} \lambda_i \langle \psi_i \otimes \psi_i \otimes \Omega_{\mathcal{R}}, [D\omega_{-t} : D\omega]_\alpha \psi_i \otimes \psi_i \otimes \Omega_{\mathcal{R}}\rangle,\end{aligned}$$

where the last equality follows from the fact that $[D\omega_{-t} : D\omega]_\alpha \in \mathfrak{M}$. Since

$$\Omega_{\nu_{\mathcal{S}}} = \sum_i \sqrt{\lambda_i}\psi_i \otimes \psi_i$$

is the vector representative of $\nu_{\mathcal{S}}$ in $\mathcal{H}^+_{\mathcal{S}}$, the last identity allows us to conclude that

$$\mathfrak{F}_{\nu,t}(\alpha) = \nu_{\mathcal{S}} \otimes \omega_{\mathcal{R}}([D\omega_{-t} : D\omega]_\alpha).$$

Finally, since $\Omega_{\nu_{\mathcal{S}}} \otimes \Omega_{\mathcal{R}} \in \mathrm{Ker}\ \log \Delta_\omega$, invoking (1.4) again yields

$$\mathfrak{F}_{\nu,t}(\alpha) = \langle \Omega_{\nu_{\mathcal{S}}} \otimes \Omega_{\mathcal{R}}, \Delta^\alpha_{\omega_{-t}|\omega}\Omega_{\nu_{\mathcal{S}}} \otimes \Omega_{\mathcal{R}}\rangle,$$

from which we can conclude that $Q_{\nu,t}$ is the spectral measure of $-\log\Delta_{\omega_{-t}|\omega}$ for the vector $\Omega_{\nu_{\mathcal S}}\otimes\Omega_{\mathcal R}$.

To prove Part (2), let B be a self-adjoint element of $\pi_{\mathcal S}(\mathcal O_{\mathcal S})'\otimes\mathbb 1$ such that $B\Omega=\Omega_{\nu_{\mathcal S}}\otimes\Omega_{\mathcal R}$. B is invertible iff $\nu_{\mathcal S}$ is faithful. For $\alpha\in\mathrm i\mathbb R$,

$$\begin{aligned}\int_{\mathbb R}\mathrm e^{-\alpha s}\mathrm dQ_{\nu,t}(s)&=\langle B\Omega,\Delta^{\alpha}_{\omega_{-t}|\omega}\Omega_{\nu_{\mathcal S}}\otimes\Omega_{\mathcal R}\rangle\\&=\langle B\Omega,\Delta^{\alpha}_{\omega_{-t}|\omega}\Delta_\omega^{-\alpha}\Omega_{\nu_{\mathcal S}}\otimes\Omega_{\mathcal R}\rangle\\&=\langle \Omega,\Delta^{\alpha}_{\omega_{-t}|\omega}\Delta_\omega^{-\alpha}B\Omega_{\nu_{\mathcal S}}\otimes\Omega_{\mathcal R}\rangle\\&=\langle \Omega,\Delta^{\alpha}_{\omega_{-t}|\omega}B\Omega_{\nu_{\mathcal S}}\otimes\Omega_{\mathcal R}\rangle,\end{aligned}\tag{2.4}$$

and similarly, if $\nu_{\mathcal S}$ is faithful,

$$\int_{\mathbb R}\mathrm e^{-\alpha s}\mathrm dQ_{\omega,t}(s)=\langle\Omega_{\nu_{\mathcal S}}\otimes\Omega_{\mathcal R},\Delta^{\alpha}_{\omega_{-t}|\omega}B^{-1}\Omega\rangle.\tag{2.5}$$

The identities (2.4) and the spectral theorem give that the measure $Q_{\nu,t}$ is absolutely continuous with respect to $Q_{\omega,t}$, with $\mathrm dQ_{\nu,t}/\mathrm dQ_{\omega,t}$ equal to the projection of $B\Omega_{\nu_{\mathcal S}}\otimes\Omega_{\mathcal R}$ onto the cyclic subspace $L^2(\mathbb R,\mathrm dQ_{\omega,t})$ generated by $-\log\Delta_{\omega_{-t}|\omega}$ and the vector Ω. Similarly, (2.5) gives that $Q_{\omega,t}$ is absolutely continuous with respect to $Q_{\nu,t}$, with $\mathrm dQ_{\omega,t}/\mathrm dQ_{\nu,t}$ equal to the projection of $B^{-1}\Omega$ onto the cyclic subspace $L^2(\mathbb R,\mathrm dQ_{\nu,t})$ generated $-\log\Delta_{\omega_{-t}|\omega}$ and the vector $\Omega_{\nu_{\mathcal S}}\otimes\Omega_{\mathcal R}$. It remains to prove the estimates (1.11) and (1.12).

By the definition of $\Omega_{\nu_{\mathcal S}}$ we have that for $\alpha\in\mathrm i\mathbb R$,

$$\begin{aligned}\langle\Omega_{\nu_{\mathcal S}}\otimes\Omega_{\mathcal R},\Delta^{\alpha}_{\omega_{-t}|\omega}\Omega_{\nu_{\mathcal S}}\otimes\Omega_{\mathcal R}\rangle&=\sum_{i=1}^N\lambda_i\langle\psi_i\otimes\psi_i\otimes\Omega_{\mathcal R},\Delta^{\alpha}_{\omega_{-t}|\omega}\Delta_\omega^{-\alpha}\psi_i\otimes\psi_i\otimes\Omega_{\mathcal R}\rangle\\&=\sum_{i=1}^N\lambda_i\langle\psi_i\otimes\psi_i\otimes\Omega_{\mathcal R},\Delta^{\alpha}_{\omega_{-t}|\omega}\psi_i\otimes\psi_i\otimes\Omega_{\mathcal R}\rangle.\end{aligned}\tag{2.6}$$

Since for $\epsilon>0$ and $x\in\mathbb R$,

$$\frac{1}{2\mathrm i}\int_{\mathrm i\mathbb R}\mathrm e^{-\alpha(s-x)}\mathrm e^{-|\alpha|\epsilon}\mathrm d\alpha=\frac{\epsilon}{\epsilon^2+(x-s)^2},$$

it follows from (2.6) that

$$\begin{aligned}&\langle\Omega_{\nu_{\mathcal S}}\otimes\Omega_{\mathcal R},\left[\epsilon^2+(x+\log\Delta_{\omega_{-t}|\omega})^2\right]^{-1}\Omega_{\nu_{\mathcal S}}\otimes\Omega_{\mathcal R}\rangle\\&=\sum_i\lambda_i\langle\psi_i\otimes\psi_i\otimes\Omega_{\mathcal R},\left[\epsilon^2+(x+\log\Delta_{\omega_{-t}|\omega})^2\right]^{-1}\psi_i\otimes\psi_i\otimes\Omega_{\mathcal R}\rangle,\end{aligned}$$

which gives that for all $x\in\mathbb R$ and $\epsilon>0$,

$$\frac{\displaystyle\int_{\mathbb R}\frac{\mathrm dQ_{\nu,t}(s)}{\epsilon^2+(x-s)^2}}{\displaystyle\int_{\mathbb R}\frac{\mathrm dQ_{\omega,t}(s)}{\epsilon^2+(x-s)^2}}\le N,\tag{2.7}$$

and if $\lambda_i \geq \gamma$ for all i, that

$$N\gamma \leq \frac{\displaystyle\int_{\mathbb{R}} \frac{\mathrm{d}Q_{\nu,t}(s)}{\epsilon^2 + (x-s)^2}}{\displaystyle\int_{\mathbb{R}} \frac{\mathrm{d}Q_{\omega,t}(s)}{\epsilon^2 + (x-s)^2}}. \tag{2.8}$$

Since[15]

$$\frac{\mathrm{d}Q_{\nu,t}}{\mathrm{d}Q_{\omega,t}}(x) = \lim_{\epsilon \downarrow 0} \frac{\displaystyle\int_{\mathbb{R}} \frac{\mathrm{d}Q_{\nu,t}(s)}{\epsilon^2 + (x-s)^2}}{\displaystyle\int_{\mathbb{R}} \frac{\mathrm{d}Q_{\omega,t}(s)}{\epsilon^2 + (x-s)^2}},$$

the estimates (1.11) and (1.12) follow from (2.7) and (2.8). □

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